(1) In any chain C, for any $\alpha \in C$ and for any subset $\{\beta_i \mid i \in I\}$ of C, the following are true:

(a) $\alpha \wedge (\vee_{i\in I} \beta_i)$ = $\vee_{i\in I} (\alpha \wedge \beta_i)$ (the infinite meet distributive law).

(b) $\alpha \vee (\wedge_{i\in I} \beta_i)$ = $\wedge_{i\in I} (\alpha \vee \beta_i)$ (the infinite join distributive law).

(2) In a $\vee$-complete poset $(P,\leq)$, $\vee_{j\in J} \vee_{i\in I} \alpha_{ij}$ = $\vee_{i\in I} \vee_{j\in J} \alpha_{ij}$, for any subset $\{\alpha_{ij} \mid i \in I, j \in J\} \subseteq P$.

(3) In any $\vee$-complete poset $(P,\leq)$, for any family $(P_i)_{i\in I}$ of subsets of P, $\vee_{i\in I} (\vee P_i) = \vee(\cup_{i\in I} P_i)$.

(4) For any subset $(t_i)_{i\in I}$ of I = $[0,1]$ of real numbers,

(a) 1 - $\vee_{i\in I} t_i$ $\leq$ $1-t_i$ $\leq$ $\vee_{i\in I} (1-t_i)$ (b) $\wedge_{i\in I} (1-t_i)$ $\leq$ $1-t_i$ $\leq$ 1 - $\wedge_{i\in I} t_i$

(c) 1 - $\wedge_{i\in I} t_i$ = $\vee_{i\in I} (1-t_i)$ (d) 1 - $\vee_{i\in I} t_i$ = $\wedge_{i\in I} (1-t_i)$.

(5) For any subset $(\overline{\alpha}_i)_{i\in I}$ of I^I,

(a) $\overline{1}$ - $\vee_{i\in I} \overline{\alpha}_i$ $\leq$ $\overline{1}-\overline{\alpha}_i$ $\leq$ $\vee_{i\in I} (\overline{1}-\overline{\alpha}_i)$ (b) $\wedge_{i\in I} (\overline{1}-\overline{\alpha}_i)$ $\leq$ $\overline{1}-\overline{\alpha}_i$ $\leq$ $\overline{1}$ - $\wedge_{i\in I} \overline{\alpha}_i$.

(c) $\overline{1}$ - $\wedge_{i\in I} \overline{\alpha}_i$ = $\vee_{i\in I} (\overline{1}-\overline{\alpha}_i)$ (d) $\overline{1}$ - $\vee_{i\in I} \overline{\alpha}_i$ = $\wedge_{i\in I} (\overline{1}-\overline{\alpha}_i)$.

Throughout this paper the capital letters $X, Y Z$ stand for arbitrary but fixed (crisp) sets, the small letters f, g stand for arbitrary but fixed (crisp) maps $f : X \to Y$ and $g : Y \to Z$, the barred italic capital letters $\overline{A},\overline{B},\overline{C},\overline{D},\overline{E},\overline{F}$ together with their suffixes stand for fuzzy subsets, the ltalic capital letters A, B, C, D, E, F and their suffixes stand for the type 2 fuzzy subsets and the capital letters I and J stand for the index sets. Note that the capital letter I is also used for the set of all real numbers between 0 and 1 and whenever there is an ambiguity we make explicit mentions of the same. Also we frequently use the standard convention that $\vee \phi$ = 0 and $\wedge \phi$ = 1.

Later on we show that for any set X, the set $Z_2(X)$ of all type-2 fuzzy subsets of X is a complete infinite distributive complete DeMorgan lattice. Let us recall that, a complete DeMorgan Lattice is a complete lattice with a unary complement operation that satisfies the complete DeMorgan identities, namely, for any subset $(\alpha_i)_{i\in I}$ of L, $(\vee_{i\in I} \alpha_i)^c$ = $\wedge_{i\in I} \alpha_i^c$ and $(\wedge_{i\in I} \alpha_i)^c$ = $\vee_{i\in I} \alpha_i^c$.

A complete infinite distributive DeMorgan lattice is a complete DeMorgan lattice which is also a complete infinite distributive lattice.

3. Type 2 Fuzzy Subsets

Definitions and Statements 3.1

(a) A *type 2 fuzzy subset*, A of a set X is any map $A : X \to I^I$, where I^I is the complete infinite distributive complete de Morgan lattice of the Zadeh fuzzy subsets of I, the interval of all real numbers between 0 and 1.

(b) For any set X, the type 2 fuzzy subset of X denoted by X, defined by Xx = $\overline{1}$ for each $x \in X$, is the *type 2 whole fuzzy subset* of X, where $\overline{1}$ is the constant map on I assuming the value 1 of I and the type 2 fuzzy subset of X denoted by ϕ, defined by ϕ x = $\overline{0}$ for each $x \in X$, is the *type 2 empty fuzzy subset* of X, where $\overline{0}$ is the constant map on I assuming the value 0 of I.

Sometimes the whole type 2 fuzzy subset is denoted by 1 and the empty type 2 fuzzy subset, by 0.

(c) The collection of all type 2 fuzzy subsets of a set X is denoted by $Z_2(X)$.

(d) For any pair of type 2 fuzzy subsets A, B of X, define $A \leq B$ if and only if $Ax \leq Bx$ in I^I for each $x \in X$ or equivalently $Ax\alpha \leq Bx\alpha$ in I for each $\alpha \in I$ and for each $x \in X$.

(e) For any type 2 fuzzy subset A of a set X, the *compliment* of A, denoted by A^c or $1-A$, is defined by $A^c x$ = $\overline{1}$ - Ax, for each $x \in X$, where $(\overline{1}-Ax)\alpha$ = $1-Ax\alpha$ for each $\alpha \in I$.

For any pair of type 2 fuzzy subsets A and B of a set X, the *compliment* of A in B, denoted by $B-A$, is defined by $(B-A)x$ = $Bx \wedge A^c x$ = $Bx \wedge (\overline{1}-Ax)$, for each $x \in X$, where $(Bx \wedge (\overline{1}-Ax))\alpha$ = $Bx\alpha \wedge (\overline{1}-Ax)\alpha$ = $Bx\alpha \wedge (1-Ax\alpha)$ for each $\alpha \in I$.

Let $(A_i)_{i\in I}$ be a family of type 2 fuzzy subsets of a set X. Then

(a) The *type 2 fuzzy union* of $(A_i)_{i\in I}$, denoted by $\vee_{i\in I} A_i$, is defined by, $(\vee_{i\in I} A_i)x$ = $\vee_{i\in I} A_i x$ for each $x \in X$.

(b) The *type 2 fuzzy intersection* of $(A_i)_{i\in I}$, denoted by $\wedge_{i\in I} A_i$, is defined by, $(\wedge_{i\in I} A_i)x$ = $\wedge_{i\in I} A_i x$ for each $x \in X$.

4. DeMorgan Algebra Of Type 2 Fuzzy Subsets

Theorem 4.1 *For any set X, the set $Z_2(X)$ of all type 2 fuzzy subsets of a set X is a complete infinite distributive complete DeMorgan lattice, since I^I is so.*

Proof: (1): $Z_2(X)$ with $\le$ defined by $A \le B$ if and only if $Ax \le Bx$ in I^I for each $x \in X$, is a poset with the largest element, the type 2 fuzzy subset X and the least element, the type 2 fuzzy subset ϕ.

(2): $Z_2(X)$ is a complete lattice with the join and meet defined for any family $F = (A_i)_{i\in I}$ of type 2 fuzzy subsets of X, by $\vee F = \vee_{i\in I} A_i$ where $\vee_{i\in I} A_i$ is the type 2 fuzzy union of $(A_i)_{i\in I}$ and $\wedge F = \wedge_{i\in I} A_i$ where $\wedge_{i\in I} A_i$ is the type 2 fuzzy intersection of $(A_i)_{i\in I}$.

(3): Since $I^I = Z(I)$ is a complete infinite distributive complete deMorgan lattice, it follows that for any type 2 fuzzy subset A of X and for any family $(B_i)_{i\in I}$ of type 2 fuzzy subsets of X, $A \wedge (\vee_{i\in I} B_i) = \vee_{i\in I} (A \wedge B_i)$ and $A \vee (\wedge_{i\in I} B_i) = \wedge_{i\in I} (A \vee B_i)$. Hence $Z_2(X)$ is a complete infinite distributive lattice.

(4): Since (a) for any subset $(\overline{\alpha}_i)_{i\in I}$ of I^I, $\overline{1}$ - $\wedge_{i\in I} \overline{\alpha}_i$ = $\vee_{i\in I} (\overline{1} - \overline{\alpha}_i)$ and $\overline{1}$ - $\vee_{i\in I} \overline{\alpha}_i$ = $\wedge_{i\in I} (\overline{1} - \overline{\alpha}_i)$ and (b) A^c = 1 - A, it follows that $Z_2(X)$ is a complete DeMorgan lattice.

5. Images and Inverse Images of type-2 fuzzy sub sets

In this section the well known notions of fuzzy image and fuzzy inverse image for a fuzzy subset of a set under a crisp map of Zadeh are extended to type 2 fuzzy image and type 2 fuzzy inverse image for a type 2 fuzzy subset of a set under a crisp map in lines similar to L-fuzzy image and L-fuzzy inverse image for an L-fuzzy subset of a set under a crisp map in Goguen's L-Fuzzy Set Theory.

Definitions and Statements 5.1 *Let X, Y be a pair of sets and let $f : X \to Y$ be an arbitrary but fixed map. Let $A: X \to I^I$ and $B: Y \to I^I$ be a pair of type 2 fuzzy subsets of X, Y respectively, where I^I is the complete infinite distributive complete de Morgan lattice of the Zadeh fuzzy subsets of I, the interval of all real numbers between 0 and 1.*

(a) The type 2 fuzzy image of A, denoted by fA, is defined by $fA: Y \to I^I$ such that $fAy = \vee Af^{-1}y$ for each $y \in Y$. Observe that

(1) whenever $y \in Y$ is such that $f^{-1}y = \phi$, fAy = $\vee Af^{-1}y = \vee \phi = \overline{0}$

(2) Whenever $y \in fX$, $(fX)^c y$ = $(Y - fX)y$ = $(Y \wedge (fX)^c)y$ = $Yy \wedge (fX)^c y$ = $Yy \wedge (\overline{1} - fXy)$, where for each $\alpha \in I$, $(Yy \wedge (\overline{1} - fXy))\alpha$ = $Yy\alpha \wedge (\overline{1} - fXy)\alpha$ = $\overline{1}\alpha \wedge (1 - fXy\alpha)$ = $1 \wedge (1 - fXy\alpha)$ = $1 \wedge (1 - (\vee Xf^{-1}y)\alpha)$ = $1 \wedge (1 - (\vee_{x\in f^{-1}y} Xx)\alpha)$ = $1 \wedge (1 - \overline{1}\alpha)$ = $1 \wedge (1 - 1)$ = $1 \wedge 0 = 0$ for each $\alpha \in I$ implying $Yy \wedge (\overline{1} - fXy) = \overline{0}$ or $(fX)^c y = \overline{0}$ and whenever $y \notin fX$, $(fX)^c y = \overline{1}$.

(b) The type 2 fuzzy inverse image of B, denoted by $f^{-1}B$, is defined by $f^{-1}B: X \to I^I$ such that $f^{-1}Bx$ = Bfx for each $x \in X$.

Lemma 5.2 *For any pair of type 2 fuzzy subsets A and C of X and for any subset E of X such that for each $e \in E$, $Ae \le Ce$, we have $\vee AE \le \vee CE$.*

Proof: $e \in E$ implies $Ae \le Ce \le \vee CE$. Therefore $\vee_{e\in E} Ae \le \vee CE$ or $\vee AE \le \vee CE$.

Lemma 5.3 *For any map $A: X \to I^I$ and for any pair of subsets P and Q of X, such that $P \subseteq Q$, we have $\vee AP \le \vee AQ$.*

Proof: If $a \in P$ then $a \in Q$. Therefore $Aa \in AQ$ which implies $Aa \le \vee AQ$, for each $a \in P$. Hence $\vee_{a\in P} Aa \le \vee AQ$ or $\vee AP \le \vee AQ$.

6. Main Results

In what follows, we show that several of the (lattice) algebraic properties that hold good for images and inverse images of crisp sets are also held good for type 2 fuzzy subsets.

Since, the proofs of almost all the statements in the following theorems are straight forward, follow from the definitions and use the standard properties of the complete lattice I^I and the results stated in this paper, we do not explicitly prove them and in stead, choose to state all of them in three relevant Theorems.

Theorem 6.1 *For any map $f : X \to Y$, for any type 2 fuzzy subsets A, C and $(A_i)_{i\in I}$ of X and B, D and $(B_i)_{i\in I}$ of Y, the following are true:*

(1) $A \leq C$ implies $fA \leq fC$.

(2) $B \leq D$ implies $f^{-1} B \leq f^{-1} D$.

(3) $A \leq f^{-1} fA$. In particular, $X = f^{-1} fX$.

(4) $ff^{-1} B \leq B$.

(5) $\vee_{i \in I} fA_i = f(\vee_{i \in I} A_i)$.

(6) $f(\wedge_{i \in I} A_i) \leq \wedge_{i \in I} (fA_i)$. The equality is true whenever f is 1-1 and strict inequality is possible otherwise.

(7) $f^{-1}(\vee_{i \in I} B_i) = \vee_{i \in I} f^{-1} B_i$.

(8) $f^{-1}(\wedge_{i \in I} B_i) = \wedge_{i \in I} f^{-1} B_i$.

(9) $fA = \phi$ iff $A = \phi$. In particular $f \phi = \phi$, and when f is 1-1, $fA = fX$ iff $A = X$.

(10)(1) $f^{-1} B = X$ iff $B \geq fX$. In particular, $f^{-1} Y = f^{-1} fX = X$.

(2) $f^{-1} B = \phi$ iff $B \leq Y - fX$. In particular, $f^{-1} \phi = \phi$.

(11) $fX - fA \leq f(X - A)$ and the equality holds whenever f is one-one.

(12) $f^{-1}(B^c) = (f^{-1} B)^c$ or $f^{-1}(Y - B) = X - f^{-1} B$.

(13) $ff^{-1} B = B \wedge fX$ and hence always $ff^{-1} B \leq B$. In particular, f is onto implies $ff^{-1} B = B$.

(14) $f^{-1} B = f^{-1}(B \wedge fX)$.

(15) $fA \leq B$ iff $A \leq f^{-1}(B)$.

(16)(1) $ff^{-1} f(A) = f(A)$ (2) $f^{-1} ff^{-1}(B) = f^{-1}(B)$.

(17)(1) $\wedge_{i \in I} fA_i = \phi$ implies $\wedge_{i \in I} A_i = \phi$. However the converse is true whenever f is one-one and it may be false otherwise.

(2) $\vee_{i \in I} fA_i = \phi$ iff $\vee_{i \in I} A_i = \phi$.

(18)(1) $\wedge_{i \in I} B_i = \phi$ implies $\wedge_{i \in I} f^{-1} B_i = \phi$. However the converse is true whenever f is onto.

(2) $\vee_{i \in I} B_i = \phi$ implies $\vee_{i \in I} f^{-1} B_i = \phi$. However the converse is true whenever f is onto.

In what follows we show that the identities

1. $(g \circ f)A = g(f(A))$ for all fuzzy subsets A of X
2. $(g \circ f)^{-1} C = f^{-1}(g^{-1} C)$, for all fuzzy subsets C of Z

remain valid even for type 2 fuzzy subsets whenever $f : X \to Y$ and $g : Y \to Z$ are crisp maps. But first we make the following simple observations which will be used in the proofs of the above:

Let X, Y, Z and f, g be as above. Then

- $f^{-1} g^{-1} z \neq \phi$ iff $g^{-1} z \neq \phi$ and $f^{-1} y \neq \phi$ for some $y \in g^{-1} z$ or equivalently
- $f^{-1} g^{-1} z = \phi$ iff $g^{-1} z = \phi$ or $g^{-1} z \neq \phi$ and $f^{-1} y = \phi$ for all $y \in g^{-1} z$.

Theorem 6.2 *For any pair of maps $f : X \to Y$ and $g : Y \to Z$ and for any pair of type 2 fuzzy subset A of X and C of Z, the following are true:*

(1) $(g \circ f)(A) = g(f(A))$.

(2) $(g \circ f)^{-1} (C) = f^{-1}(g^{-1}(C))$.

In what follows, as in the crisp set up, we characterize injectivity and surjectivity of maps in terms of some lattice algebraic properties of type 2 fuzzy images and type 2 fuzzy inverse images.

Theorem 6.3 *For any map $f : X \to Y$, the followimg are true:*

(1) f is injective iff for any type 2 fuzzy subset A of X, $f^{-1} fA = A$.

(2) f is injective iff for any pair of type 2 fuzzy subsets A_1 and A_2 of X, $A_1 < A_2$ implies $fA_1 < fA_2$.

(3) f is injective iff for any pair of type 2 fuzzy subsets A_1 and A_2 of X, $fA_1 \leq fA_2$ implies $A_1 \leq A_2$.

(4) f is injective iff for any family of type 2 fuzzy subsets $(A_i)_{i \in I}$ of X, $f(\wedge_{i \in I} A_i) = \wedge_{i \in I} fA_i$.

(5) f is surjective iff for any type 2 fuzzy subset B of Y, $B = ff^{-1} B$.

(6) f is surjective iff for any pair of type 2 fuzzy subsets B_1 and B_2 of Y, $B_1 < B_2$ implies $f^{-1} B_1 < f^{-1} B_2$.

(7) f is surjective iff for any type 2 fuzzy subset B of Y, $f^{-1} B_1 \leq f^{-1} B_2$ implies $B_1 \leq B_2$.

(8) f is surjective iff for any type 2 fuzzy subset B of Y, $f^{-1} B = 0$ implies $B = 0$.

References

[1] D. Dubois, and H. Prade, Operations on Fuzzy Numbers International Journal of Systems Science, Vol. 9(6), 1978, Pages 613-626.

[2] D. Dubois, and H. Prade, Operations on Fuzzy-valued Logic, Information and Control, Vol. 43, 1979, Pages 224-240.

[3] D. Dubois, and H. Prade, Fuzzy Sets and Systems: Theory and Applications Acadamic Press Inc., New York, 1980.

[4] H. Hamrawi, Type-2 Fuzzy Alpha Cuts, Ph.D. Thesis, De Montfort University, April 2011.

[5] H. Hamrawi and S. Coupland, (2009a), Non-specifici ty measures for type-2 fuzzy sets, in 'Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE International Conference on', Korea, pp. 732-737.

[6] H. Hamrawi and S. Coupland, (2009b), Type-2 fuzzy arithmetic using alpha-planes, in 'Fuzzy Systems, 2009. IFSA/EUSFLAT 2009. IFSA International Congress on', Portugal, pp.606-611.

[7] H. Hamrawi and S. Coupland, (2010), Measures of uncertainty for type-2 fuzzy sets, in 'Computational Intelligence (UKCI), 2010 UK Workshop on', Essex, UK, pp. 1-6.

[8] H. Hamrawi, S. Coupland, and R. John, (2009), Extending operations on type-2 fuzzy sets, in 'Computational Intelligence (UKCI), 2009 UK Workshop on', Nottingham, UK.

[9] H. Hamrawi, S. Coupland, and R. John, (2010), A novel alpha-cut representation for type-2 fuzzy sets, in 'Fuzzy Systems (FUZZ), 2010 IEEE International Conference on', Barcelona, Spain, pp. 1-8.

[10] E. Hisdal, The IF THEN ELSE Statement and Interval Valued Fuzzy Sets of Higher Type, International Journal of Man-Machine Studies, Vol. 15, 1981, P385-455.

[11] N.N. Karnik and J.M. Mendel, Introduction to Type-2 Fuzzy Logic Systems, Proc. 7th Intl. Conf. on Fuzzy Systems FUZZY-IEEE'98, Pages 915-920, 1998.

[12] J.M. Mendel, and R. John, Type-2 fuzzy sets made simple, IEEE Transaction on Fuzzy Systems, Vol.10(2), 2002, Pages 117-127.

[13] M. Mizumoto, and K. Tanaka, Fuzzy Tets of Type-2 Under Algebraic Product and Algebraic Sum, Fuzzy Sets and Systems, Vol.5, 1981, Pages 277-290.

[14] M. Mizumoto and K.Tanaka, Some Properties of Fuzzy Sets of Type-2, Information and Control, Vol.31, 1976, Pages 312-340.

[15] J. Nieminen, On the algebraic structure of fuzzy sets of type-2, Kybernetica, Vol. 13(4), 1977.

[16] Nistala V.E.S. Murthy and Lokavarapu Sujatha, Lattice Algebraic Properties Of (Inverse) Images Of Type-2 Fuzzy Subsets, MACS-2K12, 6-7 July 2012, Department of Mathematics, A.K.N. University, Rajamandry, A.P. State, INDIA.

[17] Nistala V.E.S. Murthy and Jami L. Prasanna, A Theory of Lattice Interval Valued Fuzzy Sets and Fuzzy Maps Between Different Lattice Interval Valued Fuzzy Sets, International Journal of Advanced Research in Computer Science, Vol. 4, No. 2, P227-250, 2013.

[18] L.A. Zadeh, Calculus of Fuzzy Restriction, Fuzzy Sets and Their Applications to Cognitive and Decision Processes, L.A. Zadeh, K.S. Fu, K. Tanaka and M. Shimura editors, Academic Press, New York, 1975, Pages 1-39.

[19] L.A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning, Information Sciences, Vol.8, 1975, Pages 199-249.

[20] L.A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning-II, Information Sciences, Vol.8, 1975, Pages 301-357.

Author Profile

Nistala V.E.S. Murthy received his M.S. in Mathematics from University of Hyderabad, Hyderabad, A.P. State, India, in 1981, Ph.D. from the University of Toledo, Toledo, OH-43606, U.S.A. in 1989 and an M.S. in Software Systems from B.I.T.S., Pilani Rajasthan State, India, in 2000; taught a wide range of courses both in Mathematics and Computer Science in various levels and is currently professor in the department of Mathematics, AUCST, Andhra University, Vizag-53003, A.P. State, India. His areas of publications include Various Fuzzy Set Theories and Their Applications in Mathematics (Set Theory, Algebra and Topology) and Computer Science (Data Security/Warehousing/Mining/ Hiding) and Natural Language Modeling (Reprints/ Preprints are Available on Request at drnvesmurthy@rediffmail.com or at http://andhrauniversity.academia.edu /NistalaVES Murthy). In his little own way, he (1) developed f-Set Theory generalizing L-fuzzy set Theory of Goguen which generalized the [0,1]-fuzzy set theory of Zadeh, the Father of Fuzzy Set Theory (2) imposed and studied algebraic /topological structures on f-sets (3) proved Representation Theorems for f-Algebraic and f-Topological objects in general.

Lokavarapu Sujatha received her M.Sc. in mathematics from Andhra University in 2008. Currently she is working on her doctoral thesis in Lattice (algebraic) properties of representations of type 2 fuzzy sets.