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On Certain Inequalities Pertaining to I-Function

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Abstract: I*n this paper, we establish four inequalities pertaining to I-function . On account of general nature of the results established here, a number of known and new results follow as their particular cases on suitable specifications of the parameters involved there in. To illustrate, we have recorded some particular cases of our main results.*

Mathematics Subject classification: 33C60, 26D20

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1. Introduction

The I-function occurring in the present paper was studied by Saxena[3] and defined as

$$I^{M,N}_{P_i,Q_i:r}[z] = I^{M,N}_{P_i,Q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \varphi(\xi)\, z^{\xi} d\xi \qquad (1)$$

where

$$\varphi(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}} \qquad (2)$$

and $P_i(i=1,2,...,r)$, $Q_i(i=1,2,...,r)$, M, N are integers satisfying $0\le N\le P_i$, $0\le M\le Q_i$ $(i=1,2,...,r)$; r is finite, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are complex numbers such that $a_j(b_h+v)\ne\beta_h(a_j-1-k)$, for v,k = 1,2,…; h = 1,2,…,M; i =1,2,…,r.L is the contour running from $\sigma-i\infty$ to $\sigma+i\infty$ (σ is real) in the complex ξ-plane such that the points

$$\xi=(a_j-1-v)/a_j, \quad j=1,2,...,N; v=0,1,2,...$$

$$\xi=(b_j+v)/\beta_j, \quad j=1,2,...,M; v=0,1,2,...$$

lie to the left hand and right hand sides of L, respectively.
For the present study we shall use the following result due to [2]

$$\frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i)}{\Gamma\left(\beta+\sum_{i=1}^{n}\alpha_i\right)} \le \frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i x)}{\Gamma\left(\beta+x\sum_{i=1}^{n}\alpha_i\right)} \le \frac{1}{\Gamma(\beta)} \qquad (3)$$

where $x\in[0,1]$, $\beta\ge1$, $\alpha_i>0$, $n\in N$

2. Main Results

Inequality 1 By putting $\alpha_i=\mu_i\xi-a_i$, $\beta=\upsilon+1+\mu\xi$ we get

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+\mu_i\xi-a_i\right)}{\Gamma\left(\upsilon+1+\mu\xi+\sum_{i=1}^{n}\left(\mu_i\xi-a_i\right)\right)}\leq\frac{\prod_{i=1}^{n}\Gamma\left(1+\left(\mu_i\xi-a_i\right)x\right)}{\Gamma\left(\upsilon+1+\mu\xi+x\sum_{i=1}^{n}\left(\mu_i\xi-a_i\right)\right)}\leq\frac{1}{\Gamma\left(\upsilon+1+\mu\xi\right)}$$

multipying all three terms with

$$\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}$$

and then integrating along the contour L

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{\prod_{i=1}^{n}\Gamma\left(1+\mu_i\xi-a_i\right)}{\Gamma\left(\upsilon+1+\mu\xi+\sum_{i=1}^{n}\left(\mu_i\xi-a_i\right)\right)}d\xi$$

$$\leq\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{\prod_{i=1}^{n}\Gamma\left(1+\left(\mu_i\xi-a_i\right)x\right)}{\Gamma\left(\upsilon+1+\mu\xi+x\sum_{i=1}^{n}\left(\mu_i\xi-a_i\right)\right)}d\xi$$

$$\leq\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{1}{\Gamma\left(\upsilon+1+\mu\xi\right)}d\xi$$

=

$$I^{M,N+n}_{P_i+n,Q_i+1:r}\left[z\left|\begin{matrix}(a_i,\mu_i)_{1,n};(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};\ (-\nu+\sum_{i=1}^{n}a_i,\ \mu+\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right]$$

$$\leq\ I^{M,N+n}_{P_i+n,Q_i+1:r}\left[z\left|\begin{matrix}(a_ix,\ x\mu_i)_{1,n};(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};\ (-\nu+x\sum_{i=1}^{n}a_i,\ \mu+x\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right]$$

$$\leq\ I^{M,N}_{P_i,Q_i+1:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};(-\nu,\ \mu)\end{matrix}\right.\right] \quad (4)$$

provided that $x\in[0,1]$

$$\mathrm{Re}\left[\nu+\min_{1\le j\le M}\mu\left(\frac{b_j}{\beta_j}\right)\right]>1,\ \mathrm{Re}\left[a_i+\min_{1\le j\le M}\mu_i\left(\frac{b_j}{\beta_j}\right)\right]>0\ ,\ |\arg(z)|<\frac{1}{2}\pi\phi,$$

and $j=1,\dots,M,\ i=1,\dots,n\ and\ n\in N$ (5)

<u>Inequality 2</u> By putting $\alpha_i=a_i-\mu_i\xi,\ \beta=\upsilon-\mu\xi$ we get

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+a_i-\mu_i\xi\right)}{\Gamma\left(\upsilon-\mu\xi+\sum_{i=1}^{n}a_i-\mu_i\xi\right)}\le\frac{\prod_{i=1}^{n}\Gamma\left(1+\left(a_i-\mu_i\xi\right)x\right)}{\Gamma\left(\upsilon-\mu\xi+x\sum_{i=1}^{n}\left(a_i-\mu_i\xi\right)\right)}\le\frac{1}{\Gamma\left(\upsilon-\mu\xi\right)}$$

multipying all three terms with

$$\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}$$

and then integrating along the contour L we get

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{\prod_{i=1}^{n}\Gamma\left(1+a_i-\mu_i\xi\right)}{\Gamma\left(\upsilon-\mu\xi+\sum_{i=1}^{n}(a_i-\mu_i\xi)\right)}d\xi$$

$$\le\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{\prod_{i=1}^{n}\Gamma\left(1+\left(a_i-\mu_i\xi\right)x\right)}{\Gamma\left(\upsilon-\mu\xi+x\sum_{i=1}^{n}\left(a_i-\mu_i\xi\right)\right)}d\xi$$

$$\le\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{1}{\Gamma\left(\upsilon-\mu\xi\right)}d\xi$$

=

$$I_{P_i+1,Q_i+n:r}^{M+n,N}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i};(\nu+\sum_{i=1}^{n}a_i,\ \mu+\sum_{i=1}^{n}\mu_i)\\ \left(1+a_i,\mu_i\right)_{1,n};(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\le I_{P_i+1,Q_i+n:r}^{M+n,N}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i};(\nu+x\sum_{i=1}^{n}a_i,\ \mu+x\sum_{i=1}^{n}\mu_i)\\ \left(1+a_ix,\mu_ix\right)_{1,n};(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$\le I_{P_i+1,Q_i:r}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i};(\nu,\ \mu)\\ (b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{matrix}\right.\right]\quad(6)$$

provided that

$x \in [0,1]$

$\mathrm{Re}\left[a_i - \min_{1 \le j \le M} \mu_i \left(\frac{b_j}{\beta_j}\right)\right] > 0,\ \mathrm{Re}\left[\nu - \min_{1 \le j \le M} \mu \left(\frac{b_j}{\beta_j}\right)\right] > 0\ ,\ |\arg(z)| < \frac{1}{2}\pi\phi,$

For $j = 1,\ldots,M,\ i = 1,\ldots,n\ and\ n \in N$ (7)

Inequality 3 By putting $\alpha_i = \mu_i \xi + a_i,\ \beta = \upsilon + 1 + \mu\xi$ we get

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+\mu_i\xi+a_i\right)}{\Gamma\left(\upsilon+1+\mu\xi+\sum_{i=1}^{n}(\mu_i\xi+a_i)\right)} \le \frac{\prod_{i=1}^{n}\Gamma\left(1+\left(\mu_i\xi+a_i\right)x\right)}{\Gamma\left(\upsilon+1+\mu\xi+x\sum_{i=1}^{n}(\mu_i\xi+a_i)\right)} \le \frac{1}{\Gamma(\upsilon+1+\mu\xi)}$$

multipying all three terms with

$$\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)\Gamma\left(\left(\upsilon+\sum_{i=1}^{n}a_i\right)-\left(\mu+\sum_{i=1}^{n}\mu_i\right)\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}$$

and then integrating along the contour L

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)\Gamma\left(\left(\upsilon+\sum_{i=1}^{n}a_i\right)-\left(\mu+\sum_{i=1}^{n}\mu_i\right)\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}$$

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+\mu_i\xi+a_i\right)}{\Gamma\left(\upsilon+1+\mu\xi+\sum_{i=1}^{n}(\mu_i\xi+a_i)\right)}d\xi$$

$$\le \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)\Gamma\left(\left(\upsilon+\sum_{i=1}^{n}a_i\right)-\left(\mu+\sum_{i=1}^{n}\mu_i\right)\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}$$

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+\left(\mu_i\xi+a_i\right)x\right)}{\Gamma\left(\upsilon+1+\mu\xi+x\sum_{i=1}^{n}\left(\mu_i\xi+a_i\right)\right)}d\xi$$

$$\leq \frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)\Gamma\left(\left(\upsilon+\sum_{i=1}^{n}a_i\right)-\left(\mu+\sum_{i=1}^{n}\mu_i\right)\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\frac{1}{\Gamma(\upsilon+1+\mu\xi)}d\xi$$

=

$$I^{M+1,N+n}_{P_i+n,Q_i+2:r}\left[z\left|\begin{matrix}(-a_i,\mu_i)_{1,n};(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\(\nu+\sum_{i=1}^{n}a_i^{'},\mu+\sum_{i=1}^{n}\mu_i);(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};(-\nu-\sum_{i=1}^{n}a_i,\mu+\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right]$$

$$\leq I^{M+1,N+n}_{P_i+n,Q_i+2:r}\left[z\left|\begin{matrix}(-a_i x,x\mu_i)_{1,n};(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\\left(\left(\nu+\sum_{i=1}^{n}a_i^{'},\left(\mu+\sum_{i=1}^{n}\mu_i\right)\right)\right);(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};\left(-\nu-x\sum_{i=1}^{n}a_i^{'},\left(\mu+x\sum_{i=1}^{n}\mu_i\right)\right)\end{matrix}\right.\right]$$

$$\leq I^{M+1,N}_{P_i,Q_i+2:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i}\\\left(\left(\nu+\sum_{i=1}^{n}a_i^{'},\left(\mu+\sum_{i=1}^{n}\mu_i\right)\right)\right);(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};(-\nu,\mu)\end{matrix}\right.\right] \quad (8)$$

provided that

$x\in[0,1]$

$\mathrm{Re}\left[\nu-\min_{1\leq j\leq M}\mu_i\left(\frac{b_j}{\beta_j}\right)\right]>1$, $\mathrm{Re}\left[a_i+\min_{1\leq j\leq M}\mu\left(\frac{b_j}{\beta_j}\right)\right]>0$, $|\arg(z)|<\frac{1}{2}\pi\phi$,

For $j=1,\ldots,M,\ i=1,\ldots,n$ *and* $n\in N$ (9)

<u>Inequality 4</u> By putting $\alpha_i=a_i-\mu_i\xi,\ \beta=\upsilon-\mu\xi$ we get

$$\frac{\prod_{i=1}^{n}\Gamma(1+a_i-\mu_i\xi)}{\Gamma\left(\upsilon-\mu\xi+\sum_{i=1}^{n}(a_i-\mu_i\xi)\right)}\leq\frac{\prod_{i=1}^{n}\Gamma(1+(a_i-\mu_i\xi)x)}{\Gamma\left(\upsilon-\mu\xi+x\sum_{i=1}^{n}(a_i-\mu_i\xi)\right)}\leq\frac{1}{\Gamma(\upsilon-\mu\xi)}$$

multipying all three terms with

$$\frac{1}{2\pi\omega}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\times\frac{1}{\Gamma\left(1+\upsilon+\sum_{i=1}^{n}a_i+\left(\mu-\sum_{i=1}^{n}\mu_i\right)\xi\right)}$$

and then integrating along the contour L

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\times\frac{1}{\Gamma(1+\upsilon+\sum_{i=1}^{n}a_i+\left(\mu-\sum_{i=1}^{n}\mu_i\right)\xi)}$$

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+a_i-\mu_i\xi\right)}{\Gamma\left(\upsilon-\mu\xi+\sum_{i=1}^{n}a_i-\mu_i\xi\right)}d\xi$$

$$\le\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\times\frac{1}{\Gamma(1+\upsilon+\sum_{i=1}^{n}a_i+\left(\mu-\sum_{i=1}^{n}\mu_i\right)\xi)}$$

$$\frac{\prod_{i=1}^{n}\Gamma\left(1+\left(a_i-\mu_i\xi\right)x\right)}{\Gamma\left(\upsilon-\mu\xi+x\sum_{i=1}^{n}\left(a_i-\mu_i\xi\right)\right)}d\xi$$

$$\le\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}\times\frac{1}{\Gamma(1+\upsilon+\sum_{i=1}^{n}a_i+\left(\mu-\sum_{i=1}^{n}\mu_i\right)\xi)}$$

$$\frac{1}{\Gamma\left(\upsilon-\mu\xi\right)}d\xi$$

=

$$I^{M+n,N}_{P_i+1,Q_i+n+1:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i};(-\nu-\sum_{i=1}^{n}a_i,\ \mu-\sum_{i=1}^{n}\mu_i))\\ \left(1+a_i,\mu_i\right)_{1,n};(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};(\nu+\sum_{i=1}^{n}a'_i,\ \mu+\sum_{i=1}^{n}\mu_i\end{matrix}\right.\right]$$

$$\le I^{M+n,N}_{P_i+1,Q_i+n+1:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i};(-\nu-x\sum_{i=1}^{n}a_i,\ \mu-x\sum_{i=1}^{n}\mu_i)\\ \left(1+a_ix,\mu_ix\right)_{1,n};(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};(\nu+\sum_{i=1}^{n}a'_i,\ \mu+\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right]$$

$$\le I^{M,N}_{P_i+1,Q_i+1:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i};(\nu,\mu)\\ (b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i};(\nu+\sum_{i=1}^{n}a'_i,\ \mu+\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right] \quad (10)$$

provided that $x\in[0,1]$

$$\operatorname{Re}\left[a_i - \min_{1\le j\le M} \mu_i\,(\frac{b_j}{\beta_j})\right] > 0,\ \operatorname{Re}\left[\nu + \min_{1\le j\le M} \mu\,(\frac{b_j}{\beta_j})\right] > 1\ ,\ |\arg(z)| < \frac{1}{2}\pi\phi\,,$$

For $j = 1,\ldots,M\,,\, i = 1,\ldots,n \text{ and } n \in N$ (11)

Particular cases :-

(i) On setting r = 1 in the inequalities, the I-function occurring therein reduces to H-function and we arrive at the results due to[1].

(ii) If we put $r = 1,\ \alpha_j = \beta_j = \alpha_{ji} = \beta_{ji} = \alpha$ in the inequalities, the I-function reduces to a Meijer's G-function[3].

$$G^{M,N+n}_{P+n,Q+1}\left[z\left|\begin{matrix}(a_i,\mu_i)_{1,n};(a_j,\alpha)_{1,N};(a_j,\alpha)_{N+1,P}\\ (b_j,\alpha)_{1,M};(b_j,\alpha)_{M+1,Q};\ (-\nu+\sum_{i=1}^{n} a_i,\ \mu+\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right]$$

$$\le G^{M,N+n}_{P+n,Q+1}\left[z\left|\begin{matrix}(a_i x,\mu_i x)_{1,n};(a_j,\alpha)_{1,N};(a_j,\alpha)_{N+1,P}\\ (b_j,\alpha)_{1,M};(b_j,\alpha)_{M+1,Q};\ (-\nu+x\sum_{i=1}^{n} a_i,\ \mu+x\sum_{i=1}^{n}\mu_i)\end{matrix}\right.\right]$$

$$\le G^{M,N}_{P,Q+1}\left[z\left|\begin{matrix}(a_j,\alpha)_{1,N};(a_j,\alpha)_{N+1,P}\\ (b_j,\alpha)_{1,M};(b_j,\alpha)_{M+1,Q};\ (-\nu,\mu)\end{matrix}\right.\right]$$

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