Lattice Points on the Cone $X^2 + 9Y^2 = 50Z^2$

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Abstract: The ternary quadratic homogeneous equation representing cone given by $x^2 + 9y^2 = 50z^2$ is analyzed for its non-zero distinct integer points on it. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns are presented.

Keywords: Diophantine equation, Ternary quadratic, integral solutions, special numbers, a few interesting Relation

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Notations:

- $P_n^m$ = Pyramid number of rank n with size m
- $T_{m,n}$ = Polygonal number of rank n with size m

1. Introduction

The ternary quadratic Diophantine equation offers an unlimited field for research due to their variety [1, 10]. For an extensive review of various problems one may refer [2, 9]. This communication concerns with yet another interesting ternary quadratic equation representing $x^2 + 9y^2 = 50z^2$ a cone for determining its infinitely many non-zero integral points. Also a few interesting relations among the solutions and special number are presented.

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solutions is

$$x^2 + 9y^2 = 50z^2 \quad (1)$$

Assume $z(a, b) = a^2 + 9b^2$, Where $a, b > 0 \quad (2)$

We illustrate below five different patterns of non-zero distinct integer solutions to (1).

2.1 Pattern: 1

Write 50 as $50 = (7 + i) (7 - i) \quad (3)$

Following the procedure presented in pattern: 1, the corresponding values of x and y are

$$x = x(a, b) = a^2 - 9b^2 - 42ab \quad (4)$$

$$y = y(a, b) = \frac{1}{3} [a^2 - 9b^2 + 42ab] \quad (5)$$

Thus (2), (4), (5) represents non-zero distinct integral solutions of (1) in two parameters As our interest is on finding integer solutions, we choose a and b suitably so that the value of x, y and z are in integers. In what follows the values of a, b and the corresponding integer solutions are exhibited.

Case 1: Let $a = 3A$, $b = 3B$.

The corresponding solutions of (1) are

$$x = x(A, B) = 63A^2 - 567B^2 - 54AB$$

$$y = y(A, B) = 3A^2 - 27B^2 + 126AB$$

$$z = z(A, B) = 9A^2 + 81B^2$$

Properties:

1. $x(A, 1) + T_{128, A} \equiv 1 \pmod{8}$
2. $2y(A, A+1) - x(A, A+1) = 2700T_3, A$
3. $y(A, 1) - T_{8, A} \equiv -27 \pmod{128}$
4. $6(z(A, A+1)) - 204T^2_{3, A}$ is a nasty number

Case 2: Let $a = 3A$, $b = B$.

The corresponding solutions of (1) are

$$x = x(A, B) = 63A^2 - 63B^2 - 18AB$$

$$y = y(A, B) = 3A^2 - 3B^2 + 42AB$$

$$z = z(A, B) = 9A^2 + 9B^2$$

Properties:

1. $x(A(A+1), A+2) - 21y(A(A+1), A+2) + 5400T^3_{5, A} = 0$
2. $x(A, A+1) + 9z(A, A+1) - 6T_4, A + 18P_{A^5} a nasty number$
3. $T_{128, A} - x(A, 1) \equiv 19 \pmod{44}$
4. $x(A, A(A+1)) - 12P^5_{A^5} + 12T^2_{3, A} - 3T_4, A a nasty number.$

Case 3: Let $a = 3A$, $b = 3A + 1$.

The corresponding solutions of (1) are

$$x = x(A, A) = -558A^2 - 396A - 63$$

$$y = y(A, A) = 102A^2 + 24A - 3$$

$$z = z(A, A) = 90A^2 + 54A + 9$$

Properties:

1. $z(A(A+1), 1) - 108T_3, A - 360T^2_{3, A} = 0 \pmod{9}$
2. $T_{206, A} - y(A, 1) \equiv 3 \pmod{24}$
3. $T_{128, A} - z(A, A) - 12T_4, A \equiv 0 \pmod{6}$

2.2 Pattern: 2

Instead of (3), write 50 as

$$50 = (1 + 7i)(1 - 7i) \quad (6)$$

Following the procedure presented in pattern: 1, the corresponding values of x and y are

$$x = x(a, b) = a^2 - 9b^2 - 42ab \quad (7)$$
\[ y = y(a, b) = \frac{1}{3}[7a^2 - 63b^2 + 6ab] \quad (8) \]

**Case 1:** Let \( a = 3A, b = 3B. \)

The corresponding solutions of (1) are

\[
\begin{align*}
  x &= x(A, B) = 9A^2 - 81B^2 - 126AB \\
  y &= y(A, B) = 21A^2 - 189B^2 + 18AB \\
  z &= z(A, B) = 9A^2 + 81B^2
\end{align*}
\]

**Properties:**

1. \( 7x(A, A+1) - 3y(A, A+1) + 5400T_{4A} \equiv 0 \mod 2 \)
2. \( x(A, A) + 450T_{4A} = 0 \)
3. \( y(A, A) + z(A, A) \equiv 0 \mod 4 \)
4. \( 6\{x(A, A+1) + 736T_{4A} \equiv 0 \mod 6 \)
5. \( y(2A, A) + 150T_{4A} = 0 \)
6. \( y(A, A) + 3T_{4A} \equiv 0 \mod 3 \)

**Case 2:** Let \( a = 3A, b = B. \)

The corresponding solutions of (1) are

\[
\begin{align*}
  x &= x(A, A) = 450A^2 - 9B^2 \\
  y &= y(A, B) = 1050A^2 + 189B^2 + 900AB \\
  z &= z(A, B) = 450A^2 + 81B^2 + 378AB
\end{align*}
\]

**Properties:**

1. \( y(A, A(A+1)) - 756T_{4A} = 0 \)
2. \( x(A, A) - 369T_{4A} = 0 \)
3. \( y(A, A) - 441T_{4A} = 0 \)
4. \( x(A, 1) - z(A, 1) - 252 \equiv 0 \mod 2 \)
5. \( x(A, 1) - 3T_{4A} \equiv 0 \mod 3 \)
6. \( y(A, 1) - T_{16, A} \equiv 0 \mod 4 \)

**2.4 Pattern: 4**

(1) is written as \( 50z^2 - 9y^2 = x^2 \equiv x^2*1 \) \quad (14)

Assume \( x(a, b) = 50a^2 - 9b^2 \) \quad (15)

Write (1) as \( 1 = (\sqrt{50} + 7)(\sqrt{50} - 7) \) \quad (16)
Substituting (15) and (16) in (14) and applying the method of factorization, define

\[
\begin{align*}
  y &= y(a, b) = \frac{1}{2}[350a^2 + 63b^2 + 300ab] \\
  z &= z(a, b) = 50a^2 + 9b^2 + 42ab
\end{align*}
\]

**Case 1:** let \( a = 3A, b = 3B. \)

The corresponding solutions of (1) are

\[
\begin{align*}
  x &= x(A, B) = 450A^2 - 81B^2 \\
  y &= y(A, B) = 1050A^2 + 189B^2 + 900AB \\
  z &= z(A, B) = 450A^2 + 81B^2 + 378AB
\end{align*}
\]

**Properties:**

1. \( x(A, A(A+1)) - 756T_{4A} = 0 \)
2. \( x(A, A) + 3T_{4A} = 0 \)
3. \( y(A,1) - T_{2102, A} = 189 \mod 1949 \)
4. \( z(A, 1) - T_{902, A} = 81 \mod 827 \)

**Case 2:** Let \( a = 3A, b = B. \)

The corresponding solutions of (1) are

\[
\begin{align*}
  x &= x(A, A) = 709A^2 + 180A + 9 \\
  y &= y(A, B) = 450A^2 + 9B^2 \\
  z &= z(A, B) = 450A^2 + 9B^2 + 126AB
\end{align*}
\]

**Properties:**

1. \( x(A, A(A+1)) - 84T_{4A} - 1800T_{5A} - 1050T_{4A} = 0 \)
2. \( x(A, A) + 369T_{4A} = 0 \)
3. \( y(A, 1) - T_{2102, A} = 189 \mod 575 \)
4. \( z(A, 1) - T_{902, A} = 81 \mod 625 \)

**Case 3:** Let \( a = 3A, b = 3A + 1. \)

The corresponding solutions of (1) are

\[
\begin{align*}
  x &= x(A, A) = -150A^2 - 120A - 21 \\
  y &= y(A, A) = 90A^2 + 54A + 9 \\
  z &= z(A, A) = 90A^2 + 54A + 9
\end{align*}
\]

**Properties:**

1. \( z(A, A) + 6T_{4A} \equiv 0 \mod 3 \)
2. \( \frac{1}{3}(A, A) = y(A, A) \equiv 0 \mod 6 \)
3. \( y(A, A) \equiv 0 \mod 3 \)

**2.3: Pattern: 3**

(1) is written in the form of ratio as

\[
\begin{align*}
  \frac{x+z}{2x-3y} &= \frac{A}{B} \mod 2 \neq 0, \\
  \text{which is equivalent to the system of equations,}
\end{align*}
\]

\[
\begin{align*}
  Bx - 3Ay + (B-7A)z &= 0 \quad (9) \\
  Ax + 3By - (A+7B)z &= 0 \quad (10)
\end{align*}
\]

Applying the method of cross multiplication the integer solutions (1) are given by

\[
\begin{align*}
  x &= x(A, B) = 3A^2 - 3B^2 + 42AB \\
  y &= y(A, B) = 7A^2 - 7B^2 - 2AB \\
  z &= z(A, B) = 3A^2 + 3B^2
\end{align*}
\]

which represents non-zero distinct integral solutions of (1) in two parameters.
4. \( x(1, 1) - 136 \text{ Nasty Number} \)

2.5 Pattern: 5

Assume \( x(a, b) = 50a^2 - 9b^2 \) \hspace{1cm} (19)

Substituting (19) in (1) and applying the method of factorization, define

\( (\sqrt{50} a + 3b)^2 = (\sqrt{50} z + 3y) \)

Equate rational and irrational we get.

\( y = y(a, b) = \frac{1}{3} [50a^2 + 9b^2] \)

\( z = z(a, b) = 6ab \) \hspace{1cm} (20)

As our interest is on finding integer solutions, we choose a and b suitably so that the values of x, y and z are in integer. In what follows the values of a, b and the corresponding integer solutions are exhibited.

Case 1: Let \( a = 3A, b = 3B \)

The corresponding solutions of (1) are

\( x = x(A, B) = 450A^2 - 81B^2 \)
\( y = y(A, B) = 150A^2 + 27B^2 \)
\( z = z(A, B) = 54AB \)

Properties:

1. \( 3y(1, B) - x(1, B) = 23 \text{ (mod 269)} \)
2. \( x(A, 1) = z(A, 1) = 25 \text{ (mod 23)} \)
3. \( x(1, A) + z(1, A) = 61 \text{ (mod 25)} \)

Case 2: Let \( a = 3A, b = 3B + 1 \)

The corresponding solutions of (1) are

\( x = x(A, B) = 450A^2 - 81B^2 - 54B - 9 \)
\( y = y(A, B) = 150A^2 + 27B^2 + 18B + 3 \)
\( z = z(A, B) = 162A + 486AB \)

Properties:

1. \( 3y(A, B) + 9x(A, B) = 75 \text{ (mod 269)} \)
2. \( x(A, 1) + z(A, 1) = 39 \text{ (mod 134)} \)

Case 3: Let \( a = 3A, b = B \)

The corresponding solutions of (1) are

\( x = x(A, 1) - 9 \text{ (mod 449)} \)
\( y = y(A, 1) = 150A^2 + 3B^2 \)
\( z = z(A, 1) = 18AB \)

Properties:

1. \( x(A, 1) - z(A, 1) = 9 \text{ (mod 449)} \)
2. \( z(A, A) = 0 \text{ (mod 17)} \)
3. \( z(A, A + 1) = 24 P_5^3 \text{ a Nasty number} \)

3. Remarkable Observation

Employing the solutions \((x, y, z)\) of (1) each of the following expression among the special polygonal and pyramidal numbers are observed.

\[ \left(\frac{3p^2_5}{\gamma_{3,2}}\right)^2 + 9 \left(\frac{p^5_5}{\gamma_{3,2}-1}\right)^2 = 50 \left(\frac{p^5_5}{\gamma_{3,2}}\right)^2 \]

\[ \left(\frac{p^5_5}{\gamma_{3,2}}\right)^2 + 9 \left(\frac{3p^2_5}{\gamma_{3,2}-2}\right)^2 \equiv 0 \text{ (mod 50)} \]

\[ \left(\frac{p^5_5}{\gamma_{3,2}-1}\right)^2 + 9 \left(\frac{p^5_5}{\gamma_{3,2}}\right)^2 = 50 \left(\frac{p^5_5}{\gamma_{3,2}+1}\right)^2 \]

4. Conclusion

In this work, the ternary quadratic Diophantine equations referring conies \( x^2 + 9y^2 = 50z^2 \) is analyzed for is non-zero distinct integral points. A few interesting properties between the solutions and special numbers are presented. To conclude, one may search for other patterns of solutions and their corresponding properties for the cone under consideration.

References


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