Contra qs\(I\)-Continuous Functions in Ideal Bitopological Spaces

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Abstract: In this paper, we apply the notion of qs\(I\)-open sets and qs\(I\)-continuous functions to present and study a new class of functions called contra qs\(I\)-continuous functions in ideal bitopological spaces.

Keywords: Ideal bitopological space, qs\(I\)-open sets, qs\(I\)-continuous functions contra qs\(I\)-continuous functions

AMS Subject classification: 54C08, 54A05

1. Preliminaries

In 1961 Kelly [5] introduced the concept of bitopological spaces as an extension of topological spaces. A bitopological space \((X, \tau_1, \tau_2)\) is a nonempty set \(X\) equipped with two topologies \(\tau_1\) and \(\tau_2\)[5]. The study of quasi open sets in bitopological spaces was initiated by Datta[1] in 1971. In a bitopological space \((X, \tau_1, \tau_2)\) a set \(A\) of \(X\) is said to be quasi open [9] if it is a union of a \(\tau_1\)-open set and a \(\tau_2\)-open set. Complement of a quasi open set is termed quasi closed. Every \(\tau_1\)-open (resp., \(\tau_2\)-open) set is quasi open but the converse may not be true. Any union of quasi open sets of \(X\) is quasi open in \(X\). The intersection of all quasi closed sets which contains \(A\) is called quasi closure of \(A\). It is denoted by \(qc(A)\). The union of quasi open subsets of \(A\) is called quasi interior of \(A\). It is denoted by \(qsi(A)[1]\).

In 1963 N. Levine [8] introduced the concept of semi open sets in topology. A subset \(A\) of a topological space \((X, \tau)\) is called semi open if there exists an open set \(O\) in \(X\) such that \(O \subset A \subset Cl(O)\). Every open set is semi open but the converse may not be true. In 1985, Maheshwari, Chae and Thakur[10] introduced quasi semi open sets in bitopological spaces. A set \(A\) in a bitopological space \((X, \tau_1, \tau_2)\) is called quasi semi open[10] if it is a union of a \(\tau_1\)-semi open set and a \(\tau_2\)-semi open set. Complement of a quasi semi open set is called quasi semi closed. Every \(\tau_1\)-semi open (\(\tau_2\)-semiopen, quasi open) set is quasi semi open but the converse may not be true. Any union of quasi semi open sets of \(X\) is quasi semi open set in \(X\). The intersection of all quasi semi closed sets which contains \(A\) is called quasi semi closure of \(A\). It is denoted by \(qsccl(A)\). The union of quasi semi open subsets of \(A\) is called quasi semi interior of \(A\). It is denoted by \(qsi(A)\).

In 1996 Dontchev[2] introduced a new class of functions called contra-continuous functions. A function \(f: X \to Y\) to be contra continuous if the pre image of every open set of \(Y\) is closed in \(X\). The study of ideal topological spaces was initiated by Kuratowski [7] and Vaidyanathaswamy [13]. An Ideal \(I\) on a topological space \((X, \tau)\) is a non empty collection of subsets of \(X\) which satisfies: \(\forall A \in I\) and \(B \subset A \Rightarrow B \in I\) and \(A \in I\) and \(B \in I\) \(\Rightarrow A \cup B \in I\). If \(\mathcal{P}(X)\) is the set of all subsets of \(X\), in a topological space \((X, \tau)\) a set operator \((\_): \mathcal{P}(X) \to \mathcal{P}(X)\) is called the local mapping[6] of \(A\) with respect to \(\tau\) and \(I\) and is defined as follows: \(A^{\tau}(\_): \mathcal{P}(X) \to \mathcal{P}(X)\) (in short \(A^{\tau}\)) = \(\{x \in X \mid U \cap A \neq \emptyset \} \forall U \in \tau(x)\) where \(\tau(x) = \{U \in \tau \mid x \in U\}[4]\). Given an ideal bitopological space \((X, \tau_1, \tau_2, I)\) the quasi local mapping[3] of \(A\) with respect to \(\tau_1, \tau_2\) and \(I\) denoted by \(A^{\tau_1}(\_): \mathcal{P}(X) \to \mathcal{P}(X)\) (in short \(A^{\tau_1}\)) is defined as follows: \(A^{\tau_1}(\_): \mathcal{P}(X) \to \mathcal{P}(X)\) = \(\{x \in X \mid U \cap A \neq \emptyset \} \forall quasi open set U containing x\).

A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be q-I- open [3] if \(A \subset qsi(A)\). A mapping \(f: (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)\) is called q-I- continuous[3] if \(f^{-1}(\_): \mathcal{P}(Y) \to \mathcal{P}(X)\) is q-I- open in \(X\) for every quasi open set \(V\) of \(Y\). Recently the authors of this paper[11] defined q-I- open sets and q-I- continuous mappings in ideal bitopological spaces.

**Definition 1.1.** [11] Given an ideal bitopological space \((X, \tau_1, \tau_2, I)\) the quasi semi local mapping of \(A\) with respect to \(\tau_1, \tau_2\) and \(I\) denoted by \(A^{\tau_1, \tau_2}(\_): \mathcal{P}(X) \to \mathcal{P}(X)\) (more generally as \(A^{\tau_1, \tau_2}\)) is defined as \(A^{\tau_1, \tau_2}(\_): \mathcal{P}(X) \to \mathcal{P}(X)\) = \(\{x \in X \mid U \cap A \neq \emptyset \} \forall quasi semi-open set U containing x\).

**Definition 1.2.** [11] A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is q-I- open if \(A \subset qsi(A)\) and q-I- closed if its complement is q-I- open. If the set \(A\) is q-I- open and q-I- closed, then it is called q-I-clopen.

**Remark 1.1.** [11] Every q-I- open set is q-I- open but the converse is not true
**Remark 1.2.** [11] The concepts of q-I- open sets and quasi semi open sets are independent.

**Definition 1.3.** [11] A mapping \(f: (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)\) is called a q-I- continuous if \(f^{-1}(\_): \mathcal{P}(Y) \to \mathcal{P}(X)\) is q-I- open set in \(X\) for every quasi open set \(V\) of \(Y\).

**Remark 1.3.** [11] Every q-I- continuous mapping is q-I- continuous but the converse is not true.
Definition 1.4. [11] In an ideal bitopological space \((X, \tau_1, \tau_2, I)\) the quasi semi closure of \(A\) of \(X\) denoted by \(qscl (A)\) is defined by \(qscl (A) = A \cup A_{qs}\).

Definition 1.5. [11] A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be a quasi \(I\)-neighbourhood of a point \(x\) if \(x\) is an \(I\)-open set O such that \(x\in O \subseteq A\).

Definition 1.6. [11] Let \(A\) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) and \(x\in X\). Then \(x\) is called a quasi \(I\)-interior point of \(A\) if \(\exists V\) a quasi \(I\)-open set in \(X\) such that \(x\in V \subseteq A\).

Definition 1.9. [12] The set of all quasi \(I\)-open sets \(O\) such that \(x\in O \subseteq A\) is called the quasi \(I\)-closure of \(A\).

Definition 1.10. [11] A mapping \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) is called quasi \(I\)-irresolute if \(f^{-1}(V)\) is a quasi \(I\)-open set in \(X\) for every quasi \(I\)-open set \(V\) of \(Y\).

Definition 1.12. [11] A mapping \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) is called quasi \(I\)-irresolute if \(f^{-1}(V)\) is a quasi \(I\)-open set in \(X\) for every quasi semi open set \(V\) of \(Y\).

2. Contra quasi \(I\)-continuous functions

Definition 2.1. A function \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) is called contra quasi \(I\)-continuous for \(f^{-1}(V)\) is quasi \(I\)-closed in \(X\) for each quasi open set \(V\) of \(Y\).

Theorem 2.1. For a function \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\), the following are equivalent:

a) \(f\) is contra quasi \(I\)-continuous.

b) For every quasi closed subset \(F\) of \(Y\), \(f^{-1}(F)\) is quasi \(I\)-open in \(X\).

c) For each \(x \in X\) and each quasi closed subset \(F\) of \(Y\) with \(f(x) \in F\), there exists a quasi \(I\)-open subset \(U\) of \(X\) with \(x \in U\) such that \(f(U) \subseteq F\).

Proof: (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (c) are obvious.

(c) \(\Rightarrow\) (b) Let \(F\) be any quasi closed subset of \(Y\). If \(x \in F\), then \(f^{-1}(F)\) is quasi \(I\)-open in \(X\). Therefore, \(f^{-1}(F) = U_x\), where \(U_x\) is a quasi \(I\)-open subset of \(X\) with \(x \in U_x\) such that \(f(U_x) \subseteq F\). Hence, we get \(f^{-1}(F)\) is quasi \(I\)-open. [11]

Remark 2.1. Every contra quasi \(I\)-continuous function is contra quasi \(I\)-continuous, but the converse need not be true.

Remark 2.2. The concepts of quasi \(I\)-continuity and contra quasi \(I\)-continuity are independent of each other.

Theorem 2.2. If a function \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) is contra quasi \(I\)-continuous and \(Y\) is regular, then \(f\) is quasi \(I\)-continuous.

Proof: Let \(x \subseteq X\) and let \(V\) be a quasi open subset of \(Y\) with \(f(x) \subseteq V\). Since \(Y\) is regular, there exists a quasi open set \(W\) in \(Y\) such that \(f(x) \subseteq W \subseteq cl(W) \subseteq V\). Since \(f\) is contra quasi \(I\)-continuous, by Theorem 2.1, there exists a quasi \(I\)-open set \(U\) in \(X\) with \(x \subseteq U\) such that \(f(U) \subseteq cl(W) \subseteq V\). Hence, \(f\) is quasi \(I\)-continuous. [11]

Remark 2.3. If \(f\) is contra quasi \(I\)-continuous and \(Y\) is regular, then \(f\) need not be contra quasi \(I\)-continuous.

Definition 2.2. A topological space \((X, \tau_1, \tau_2, I)\) is said to be quasi \(I\)-connected if \(X\) is not the union of two disjoint non-empty quasi \(I\)-open subsets of \(X\).

Theorem 2.3. If \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) is a contra quasi \(I\)-continuous function from a quasi \(I\)-connected space \(X\) onto any space \(Y\), then \(Y\) is not a discrete space.

Proof: Suppose that \(Y\) is discrete. Let \(A\) be a proper non-empty quasi clopen set in \(Y\). Then \(f^{-1}(A)\) is a proper non-empty quasi \(I\)-clopen subset of \(X\), which contradicts the fact that \(X\) is quasi \(I\)-connected.

Theorem 2.4. A contra quasi \(I\)-continuous image of a quasi \(I\)-connected space is connected.

Proof: Let \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) be a contra quasi \(I\)-continuous function from a quasi \(I\)-connected space \(X\) onto a space \(Y\). Assume that \(Y\) is disconnected. Then \(Y = A \cup B\), where \(A\) and \(B\) are non-empty quasi clopen sets in \(Y\) with \(A \cap B = \emptyset\). Since \(f\) is contra quasi \(I\)-continuous, we have that \(f^{-1}(A)\) and \(f^{-1}(B)\) are quasi \(I\)-open non-empty sets in \(X\) with \(f^{-1}(A) \cap f^{-1}(B) = \emptyset\). This means that \(X\) is not semi-connected, which is a contradiction. Then \(Y\) is connected.

Definition 2.5. A space \((X, \tau_1, \tau_2, I)\) is said to be quasi \(I\)-normal if each pair of non-empty quasi clopen sets can be separated by disjoint quasi \(I\)-open sets.

Definition 2.6. A space \((X, \tau_1, \tau_2, I)\) is said to be quasi \(I\)-normal if each pair of non-empty quasi clopen sets can be separated by disjoint quasi clopen sets.

Theorem 2.7. If \(f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)\) is contra quasi \(I\)-continuous, closed and one-one, \(Y\) is ultra normal, then \(X\) is quasi \(I\)-normal.

Proof: Let \(C_1\) and \(C_2\) be disjoint quasi clopen subsets of \(X\). Since \(f\) is closed and one-one \(f(C_1)\) and \(f(C_2)\) are disjoint quasi closed subsets of \(Y\). But \(Y\) is ultra normal, so \(f(C_1)\) and \(f(C_2)\) are separated by disjoint quasi clopen sets \(V_1\) and \(V_2\), respectively.

Since \(f\) is contra quasi \(I\)-continuous, \(f^{-1}(V_1)\) and \(f^{-1}(V_2)\) are quasi \(I\)-open, where \(C_1 \subseteq f^{-1}(V_1)\), \(C_2 \subseteq f^{-1}(V_2)\) and \(f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset\). Hence, \(X\) is quasi \(I\)-normal.
**Definition 2.5.** A space $(X, \tau_1, \tau_2, I)$ is said to be $qsI$-compact if every $qsI$-open cover of $X$ has a finite subcover.

**Definition 2.6.** A mapping $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is called contra $qsI$-irresolute if $f^{-1}(V)$ is a $qsI$-closed set in $X$ for every quasi semi open set $V$ of $Y$.

**Remark 2.5.** Contra $qsI$-irresoluteness and $qsI$-irresoluteness are independent.

**Definition 2.7** A function $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ is called quasi-irresolute if $f^{-1}(V)$ is $qsI_2$-open in $X$ for each $qsI_2$-open set $V$ of $Y$.

**Definition 2.8.** A function $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ is called contra quasi-irresolute if $f^{-1}(V)$ is $qsI_2$-closed in $X$ for each $qsI_2$-open set $V$ of $Y$.

The following two remarks are evident from the definition

**Remark 2.5.** Contra quasi-irresoluteness and quasi-irresoluteness are independent.

**Remark 2.6.** Contra quasi-irresolute function is contra $qsI$-continuous, but the converse is not true.

**Theorem 2.6.** A function $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ is quasi-irresolute if and only if the inverse image of each $qsI_2$-closed set in $Y$ is $qsI_1$-open in $X$.

**Theorem 2.7.** Let $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ and $g: (Y, \sigma_1, \sigma_2, I_2) \rightarrow (Z, \rho_1, \rho_2, I_3)$ Then,

1. $gof$ is contra quasi-irresolute if $g$ is quasi-irresolute and $f$ is contra quasi-irresolute.
2. $gof$ is contra quasi-irresolute if $g$ is contra quasi-irresolute and $f$ is quasi-irresolute.

**Theorem 2.8.** Let $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ and $g: (Y, \sigma_1, \sigma_2, I_2) \rightarrow (Z, \rho_1, \rho_2, I_3)$ Then,

1. $gof$ is contra $qsI$-continuous if $g$ is continuous and $f$ is contra $qsI$-continuous.
2. $gof$ is contra $qsI$-continuous if $g$ is $qsI$-continuous and $f$ is contra quasi-irresolute

The next theorem follows from the fact that a function $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ is $qsI$-open [11] if for each quasi open set $U$ of $X$, $f(U)$ is $qsI$-open in $Y$.

**Theorem 2.9.** Let $f: (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2, I_2)$ be onto, quasi-irresolute and $qsI$-open and let $g: (Y, \sigma_1, \sigma_2, I_2) \rightarrow (Z, \rho_1, \rho_2, I_3)$ be any function. Then $gof$ is contra $qsI$-continuous if and only if $g$ is contra $qsI$-continuous.

**Proof:** Necessary: Let $gof$ be contra $qsI$-continuous and $C$ a quasi closed subset of $Z$. Then $(gof)^{-1}(C)$ is a $qsI$-open subset of $X$. Thus $f^{-1}(g^{-1}(C))$ is $qsI$-open in $X$. Since $f$ is $qsI$-open, $f(f^{-1}(g^{-1}(C)))$ is $qsI$-open subset of $Y$. So $g^{-1}(C)$ is $qs$-open in $Y$. Therefore, $g$ is contra $qsI$-continuous.

Sufficient: Obvious.

**References**


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**Mandira Kar** has completed her M. Sc & M. Phil (Mathematics) degree from RDVV, Jabalpur in 1983 and 1984 respectively. She has worked as a Lecturer from 1984-87 in St. Joseph’s Convent. Asst. Prof. Govt. P.G College, Chhindwara from 1987-1992. Presently she is Prof. in St. Aloysius College, Jabalpur. She has presented more than 45 Research papers in International and National conferences. She has also been the Resource person in 06 National Conferences and UGC scheme (coaching) NET Exam. She has been a Visiting Faculty of FOMS, CMM, Jabalpur. She has 20 publications to her credit. Her specialization is Ideal Bitopology. She is the Assistant Editor of Global Research Journal on Mathematics and Science Education ISSN 2278-0769. She is also a Guest lecturer in esteemed institutions. She is the Member of many Academic bodies.