Lattice Points on the Cone $X^2 + 9Y^2 = 50Z^2$

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Abstract: ***The ternary quadratic homogeneous equation representing cone given by $x^2 + 9y^2 = 50z^2$ is analyzed for its non- zero distinct integer points on it. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns are presented.***

Keywords: Diophantine equation, Ternary quadratic, integral solutions, special numbers, a few interesting Relation

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Notations:
P_n^m = Pyramid number of rank n with size m
$T_{m,n}$ = Polygonal number of rank n with size m

1. Introduction

The ternary quadratic Diophantine equation offers an unlimited field for research due to their variety [1, 10]. For an extensive review of various problems one may refer [2, 9]. This communication concerns with yet another interesting ternary quadratic equation representing $x^2 + 9y^2 = 50z^2$ a cone for determining its infinitely many non- zero integral points. Also a few interesting relations among the solutions and special number are presented.

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solutions is

$$x^2 + 9y^2 = 50z^2 \quad (1)$$

Assume $z(a, b) = a^2 + 9b^2$, Where $a, b > 0$ (2)

We illustrate below five different patterns of non-zero distinct integer solutions to (1).

2.1 Pattern: 1

Write 50 as $50 = (7 + i)(7 - i)$ (3)

Substituting (2) and (3) in (1), employing the method of factorization, define

$$(x + 3iy)(x - 3iy) = (7 + i)(7 - i)(a + 3ib)^2 (a - 3ib)^2$$

Equating real and imaginary parts, we get

$$x = x(a, b) = 7a^2 - 63b^2 - 6ab \quad (4)$$

$$y = y(a, b) = \frac{1}{3}[a^2 - 9b^2 + 42ab] \quad (5)$$

Thus (2), (4), (5) represents non-zero distinct integral solutions of (1) in two parameters As our interest is on finding integer solutions, we choose a and b suitably so that the value of x, y and z are in integers. In what follows the values of a, b and the corresponding integer solutions are exhibited.

Case 1: Let a = 3A, b = 3B.
The corresponding solutions of (1) are

$x = x(A, B) = 63A^2 - 567B^2 - 54AB$
$y = y(A, B) = 3A^2 - 27B^2 + 126AB$
$z = z(A, B) = 9A^2 + 81B^2$

Properties:
1. $x(A,1) + T_{128, A} \equiv 1 \pmod 8$
2. $2y(A, A+1) - x(A, A+1) = 2700\, T_{3, A}$
3. $y(A,1) - T_{8, A} \equiv -27 \pmod{128}$
4. $6\{z(A, A(A+1)) - 204T^2_{3,A}\}$ is a nasty number

Case 2: Let a = 3A, b = B.
The corresponding solutions of (1) are
$x = x(A, B) = 63A^2 - 63B^2 - 18AB$
$y = y(A, B) = 3A^2 - 3B^2 + 42AB$
$z = z(A, B) = 9A^2 + 9B^2$

Properties:
1. $x(A(A+1), A+2) - 21y(A(A+1), A+2) + 5400P_A^5 = 0$
2. $x(A, A+1) + 9z(A, A+1) - 6T_{4, A} + 18Pr_A$ a nasty number
3. $T_{128, A} - x(A, 1) \equiv 19 \pmod{44}$
4. $x(A, A(A+1) - 12P_A^5 + 12T^2_{3,A} - 3\, T_{4, A}$ a nasty number.

Case 3: Let a = 3A, b = 3A + 1.
The corresponding solutions of (1) are

$x = x(A, A) = -558A^2 - 396A - 63$
$y = y(A, A) = 102A^2 + 24A - 3$
$z = z(A, A) = 90A^2 + 54A + 9$

Properties:
1. $z(A(A+1),1) - 108T_{3, A} - 360T^2_{3,A} \equiv 0 \pmod 9$
2. $T_{206, A} - y(A, 1) \equiv 3 \pmod{24}$
3. $y(A,A) - z(A,A) - 12\, T_{4, A} \equiv 0 \pmod 6$

2.2 Pattern: 2

Instead of (3), write 50 as

$$50 = (1 + 7i)(1 - 7i) \quad (6)$$

Following the procedure presented in pattern: 1, the corresponding values of x and y are

$$x = x(a, b) = a^2 - 9b^2 - 42ab \quad (7)$$

$$y = y(a, b) = \frac{1}{3}[7a^2 - 63b^2 + 6ab] \quad (8)$$

Case 1: Let a = 3A, b = 3B.

The corresponding solutions of (1) are

$x = x(A, B) = 9A^2 - 81B^2 - 378AB$
$y = y(A, B) = 21A^2 - 189B^2 + 18AB$
$z = z(A, B) = 9A^2 + 81B^2$

Properties:
1. $7x(A, A+1) - 3y(A, A+1) + 5400 T_{3, A} = 0$
2. $x(A, A) + 450 T_{4, A} = 0$
3. $y(A, A) + 150 T_{4, A} = 0$
4. $6\{x(A, A(A+1)) + 736P_A^5 + 344T_{3,A}^2\}$ is a nasty number
5. $y(2A, A) + T_{12, A} \equiv 0 \pmod 4$
6. $y(A, A(A+1)) + 12P_A^5 + 36T_{3,A}^2 + 3T_{4, A}$ is a nasty number

Case 2: Let a = 3A, b = B.

The corresponding solutions of (1) are
$x = x(A, B) = 9A^2 - 9B^2 - 126AB$
$y = y(A, B) = 21A^2 - 21B^2 + 6AB$
$z = z(A, B) = 9A^2 + 9B^2$

Properties:
1. $x(A, A) + 126 T_{4, A} = 0$
2. $y(A, A) - 6T_{4, A} = 0$
3. $x(A, A) + y(A, A)$ a Nasty number
4. $y(A, A) + z(A, A)$ a Nasty number
5. $y(2A, A) + T_{8, A} \equiv 0 \pmod 2$
6. $y(A, A(A+1)) - 12T_{3,A}^2 + 3T_{4, A}$ a Nasty number.

Case 3: Let a = 3A, b = 3A + 1.
The corresponding solutions of (1) are
$x = x(A, A) = -450A^2 - 180A - 9$
$y = y(A, A) = -150A^2 - 120A - 21$
$z = z(A, A) = 90A^2 + 54A + 9$

Properties:
1. $z(A, A) + 6T_{4, A} - 9$ a Nasty number
2. $\frac{1}{2}\{x(A, A) - y(A, A)\}$ a Nasty number
3. $y(A,A) + z(A,A) \equiv 0 \pmod 6$

2.3: Pattern: 3
(1) is written in the form of ratio as

$$\frac{x+z}{7z+3y} = \frac{7z-3y}{x-z} = \frac{A}{B}, B \neq 0.$$

which is equivalent to the system of equations,

$$Bx - 3Ay + (B-7A)z = 0 \quad (9)$$
$$Ax + 3By - (A+7B)z = 0 \quad (10)$$

Applying the method of cross multiplication the integer solutions (1) are given by

$$x = x(A, B) = 3A^2 - 3B^2 + 42AB \quad (11)$$
$$y = y(A, B) = 7A^2 - 7B^2 - 2AB \quad (12)$$
$$z = z(A, B) = 3A^2 + 3B^2 \quad (13)$$

which represents non- zero distinct integral solutions of (1) in two parameters.

Properties:

1. $x(A, A+1) - z(A, A+1) - 72T_{3, A} \equiv 0 \pmod 6$
2. $x(A(A+1), A+2) - z(A(A+1), A+2) - 252P_A^3$ a Nasty number
3. $7x(A^2, A+1) - 3y(A^2, A+1) - 600P_A^5 = 0$
4. $x(A, 1) - 3T_{4, A} \equiv -3 \pmod{42}$
5. $3\{x(A,A) + y(A,A)\}$ a Nasty number
6. $3\{z(A,A) - y(A,A)\}$ a Nasty number
7. $y(A, 1) - T_{16, A} \equiv -3 \pmod 4$

2.4 Pattern: 4

$$\text{(1) is written as } 50z^2 - 9y^2 = x^2 = x^2 * 1 \quad (14)$$
$$\text{Assume } x(a, b) = 50a^2 - 9b^2 \quad (15)$$

Write (1) as

$$1 = (\sqrt{50} + 7)(\sqrt{50} - 7) \quad (16)$$

Substituting (15) and (16) in (14) and applying the method of factorization, define

$$(\sqrt{50}z + 3y) = (\sqrt{50} + 7)(\sqrt{50}a + 3b)^2$$

Equating rational and irrational parts, we have

$y = y(a, b) = \frac{1}{3}[350a^2 + 63b^2 + 300ab]$
$z = z(a, b) = 50a^2 + 9b^2 + 42ab$

Case 1: let a = 3A, b = 3B
The corresponding solutions of (1) are
$x = x(A, B) = 450A^2 - 81B^2$

$$y = y(A, B) = 1050A^2 + 189B^2 + 900AB \quad (17)$$
$$z = z(A, B) = 450A^2 + 81B^2 + 378AB \quad (18)$$

Properties:

1. $y(A, A(A+1)) - 756T_{3,A}^2 - 1800P_A^5 - 1050T_{4,A} = 0$
2. $x(A, A) - 369T_{4, A} = 0$
3. $y(A,1) - T_{2102, A} \equiv 189 \pmod{1949}$
4. $z(A, 1) - T_{902, A} \equiv 81 \pmod{827}$

Case 2: Let a = 3A, b = B.
The corresponding solutions of (1) are
$x = x(A, B) = 450A^2 - 9B^2$
$y = y(A, B) = 1050A^2 + 21B^2 + 300AB$
$z = z(A, B) = 450A^2 + 9B^2 + 126AB$

Properties:
1. $y(A, A(A+1)) - 84T_{3,A}^2 - 600P_A^5 - 1050T_{4,A} = 0$
2. $x(A, A) - 441T_{4, A} = 0$
3. $y(A,1) - T_{2102, A} \equiv 2 \pmod{19}$
4. $z(A, 1) - T_{902, A} \equiv 9 \pmod{575}$
5. $x(A, 1) + z(A, 1) - T_{1002, A} \equiv 0 \pmod{625}$

Case 3: Let a = 3A, b = 3A + 1.
The corresponding solutions of (1) are
$x = x(A, A) = 369A^2 - 54A + 1$
$y = y(A, A) = 2139A^2 + 426A + 21$
$z = z(A, A) = 709A^2 + 180A + 9$

Properties:
1. $x(A, A) - 369T_{4, A} \equiv 1 \pmod{54}$
2. $y(A,1) - T_{4280, A} \equiv 1 \pmod 4$
3. $z(A, 1) - T_{1420, A} \equiv 1 \pmod 8$

4. x (1, 1) – 136 Nasty Number

2.5 Pattern: 5

Assume x (a, b) = $50a^2 - 9b^2$ (19)

Substituting (19) in (1) and applying the method of factorization, define

$(\sqrt{50}\, a + 3b)^2 = (\sqrt{50}\, z + 3y)$

Equating rational and irrational we get.

$y = y(a, b) = \frac{1}{3}[50a^2 + 9b^2]$ (20)

z = z (a, b) = 6ab (21)

As our interest is on finding integer solutions, we choose a and b suitably so that the values of x, y and z are in integer. In what follows the values of a, b and the corresponding integer solutions are exhibited.

Case 1: Let a = 3A, b = 3B

The corresponding solutions of (1) are

$x = x(A, B) = 450A^2 - 81B^2$

$y = y(A, B) = 150A^2 + 27B^2$

z = z (A, B) = 54AB

Properties :

1. $3y(1, B) - x(1, B) - z(1, B) - T_{324, B} \equiv 0 \pmod{107}$
2. $x(B, 1) + 3y(B, 1) + z(B, 1) - T_{1802, B} \equiv 0 \pmod{953}$
3. $x(A,1) + z(A,1) - T_{902, A} \equiv -81 \pmod{503}$
4. y (A(A+1), A) -3 $T_{4, A}$ a Nasty number

Case 2: Let a = 3A, b = 3B+1

The corresponding solutions of (1) are

$x = x(A, B) = 450A^2 - 81B^2 - 54B - 9$

$y = y(A, B) = 150A^2 + 27B^2 + 18B + 3$

z = z (A, B) = 162A + 486AB

Properties :

1. $3y(A, B) - x(A, B) - T_{326, B} \equiv 18 \pmod{269}$
2. $y(1, B) - T_{56, B} \equiv 21 \pmod{44}$
3. $x(1, B) + T_{164, B} \equiv 39 \pmod{134}$

Case 3: Let a = 3A, b = B

The corresponding solutions of (1) are

$x = x(A, B) = 450A^2 - 9B^2$

$y = y(A, B) = 150A^2 + 3B^2$

z = z (A, B) = 18AB

Properties:

1. $x(A, 1) - T_{902, A} \equiv -9 \pmod{449}$
2. $z(A, A) - T_{38, A} \equiv 0 \pmod{17}$
3. z (A, A(A+1)) – 24 P_A^5 a Nasty number

3. Remarkable Observation

Employing the solutions (x, y, z) of (1) each of the following expression among the special polygonal and pyramidal numbers are observed.

- $\left(\frac{3P_{x-2}^3}{T_{3,x-2}}\right)^2 + 9\left(\frac{P_{y-1}^5}{T_{4,y-1}}\right)^2 = 50\left(\frac{P_2^5}{T_{3,2}}\right)^2$
- $\left(\frac{P_x^5}{T_{3,x}}\right)^2 + 9\left(\frac{3P_{y-2}^3}{T_{3,y-2}}\right)^2 \equiv 0 (mod\ 50)$
- $\left(\frac{P_{x-1}^5}{T_{3,x-1}}\right)^2 + 9\left(\frac{P_y^5}{T_{3,y}}\right)^2 = 50\left(\frac{P_z^5}{T_{3,z+1}}\right)$
- $\left(\frac{P_x^3}{T_{3,x+1}}\right)^2 + 9\left(\frac{3P_{y-2}^3}{T_{3,y-2}}\right)^2 = 50\left(\frac{P_{z-1}^5}{T_{4,z+1}}\right)$

4. Conclusion

In this work, the ternary quadratic Diophantine equations referring conies $x^2 + 9y^2 = 50z^2$ is analyzed for is non-zero distinct integral points. A few interesting properties between the solutions and special numbers are presented. To conclude, one may search for other patterns of solutions and their corresponding properties for the cone under consideration.

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