and a signed dominating function we define signed Roman dominating function (SRDF) on a graph $G = (V, E)$ to be a function $f : V \rightarrow (-1, 1, 2)$ satisfying the condition that f is a dominating function (that is, the sum of the values assigned to a vertex and its neighbors is at least 1 for every vertex) and every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The signed Roman domination number denoted $\gamma_{sR}(G)$, is the minimum weight of a SRDF in G ; that is, $\gamma_{sR}(G) = \min\{w(f) is\ a\ SRDF\ in\ G$. A SRDF of weight $\gamma_{sR}(G)$, we call a $\gamma_{sR}(G) -$ function. [7] A signed Roman dominating function $f : v \rightarrow (-1, 1, 2)$ can be represented by the ordered partition $\{V_{-1}, V_1, V_2\}$ of V, where $V_i = \{v \in V / f(v) = i\}$.

2. Preliminary Results and Observations

Observation 2.1 [7] : Let $f = (V_{-1}, V_1, V_2)$ be a SRDF in a graph G. Then the following holds.

a) Every vertex in V_{-1} is dominated by a vertex in V_2

b) $w(f) = |V_1| + 2|V_2| - |V_{-1}|$.

c) $V_1 \cup V_2$ is a dominating set in G.

Proposition 2.2 [7] : $(2\Delta + 1)|V_2| + \Delta|V_1| \geq (\delta + 2)|V_{-1}|$

Corollary 2.3 [7]: For a cubic graph $|V_{-1}| \leq \frac{1}{5}[7|V_2| + 3V1$.

Corollary 2.4 [7] : For $r \geq 1$, if G is a r-regular graph order n, then $\gamma_{sR}(G) \geq \frac{n}{(r+1)}$.

Observation 2.5 [7] : For $n \geq 2$,
$\gamma_{sR}(K_{1,n-1}) = 1$ for n even
2 for n odd

Proposition 2.6 [7]: For $n \geq 1$, $\gamma_{sR}(K_n) = 1$ unless $n = 3$, in which case $\gamma_{sR}(K_n) = 2$
in which case $\gamma_{sR}(K_n) = 2$.

Proposition 2.7 [7] : For $n \geq 3$, $\gamma_{sR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Proposition 2.8 [7] : For $n \geq 1$, $\gamma_{sR}(P_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Proposition 2.9 [7]: Let G be a graph of order n., then the following holds

a). $\gamma_{sR}(G) \leq n$ with equality if and only if $G = \overline{K}_n$

b). $\gamma_{sR}(G) \geq 2\gamma(G) - n$, with equality if and only if $G = \overline{K}_n$.

3. Some exact values of $\gamma_{sR}(Pn, k)$

Theorem 3.1 : For $G = P(n, 1)$, a generalized Petersen graph then

i). $\gamma_{sR}(P(4m, 1)) = 2m$

ii) $\gamma_{sR}(P(4m+1, 1)) = \gamma_{sR}(P(4m, 1)) + 3$.

iii). $\gamma_{sR}(P(4m+2, 1)) = \gamma_{sR}(P(4m, 1)) + 4$

iv). $\gamma_{sR}(P(4m+3, 1)) = \gamma_{sR}(P(4m, 1)) + 2$

Proof : Let $G = P(n, 1)$, $n \geq 3$ be a generalized Petersen graph with $2n$ vertices whose vertex set be $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and a edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k}, i = 0, 1, 2, \ldots, n-1\}$.

Let $f : V \rightarrow \{-1, 1, 2\}$ be a signed Roman domination function.

For $r \geq 1$, if G is r - regular graph of order n then $\gamma_{sR}(G) \geq \frac{n}{r+1}$.

As G is cubic graph of order n then, $\gamma_{sR}(G) \geq 2m$

Case 1. If $n=4m$, $m \geq 1$

Let $g : v \rightarrow \{-1, 1, 2\}$ be a function defined as follows.

Let $g(u_i) = -1$ if $i \equiv 5 \bmod (4)$ and $i \equiv 2 \bmod (4)$

Let $g(u_i) = 1$ if $i \equiv -1 \bmod (4)$

Let $g(u_i) = 2$ if $i \equiv 4 \bmod (4)$ and

Let $g(v_i) = -1$ if $i \equiv 3 \bmod (4)$ and $i \equiv 4 \bmod (4)$

Let $g(v_i) = 1$ if $i \equiv 1 \bmod (4)$

Let $g(v_i) = 2$ if $i \equiv 2 \bmod (4)$

The assignment is shown in following figure

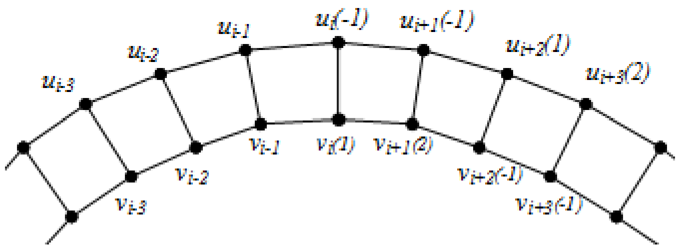

Figure 1

With this assignment the first copy of m will have weight exactly two. As we have m such copies,
$\gamma_{sR}(P(4m, 1)) \leq 2m$
Consequently $\gamma_{sR}(P(4m, 1)) = 2m$

Case 2. If $n=4m+1$, $m \geq 1$

The proof is same as case 1 for first 4m-vertices whose minimum weight is $2m$. The remaining two vertices namely u_{4m+1} and v_{4m+1} are to be assigned with 2 and 1 respectively. Otherwise the vertex u_1 will loose the SRDF property. Thus g is a SRDF of G. Hence the total weight of the graph is $2m+3$.

Thus $\gamma_{sR}(P(4m+1, 1)) = \gamma_{sR}(P(4m, 1)) + 3$.

Case 3. If $n=4m+2$

The proof is same as case 1 for first $4m$-vertices. Here we are left with 4 vertices which are to be assigned as follows. Vertices namely u_{4m+1}, u_{4m+2}, v_{4m+1}, v_{4m+2} as -1, 2, 1, 2 respectively to get the minimum weight. Thus g is a SRDF of G .Hence total weight of the graph is $2m+4$.

Thus $\gamma_{sR}(P(4m+2, 1)) = \gamma_{sR}(P(4m, 1)) + 4$.

Case 4. If $n=4m+3$

The proof is same as case 1 for first $4m$-vertices. Here we are left with six vertices namely u_{4m+1}, u_{4m+2}, u_{4m+3}, v_{4m+1}, v_{4m+2}, v_{4m+3} which are to be assigned as -1, -1, 2, 1, 2, -1 respectively to get the minimum weight. Thus g is a SRDF of G and Hence total weight of the graph is $2m+2$.

$\gamma_{sR}(P(4m+3, 1)) = \gamma_{sR}(P(4m, 1)) + 2$

Observation 3.2 : The set of vertices assigned 2 forms a dominating set in $P(n, k)$, where k is odd. Therefore $\sum N[V_2] = 2n$.

Lemma 3.3: For a graph $G = P(n, k)$
$|V_{-1}| = 2m$ for $n = 3m, 3m+1$
$|V_{-1}| = 2m + 1$ for $n = 3m+2$

Proposition 3.4: For $G = P(n, 3)$ a generalized Petersen graph then,
i). $\gamma_{sR}(P(4m, 1)) = 2m$ for all $m \geq 2$
ii) $\gamma_{sR}(P(4m+1, 3)) = \gamma_{sR}(P(4m, 3)) + 3$
iii). $\gamma_{sR}(P(4m+2, 3)) = \gamma_{sR}(P(4m, 3)) + 6$
iv). $\gamma_{sR}(P(4m+3, 3)) = \gamma_{sR}(P(4m, 3)) + 5$
Proof : Let $G=P(n, 3)$ be a generalized Petersen graph with $2n$ vertices whose vertex set is $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_i u_{i+1}, u_i v_i, v_i, v_{i+k}, i=0,1,2,3----n-1\}$ and
Let $f: V \rightarrow \{-1,1,2\}$ be a signed Roman dominating function.

For $r \geq 1$, if G is r-regular graph of order n then $\gamma_{sR}(G) \geq \frac{n}{r+1}$
Since G is cubic graph of order $2n$ then $\gamma_{sR}(G) \geq 2m$

Case 1. If $n = 4m, m \geq 2$
Let $g : V \rightarrow \{-1, 1, 2\}$ be a function defined as follows
Let $g(u_i) = -1$ if $i \equiv 5 \bmod (4)$ and $i \equiv 0 \bmod (4)$
Let $g(u_i) = 1$ if $i \equiv -2 \bmod (4)$
Let $g(u_i) = 2$ if $i \equiv -1 \bmod (4)$
Similarly Let $g(v_i) = -1$ if $i \equiv 2 \bmod (4)$ and $i \equiv 3 \bmod (4)$
Let $g(v_i) = 1$ if $i \equiv 4 \bmod (4)$
Let $g(v_i) = 2$ if $i \equiv 1 \bmod (4)$
The assignment is as shown below

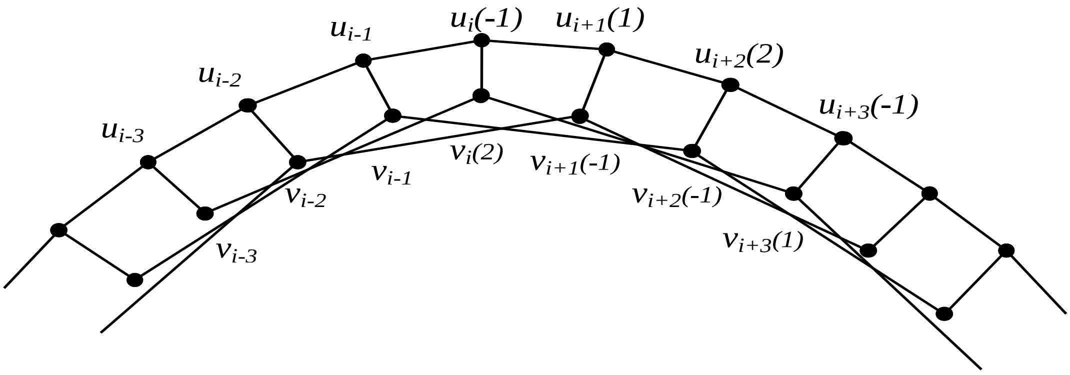

Figure 2

With this assignment the first copy of m will have weight exactly 4.
As we have m such copies $\gamma_{sR}(P(n, 3)) \leq w(g) \leq 2m = \frac{n}{2}$
Consequently $\gamma_{sR}(P(4m, 3)) = 2m$
Case 2. If $n = 4m + 1, m \geq 2$
The proof is same as case 1 for first 4m-vertices whose minimum weight is $\frac{n}{2}$.
There are two vertices left namely u_{4m+1}, v_{4m+1} which are to be assigned as 1 and 2 respectively to have minimum weight.
Hence total weight of a graph is $\frac{n}{2} + 3$.
Thus $\gamma_{sR}(P(4m+1, 3)) = \gamma_{sR}(P(4m, 3)) + 3$

Case 3. If $n = 4m + 2$
The proof is same as case 1 for first $4m$-vertices whose minimum weight is $\frac{n}{2} = 2m$.
Here we are left with $u_{4m+1}, u_{4m+2}, v_{4m+1}, v_{4m+2}$
which are to be assigned as $1, 2, 2, 1$ respectively to get minimum weight.
Hence total weight of a graph $\frac{n}{2} + 6$
Thus $\gamma_{sR}(P(4m+2, 3)) = \gamma_{sR}(P(4m, 3)) + 6$.

Case 4. If $n = 4m + 3$
The proof is same as case 1 for first $4m$-vertices whose minimum weight is $\frac{n}{2}$.
Here we are left with six vertices namely $u_{4m+1}, u_{4m+2}, u_{4m+3}, v_{4m+1}, v_{4m+2}, v_{4m+3}$ which are to be assigned as $-1, 2, -1, 2, 1, 2$ respectively to get the minimum weight.
Hence total weight of a graph is $\frac{n}{2} + 5$
Thus $\gamma_{sR}(P(4m+3, 3)) = \gamma_{sR}(P(4m, 3)) + 5$.

4. Efficient signed Roman domination

A signed Roman domination function (SRDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{-1, 1, 2\}$ satisfying the condition that,
i) The sum of its function values over closed neighborhood is atleast 1 and
ii) For every vertex u for which $f(u) = -1$ is adjacent to atleast one vertex v for which $f(v)=2$.

The weight of SRDF is sum of its function values over all vertices. The singed Roman domination number of G is a minimum weight of SRDF in G. The SRDF is said to be an efficient signed Roman domination if $f[v] = 1$, for all $v \in V(G)$.

Theorem.4.1 If $G = P(n, k)$ has an efficient signed Roman function
Then $\gamma_{sR}(G) = \gamma(G)$.
In the following lemma a useful necessary condition for $P(n, k)$ to have an efficient signed Roman dominating function is given.

Lemma 4.2: If $P(n, k)$ where k is odd has an efficient signed Roman dominating function, then $\gamma_{sR}(P(n, k)) = \gamma(P(n, k)) = \frac{n}{2}$ *and* $4/n$

Lemma 4.3: If k is odd number and $4/n$ then $\gamma_{sR}(P(n, k)) = \frac{n}{2}$ and there fore $P(n, k)$ has an efficient signed Roman dominating function.

Theorem 4.4: A generalized Petersen graph $P(n, k)$ has an efficient signed Roman domination function if and only if $n \equiv 0 \mod (4)$ and k is odd.

Proof : Sufficiency of the statement follows from lemma 4.2. For necessity, suppose f is an efficient signed Roman dominating function in $P(n, k)$, a graph must have efficient dominating set in $P(n, k)$. As in Lemma 4.1, we have $|S| = \frac{n}{2} = 2m$. Where m is a number of u vertices in S, which is equal to the number of v vertices in S.

Each u-vertex dominates three u-vertices (including itself) and one v-vertex. So there are $3m$, u-vertices dominated by u-vertices and m of them dominated by v- vertices. Let u_i and u_j be two u-vertices in S, such that on one of the u-paths from u_i to u_j there is no other u-vertex in S. Then there are exactly five u-vertices on the u-path from u_i to u_j , including u_i and u_j . For, since S is an efficient signed Roman dominating set and by lemma 4.2 the number of vertices on that path dominated by v-vertex is atmost 1, and also since there are m v- vertices in S, there must be atleast one vertex of that path dominated by a v-vertex. So there is a unique pattern for the u-vertices in S, say $\{u_{i-1}, u_{i+3}\} \underline{C}\ S$ and similarly $\{v_{i+1}, v_{i+5}\} \underline{C}\ S$, see figure for the pattern. By this unique pattern, it is clear that $P(n, k)$ does not have an efficient signed Roman dominating function for even values of k. See figure 3 for a generalized Petersen graph $P(16, 5)$ and an efficient signed Roman domination.

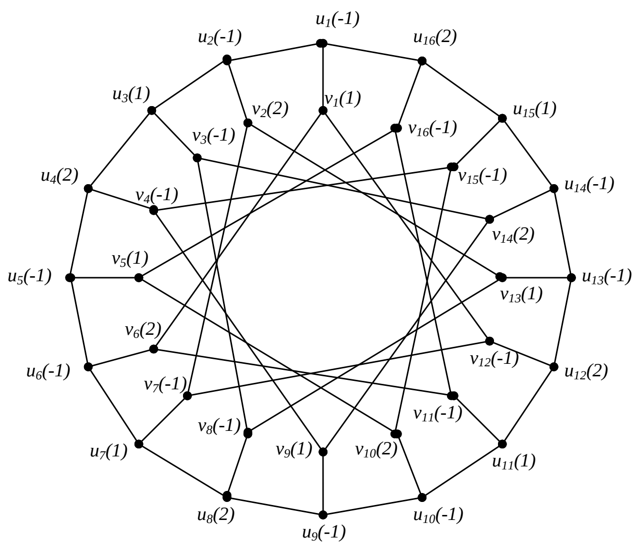

Figure 3

5. Scope and Conclusion

5.1 Scope

1) We can find signed Roman domination number for other classes of graphs.
2) We can determine the signed Roman domination number of any grid graph $G_{m,n}$.

5.2 Conclusion

We found signed Roman domination number of a generalized Petersen graph $P(n, k)$ for $k = 1$ and 3 and also found condition for a $P(n, k)$ to have efficient SRDF for Generalized Petersen graph of odd k.

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