

# On Normed Space Valued Paranormed Orlicz Space of Bounded Functions and its Topological Structures

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**Abstract:** This paper attempts to introduce and study a new class  $\ell_\infty(X, U, \Phi, p)$  of Banach space valued function space using Orlicz function  $\Phi$  as a generalization of the basic space  $\ell_\infty$  of bounded sequences. Besides the investigation of linear topological structures of the class  $\ell_\infty(X, U, \Phi, p)$ , our primarily interest is to explore the conditions pertaining to the containment relation between the classes  $\ell_\infty(X, U, \Phi, p)$  in terms of different values of  $p$ .

**Keywords:** Orlicz function, Orlicz space, Paranormed Space, solid space

## 1. Introduction

We begin with recalling some requisites that are used in this paper. The concept of paranormed space is closely related to linear metric space; see [23] and its studies on sequence spaces were initiated by Maddox [10] and many others.

**Definition 1.1:** A paranormed space  $(X, H)$  is a linear space  $X$  with zero element  $\theta$  together with a function  $H: X \rightarrow \mathbf{R}^+$  (called a paranorm on  $X$ ) which satisfies the following axioms:

$$PN_1: H(\theta) = 0;$$

$$PN_2: H(x) = H(-x) \text{ for all } x \in X;$$

$$PN_3: H(x_1 + x_2) \leq H(x_1) + H(x_2) \text{ for all } x_1, x_2 \in X; \text{ and}$$

$$PN_4: \text{Scalar multiplication is continuous.}$$

Note that the continuity of scalar multiplication is equivalent to

$$(i) \text{ if } H(x_n) \rightarrow 0 \text{ and } \alpha_n \rightarrow \alpha \text{ as } n \rightarrow \infty, \text{ then}$$

$$H(\alpha_n x_n) \rightarrow 0 \text{ as } n \rightarrow \infty; \text{ and}$$

$$(ii) \text{ if } \alpha_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } x \text{ be any element in } X, \text{ then } H(\alpha_n x) \rightarrow 0, \text{ see Wilansky [23].}$$

A paranorm is called total if  $H(x) = 0$  implies  $x = \theta$ .

Various types of paranormed sequence spaces are investigated and studied by the several workers, for instances, see [1], [2], [6] and [14].

**Definition 1.2:** By an Orlicz function, we mean a continuous, non decreasing and convex function

$$\Phi: [0, \infty) \rightarrow [0, \infty) \text{ with}$$

$$(i) \Phi(0) = 0,$$

$$(ii) \Phi(x) > 0 \text{ for } x > 0, \text{ and}$$

$$(iii) \Phi(x) \rightarrow \infty \text{ as } x \rightarrow \infty.$$

An Orlicz function  $\Phi$  is said to satisfy  $\Delta_2$ -condition for all values of  $t$ , if there exists a constant  $K > 0$  such that

$$\Phi(2t) \leq K \Phi(t), \text{ for all } t \geq 0.$$

The  $\Delta_2$ -condition is equivalent to the satisfaction of inequality

$$\Phi(Lt) \leq K L \Phi(t)$$

for all values of  $t$  for which  $L > 1$ , (see, Krasnosel'skiĭ and Rutickiĭ, [8]).

**Definition 1.3:** Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to construct the sequence space

$\ell_\Phi(X)$  of scalars  $\langle x_k \rangle$  such that  $\ell_\Phi(X)$

$$= \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{r}\right) < \infty \text{ for some } r > 0 \right\}.$$

The space  $\ell_\Phi(X)$  with the norm

$$\|x\|_\Phi = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{r}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space  $\ell_\Phi(X)$  is closely related to the space  $\ell_p$  which is an Orlicz sequence space with

$$\Phi(x) = x^p : 1 \leq p < \infty.$$

Subsequently, various types of algebraic and topological properties of sequence and function spaces using Orlicz function as the generalizations of well known sequence and function spaces have been introduced and studied in [1], [2], [3], [4], [5], [6], [7], [11], [12], [13], [14], [15], [16], [17], [19] and [20].

**Definition 1.4:** A normed space  $(X, \|\cdot\|)$  is a linear space  $X$  with zero element  $\theta$  together with the mapping  $\|\cdot\|: X \rightarrow \mathbf{R}^+$  (called norm on  $X$ ) such that for all  $x, x_1, x_2 \in X$  and  $\alpha \in \mathbf{C}$ , we have

$$N_1: \|x\| \geq 0 \text{ and } \|x\| = 0 \text{ if and only if } x = \theta;$$

$$N_2: \|\alpha x\| = |\alpha| \|x\|; \text{ and}$$

$$N_3: \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|.$$

In fact, the works [10], [18], [19],[20], [21] and [22] have been introduced and studied the algebraic and topological properties of various sequence and function spaces in normed space. All these sequence and function spaces generalize and unify various existing basic sequence spaces  $\ell_\infty$  studied in Functional Analysis.

**Definition 1.5:** Let  $U$  be a normed space and

$V(U) = \{ \phi : X \rightarrow U \}$  be the class of  $U$ -valued functions. Then  $V(U)$  is called *solid* if  $\phi \in V(U)$  and scalars  $\alpha(x), x \in X$  such that  $|\alpha(x)| \leq 1, x \in X$  implies  $\alpha(x)\phi(x) \in V(U)$ .

## 2. The Class $\ell_\infty(X, U, \Phi, p)$ of Normed Space Valued Functions

Let  $X$  be an arbitrary non empty set (not necessarily countable) and  $\mathcal{F}(X)$  be the collection of all finite subsets  $J$  of  $X$  directed by set theoretic inclusion relation. Let  $(U, \|\cdot\|)$  be a normed space over the field of complex number  $\mathbb{C}$ . Let  $p$  and  $q$  be any functions on  $X \rightarrow \mathbb{R}^+$ , the set of positive real numbers, and

$$\ell_\infty(X, \mathbb{R}^+) = \{ p : X \rightarrow \mathbb{R}^+ \text{ such that } \sup_x p(x) < \infty \}.$$

We now introduce the following new class of normed space  $U$ -valued functions:

$$\ell_\infty(X, U, \Phi, p) = \{ \phi : X \rightarrow U: \text{ for some } r > 0, \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) < \infty \} \dots (2.1)$$

Further when  $p : X \rightarrow \mathbb{R}^+$  is a function such that  $p(x) = 1$  for all  $x$ , then  $\ell_\infty(X, U, \Phi, p)$  will be denoted by  $\ell_\infty(X, U, \Phi)$ . Besides studying the class (2.1), we also deal the following class of normed space -  $U$  valued functions

$$\bar{\ell}_\infty(X, U, \Phi, p) = \{ \phi : X \rightarrow U: \text{ for every } r > 0, \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) < \infty \} \dots (2.2)$$

Actually, these classes are the generalizations of the sequence and function spaces, studied in [11], [12], [13], [19] and [20]) using norm.

## 3. Main Results

In this section, we shall investigate some results that characterize the linear topological structures of  $\ell_\infty(X, U, \Phi, p)$  of normed space  $U$ -valued functions by endowing it with suitable natural paranorm  $H$ .

Beside this, we shall also explore the conditions in terms of different  $p$  so that a class  $\ell_\infty(X, U, \Phi, p)$  is contained in or equal to another similar class and thereby derive the conditions of their equality. As far as the linear space structure of the class over the field  $\mathbb{C}$  of complex numbers is concerned, we throughout take pointwise operations i.e., for functions  $\phi, \psi$  and scalar  $\alpha, (\phi + \psi)(x) = \phi(x) + \psi(x)$  and  $(\alpha\phi)(x) = \alpha\phi(x), x \in X$

Moreover, we shall denote the zero element of this space by  $\theta$  by which we shall mean the function  $\theta : X \rightarrow U$  such that  $\theta(x) = \theta$ , for all  $x \in X$ .

We shall also frequently use the notations

$$L = \sup_x p(x) \text{ and for scalar } \alpha, A[\alpha] = \max(1, |\alpha|).$$

But when the functions  $p(x)$  and  $q(x)$  occur, then to distinguish  $L$  we use the notations  $L(p)$  and  $L(q)$  respectively. Following inequality will be used in this paper

$$|a + b|^{p(x)} \leq D\{|a|^{p(x)} + |b|^{p(x)}\},$$

where

$$a, b \in \mathbb{C} \text{ and } D = A[2^{L-1}] \text{ and for all } x, w(x) = \frac{q(x)}{p(x)}.$$

**Theorem 3.1:** If  $\Phi$  satisfies the  $\Delta_2$ -condition, then

$$\ell_\infty(X, U, \Phi, p) = \bar{\ell}_\infty(X, U, \Phi, p).$$

**Proof:**

It suffices to show that  $\ell_\infty(X, U, \Phi, p)$  is a subspace of  $\bar{\ell}_\infty(X, U, \Phi, p)$ , since the reverse inclusion is always true. Let  $\phi \in \ell_\infty(X, U, \Phi, p), r > 0$  be associated with  $\phi$ , then we have

$$\sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) < \infty$$

and hence

$$\Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) < \infty.$$

Let us consider an arbitrary  $r_1 > 0$ .

If  $r \leq r_1$ , then by non decreasing property of  $\Phi$ , we have

$$\sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r_1} \right) \leq \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) < \infty,$$

shows that  $\phi \in \bar{\ell}_\infty(X, U, \Phi, p)$ .

On the other hand, if  $r > r_1$ , then put  $t = \frac{r}{r_1} > 1$ .

Since  $\Phi$  satisfies the  $\Delta_2$ -condition. There exists a constant  $K > 0$  such that

$$\sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r_1} \right) = \sup_{x \in X} \Phi \left( \frac{t \|\phi(x)\|^{p(x)}}{r} \right) \leq K \cdot \frac{r}{r_1} \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right)$$

< ∞,

and so  $\phi \in \bar{\ell}_\infty(X, U, \Phi, p)$ . This completes the proof.

Theorem 3.2: The class  $\ell_\infty(X, U, \Phi, p)$  forms a linear space over the field  $\mathbb{C}$  with respect to the pointwise vector operations if  $\langle p(x) \rangle_{x \in X}$  is bounded above.

Proof:

Suppose  $\langle p(x) \rangle_{x \in X}$  is bounded above,  $\phi, \psi \in \ell_\infty(X, U, \Phi, p)$ ,  $r_1 > 0$  and  $r_2 > 0$  are associated with  $\phi$  and  $\psi$  respectively and  $\alpha, \beta \in \mathbb{C}$ . Then we have

$$\sup_{x \in X} \Phi \left( \frac{\|\alpha\phi(x)\|^{p(x)}}{r_1} \right) < \infty,$$

and

$$\sup_{x \in X} \Phi \left( \frac{\|\beta\psi(x)\|^{p(x)}}{r_2} \right) < \infty.$$

We now choose  $r$  such that

$$2 D r_1 A [|\alpha|^L] \leq r \text{ and } 2 D r_2 A [|\beta|^L] \leq r.$$

For such  $r$ , using non decreasing and convex properties of  $\Phi$  we have

$$\begin{aligned} & \Phi \left( \frac{\|(\alpha\phi(x) + \beta\psi(x))\|^{p(x)}}{r} \right) \\ & \leq \Phi \left[ \frac{D|\alpha|^{p(x)}}{r} \|\phi(x)\|^{p(x)} + \frac{D|\beta|^{p(x)}}{r} \|\psi(x)\|^{p(x)} \right] \\ & \leq \Phi \left[ \frac{DA [|\alpha|^M]}{r} \|\phi(x)\|^{p(x)} + \frac{DA [|\beta|^M]}{r} \|\psi(x)\|^{p(x)} \right] \\ & \leq \Phi \left( \frac{1}{2 r_1} \|\phi(x)\|^{p(x)} + \frac{1}{2 r_2} \|\psi(x)\|^{p(x)} \right) \\ & \leq \frac{1}{2} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r_1} \right) + \frac{1}{2} \Phi \left( \frac{\|\psi(x)\|^{p(x)}}{r_2} \right) \end{aligned}$$

and therefore,

$$\begin{aligned} & \sup_{x \in X} \Phi \left( \frac{\|(\alpha\phi(x) + \beta\psi(x))\|^{p(x)}}{r} \right) \\ & \leq \frac{1}{2} \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r_1} \right) + \frac{1}{2} \sup_{x \in X} \Phi \left( \frac{\|\psi(x)\|^{p(x)}}{r_2} \right) \\ & < \infty. \end{aligned}$$

This implies that  $\alpha\phi + \beta\psi \in \ell_\infty(X, U, \Phi, p)$  and so  $\ell_\infty(X, U, \Phi, p)$  forms a linear space over  $\mathbb{C}$ .

This completes the proof.

In view of Theorem 3.1 and Theorem 3.2, we have

**Corollary 3.3:** If  $\sup_{x \in X} p(x) < \infty$  and  $\Phi$  satisfies the  $\Delta_2$ -condition, then  $\bar{\ell}_\infty(X, U, \Phi, p)$  forms a linear space over  $\mathbb{C}$

with respect to the pointwise vector operations if  $\langle p(x) \rangle_{x \in X}$  is bounded above.

**Theorem 3.4:**  $\ell_\infty(X, U, \Phi, p)$  forms a solid.

Proof:

Let  $\phi \in \ell_\infty(X, U, \Phi, p)$ ,  $r > 0$  be associated with  $\phi$ . Then we have

$$\sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) < \infty.$$

Now, if we take scalars  $\alpha(x)$ ,  $x \in X$  such that  $|\alpha(x)| \leq 1$ , then

$$\begin{aligned} & \Phi \left( \frac{\|\alpha(x)\phi(x)\|^{p(x)}}{r} \right) \leq \Phi \left( \frac{|\alpha(x)|^{p(x)} \|\phi(x)\|^{p(x)}}{r} \right) \\ & \leq \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) \end{aligned}$$

and therefore

$$\begin{aligned} & \sup_{x \in X} \Phi \left( \frac{\|\alpha(x)\phi(x)\|^{p(x)}}{r} \right) \leq \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)}}{r} \right) \\ & < \infty. \end{aligned}$$

This shows that  $\alpha\phi \in \ell_\infty(X, U, \Phi, p)$  and hence  $\ell_\infty(X, U, \Phi, p)$  is a solid. This completes the proof.

Let  $\phi \in \ell_\infty(X, U, \Phi, p)$ , define a function

$$H : \ell_\infty(X, U, \Phi, p) \rightarrow \mathbb{R} \text{ by}$$

$$H_p(\phi) = \inf \{ r > 0 : \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)/L}}{r} \right) \leq 1 \} \dots (3.1)$$

We prove below that  $\ell_\infty(X, U, \Phi, p)$  forms a paranormed space with respect to  $H$ .

**Theorem 3.5:** If  $\inf_x p(x) = l > 0$ , then  $(\ell_\infty(X, U, \Phi, p), H)$  forms a paranormed space.

Proof:

Since  $\Phi(0) = 0$  and therefore  $H(\theta) = 0$ .

Also,  $H(-\phi) = H(\phi)$  and so  $PN_1$  and  $PN_2$  are obvious.

For  $PN_3$ , for any  $\phi \in \ell_\infty(X, U, \Phi, p)$ , we denote

$$R(\phi) = \{ r > 0 : \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)/L}}{r} \right) \leq 1 \} \dots (3.2)$$

Now for  $\phi, \psi \in \ell_\infty(X, U, \Phi, p)$ , consider  $r_1 \in R(\phi)$  and  $r_2 \in R(\psi)$  and  $r_3 = r_1 + r_2$ .

Then clearly by the convexity of  $\Phi$ , we have

$$\begin{aligned} & \sup_{x \in X} \Phi \left( \frac{\|\phi(x) + \psi(x)\|^{p(x)/L}}{r_1 + r_2} \right) \\ & \leq \sup_{x \in X} \Phi \left( \frac{\|\phi(x)\|^{p(x)/L}}{r_1} \times \frac{r_1}{r_3} + \frac{\|\psi(x)\|^{p(x)/L}}{r_2} \times \frac{r_2}{r_3} \right) \\ & \leq \frac{r_1}{r_3} + \frac{r_2}{r_3} = 1. \end{aligned}$$

In view of (3.2), this shows that  $r_1 + r_2 \in R(\phi + \psi)$  and therefore  $H(\phi + \psi) \leq r_1 + r_2$  for each  $r_1 \in R(\phi)$  and  $r_2 \in R(\psi)$ .

$R(\psi)$ , which implies that  $H(\phi + \psi) \leq H(\phi) + H(\psi)$  i.e.,  $PN_3$  holds.

Finally analogous to the proof of the lines for the sequence space  $\ell_\infty(X, \Phi, \bar{\lambda}, \bar{p}, L)$  studied in [11], the conditions of continuity of scalar multiplication follows:

In the forthcoming Theorems, we shall deal with the class  $\ell_\infty(X, U, \Phi, p)$  to investigate the conditions in terms of different  $p$  so that it is contained in or equal to another class of similar nature.

**Theorem 3.6:** If  $q : X \rightarrow \mathbf{R}^+$  and  $p \in \ell_\infty(X, \mathbf{R}^+)$  then

$$\ell_\infty(X, U, \Phi, p) \subset \ell_\infty(X, U, \Phi, q)$$

if  $\langle w(x) \rangle_{x \in X}$  has finite limit superior.

Proof:

Assume that  $\limsup_x \frac{q(x)}{p(x)} < \infty$ . Then there exists a constant  $d > 0$  such that  $q(x) < d p(x)$  for all but finitely many  $x \in X$ .

Now, if  $\phi \in \ell_\infty(X, U, \Phi, p)$ ,  $r > 0$  is associated with  $\phi$  then we have

$$\sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{p(x)}}{r}\right) < \infty.$$

This shows that there exists some positive real number  $\eta$  satisfying

$$\Phi\left(\frac{\|\phi(x)\|^{p(x)}}{r}\right) \leq \Phi\left(\frac{\eta}{r}\right),$$

for all but finitely many  $x \in X$ . Since  $\Phi$  is non decreasing, therefore  $\|\phi(x)\|^{p(x)} < \eta$ . Since  $q(x) < d p(x)$  and so if  $\|\phi(x)\| \leq 1$ , then obviously  $\|\phi(x)\|^{q(x)} \leq 1$ ;

and on the other hand if  $\|\phi(x)\| > 1$ , then

$$\|\phi(x)\|^{q(x)} < \|\phi(x)\|^{d p(x)} < \eta^d.$$

Therefore

$$\|\phi(x)\|^{q(x)} \leq \max(1, \eta^d),$$

for all but finitely many  $x \in X$ . This shows that for all but finitely many  $x \in X$ ,

$$\Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) \leq \Phi\left(\frac{\max(1, \eta^d)}{r}\right),$$

$$\text{and therefore } \sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) < \infty.$$

This shows that  $\phi \in \ell_\infty(X, U, \Phi, q)$  and hence  $\ell_\infty(X, U, \Phi, p) \subset \ell_\infty(X, U, \Phi, q)$ .

**Theorem 3.7:** If  $q : X \rightarrow \mathbf{R}^+$ ,  $p \in \ell_\infty(X, \mathbf{R}^+)$  and  $\ell_\infty(X, U, \Phi, p) \subset \ell_\infty(X, U, \Phi, q)$  then  $\langle w(x) \rangle_{x \in X}$  has finite limit superior.

Proof:

Assume that

$$\ell_\infty(X, U, \Phi, p) \subset \ell_\infty(X, U, \Phi, q)$$

but  $\limsup_x \frac{q(x)}{p(x)} = \infty$ . Then there exists a sequence

$\langle x_k \rangle$  of distinct points in  $X$  such that for each  $k \geq 1$ ,  $q(x_k) > k p(x_k)$ . ... (3.3)

Now, taking  $u \in U$  with  $\|u\| = 1$ . We define  $\phi : X \rightarrow U$  by

$$x_k = \begin{cases} 2^{1/p(x_k)} u, & \text{for } x = x_k, k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \dots(3.4)$$

Let  $r > 0$ . Then we have

$$\begin{aligned} \sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{p(x)}}{r}\right) &= \sup_{k \geq 1} \Phi\left(\frac{\|2^{1/p(x_k)} u\|^{p(x_k)}}{r}\right) \\ &= \sup_{k \geq 1} \Phi\left(\frac{\|2 u\|^{p(x_k)}}{r}\right) \\ &\leq \Phi\left(\frac{2A [\|u\|^{L(p)}]}{r}\right). \end{aligned}$$

This shows that  $\phi \in \ell_\infty(X, U, \Phi, p)$  but in view of (3.3) and (3.4), we have

$$\begin{aligned} \sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) &= \sup_{k \geq 1} \Phi\left(\frac{2^{q(x_k)/p(x_k)} \|u\|^{q(x_k)}}{r}\right) \\ &\geq \sup_{k \geq 1} \Phi\left(\frac{2^k}{r}\right) \\ &= \infty \end{aligned}$$

and hence  $\phi \notin \ell_\infty(X, U, \Phi, q)$ , a contradiction. This completes the proof.

After combining Theorem 3.5 and Theorem 3.6, we get the following theorem:

**Theorem 3.8:** If  $q : X \rightarrow \mathbf{R}^+$  and  $p \in \ell_\infty(X, \mathbf{R}^+)$  then

$$\ell_\infty(X, U, \Phi, p) \subset \ell_\infty(X, U, \Phi, q)$$

if and only if  $\langle w(x) \rangle_{x \in X}$  has finite limit superior.

In the following example, we show that inspite of satisfaction of the condition of the Theorem 3.8,

$\ell_\infty(X, U, \Phi, p)$  may strictly be contained in  $\ell_\infty(X, U, \Phi, q)$ .

**Example 3.9:**

Let  $X$  be any set and  $\langle x_k \rangle$  be a sequence of distinct points of  $X$ . Take  $u \in U$  such that  $\|u\| = 1$  and define  $\phi : X \rightarrow U$  by

$$\phi(x) = \begin{cases} k^k u, & \text{for } x = x_k, k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \dots (3.5)$$

Further if  $x = x_k$ , then we take  $p(x_k) = k^{-1}$ , if  $k$  is odd integer and  $p(x_k) = k^{-2}$ , if  $k$  is even integer.

Also,  $q(x_k) = k^{-2}$  for all values of  $k$ ,  $p(x) = \frac{1}{2}$ ,

$q(x) = 1$  otherwise.

Then we have,  $\frac{q(x_k)}{p(x_k)} = \begin{cases} k^{-1}, & \text{if } k \text{ is odd integer} \\ 1, & \text{if } k \text{ is even integer.} \end{cases}$

and  $\frac{q(x)}{p(x)} = 2$ , otherwise.

Thus,  $\limsup_x w(x) = 1 < \infty$  i.e., condition of the Theorem 3.8 is satisfied. Let  $r > 0$ . Then we have

$$\begin{aligned} \Phi\left(\frac{\|\phi(x_k)\|^{q(x_k)}}{r}\right) &= \Phi\left(\frac{\|k^k u\|^{1/k^2}}{r}\right) \\ &= \Phi\left(\frac{k^{1/k} \|u\|^{1/k^2}}{r}\right) \\ &\leq \Phi\left(\frac{2 \cdot A \|u\|}{r}\right) \end{aligned}$$

and

$$\Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) = \Phi\left(\frac{\|\theta\|^{q(x)}}{r}\right) = 0, \text{ otherwise.}$$

This shows that  $\phi \in \ell_\infty(X, U, \Phi, q)$ . But on the other hand in view of (3.5) for  $k$  an odd integer,

$$\begin{aligned} \sup_{k \geq 1} \Phi\left(\frac{\|\phi(x_k)\|^{p(x_k)}}{r}\right) &= \sup_{k \geq 1} \Phi\left(\frac{\|k^k u\|^{1/k}}{r}\right) \\ &= \sup_{k \geq 1} \Phi\left(\frac{k \|u\|^{1/k}}{r}\right) \\ &= \sup_{k \geq 1} \Phi\left(\frac{k}{r}\right) = \infty, \end{aligned}$$

implies that  $\phi \notin \ell_\infty(X, U, \Phi, p)$ .

This shows that the condition of Theorem 3.7 is satisfied but  $\ell_\infty(X, U, \Phi, p)$  is strictly contained in  $\ell_\infty(X, U, \Phi, q)$ . This completes the proof.

**Theorem 3.10:** If  $p : X \rightarrow \mathbf{R}^+$  and  $q \in \ell_\infty(X, \mathbf{R}^+)$ , then  $\ell_\infty(X, U, \Phi, q) \subset \ell_\infty(X, U, \Phi, p)$  if  $\langle w(x) \rangle_{x \in X}$  has positive limit inferior.

Proof:

Suppose that  $\langle w(x) \rangle_{x \in X}$  has positive limit inferior i.e.,  $\liminf_x \frac{q(x)}{p(x)} > 0$ . Then there exists  $m > 0$  such that  $q(x) > m p(x)$

for all but finitely many  $x \in X$ . Let  $\phi \in \ell_\infty(X, U, \Phi, q)$ ,  $r > 0$  is associated with  $\phi$ . Then we have

$$\sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) < \infty.$$

This shows that we can find some positive real number  $\eta$  satisfying

$$\Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) < \Phi\left(\frac{\eta}{r}\right),$$

for all but finitely many  $x \in X$ . Since  $\Phi$  is non decreasing, we have

$$\|\phi(x)\|^{q(x)} \leq \eta.$$

Since  $q(x) > m p(x)$  and so if  $\|\phi(x)\| \geq 1$ , then

$$\begin{aligned} \|\phi(x)\|^{p(x)} &\leq \|\phi(x)\|^{q(x)/m} \\ &\leq \eta^{1/m}, \end{aligned}$$

and on the other hand if  $\|\phi(x)\| < 1$ , then obviously

$$\|\phi(x)\|^{p(x)} < 1.$$

Therefore

$$\|\phi(x)\|^{p(x)} \leq \max(1, \eta^{1/m}),$$

for all but finitely many  $x \in X$ . This shows that for all but finitely many  $x \in X$ ,

$$\begin{aligned} \sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{p(x)}}{r}\right) &\leq \Phi\left(\frac{\max(1, \eta^{1/m})}{r}\right) \\ &< \infty. \end{aligned}$$

It follows that  $\phi \in \ell_\infty(X, U, \Phi, p)$  and hence

$$\ell_\infty(X, U, \Phi, q) \subset \ell_\infty(X, U, \Phi, p).$$

This completes the proof.

**Theorem 3.11:** If  $p : X \rightarrow \mathbf{R}^+$ ,  $q \in \ell_\infty(X, \mathbf{R}^+)$  and  $\ell_\infty(X, U, \Phi, q) \subset \ell_\infty(X, U, \Phi, p)$  then  $\langle w(x) \rangle_{x \in X}$  has positive limit inferior.

Proof:

Assume that

$$\ell_\infty(X, U, \Phi, q) \subset \ell_\infty(X, U, \Phi, p)$$

holds but  $\langle w(x) \rangle_{x \in X}$  has zero limit inferior i.e.,

$$\liminf_x \frac{q(x)}{p(x)} = 0. \text{ Then there exists a sequence } \langle x_k \rangle$$

of distinct points in  $X$  such that for  $k \geq 1$ ,

$$k q(x_k) < p(x_k). \dots (3.6)$$

Now, taking  $u \in U$  with  $\|u\| = 1$ , define  $\phi : X \rightarrow U$  by

$$x_k = \begin{cases} 2^{1/q(x_k)} u, & \text{for } x = x_k, k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \dots (3.7)$$

Let  $r > 0$ . Then we have

$$\begin{aligned} \sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{q(x)}}{r}\right) &= \sup_{k \geq 1} \Phi\left(\frac{\|\phi(x_k)\|^{q(x_k)}}{r}\right) \\ &= \sup_{k \geq 1} \Phi\left(\frac{2 \|u\|^{q(x_k)}}{r}\right) \\ &\leq \Phi\left(\frac{2A [\|u\|^{L(q)}]}{r}\right). \end{aligned}$$

This implies that  $\phi \in \ell_\infty(X, U, \Phi, q)$ . But on the other hand, in view of (3.6) and (3.7), we get

$$\begin{aligned} \sup_{x \in X} \Phi\left(\frac{\|\phi(x)\|^{p(x)}}{r}\right) &= \sup_{k \geq 1} \Phi\left(\frac{\|\phi(x_k)\|^{p(x_k)}}{r}\right) \\ &= \sup_{k \geq 1} \Phi\left(\frac{2^{p(x_k)/q(x_k)} \|u\|^{p(x_k)}}{r}\right) \\ &\geq \sup_{k \geq 1} \Phi\left(\frac{2^k}{r}\right) = \infty, \end{aligned}$$

shows that  $\phi \notin \ell_\infty(X, U, \Phi, p)$ , a contradiction. This completes the proof.

After combining Theorem 3.10 and Theorem 3.11, we get the following theorem:

**Theorem 3.12:** If  $p : X \rightarrow \mathbf{R}^+$  and  $q \in \ell_\infty(X, \mathbf{R}^+)$ , then  $\ell_\infty(X, U, \Phi, q) \subset \ell_\infty(X, U, \Phi, p)$

if and only if  $\langle w(x) \rangle_{x \in X}$  has positive limit inferior.

After combining Theorem 3.7 and Theorem 3.10, we get the following theorem:

**Theorem 3.13:** If  $p, q \in \ell_\infty(X, \mathbf{R}^+)$ , then

$\ell_\infty(X, U, \Phi, p) = \ell_\infty(X, U, \Phi, q)$  if and only if  $0 < \liminf_x w(x) \leq \limsup_x w(x) < \infty$ .

**Theorem 3.14:** If  $p \in \ell_\infty(X, \mathbf{R}^+)$ , then

(i)  $\ell_\infty(X, U, \Phi) \subset \ell_\infty(X, U, \Phi, p)$  if and only if  $\limsup_x p(x) < \infty$ ;

(ii)  $\ell_\infty(X, U, \Phi, p) \subset \ell_\infty(X, U, \Phi)$  if and only if  $\liminf_x p(x) > 0$ ; and

(iii)  $\ell_\infty(X, U, \Phi, p) = \ell_\infty(X, U, \Phi)$  if and only if  $0 < \liminf_x p(x) \leq \limsup_x p(x) < \infty$ .

Proof:

If we consider  $p : X \rightarrow \mathbf{R}^+$  such that  $p(x) = 1$  for all  $x \in X$  and  $q$  is replaced by  $p$  in Theorems 3.7, 3.10 and 3.11, we can easily prove the assertions (i), (ii) and (iii) respectively.

#### 4. Conclusion

This paper establishes some of the results that characterize the linear topological structures of the class  $\ell_\infty(X, U, \Phi, p)$  of normed space valued function space using Orlicz function. In fact, these results can be used for further generalization and unification to investigate the properties of the various existing Orlicz function spaces studied in Functional Analysis.

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