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Lattice (Algebraic) Properties of (Inverse) Images of Type-2 Fuzzy Subsets

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Abstract: ***The aim of this paper is basically to study some of the standard lattice algebraic properties for type-2 fuzzy images of type-2 fuzzy sub sets and type-2 fuzzy inverse images of type-2 fuzzy sub sets under a crisp map.***

Keywords: Type-2 fuzzy subset, type-2 fuzzy image of type-2 fuzzy sub set and type-2 fuzzy inverse image of type-2 fuzzy subset.

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1. Introduction

The traditional view in science, especially in mathematics, is to avoid uncertainty at all levels at any cost. Thus "being uncertain" is regarded as "being unscientific". But unfortunately in real life most of the information that we have to deal with is mostly uncertain.

One of the paradigm shifts in science and mathematics in this century is to accept uncertainty as part of science and the desire to be able to deal with it, as there is very little left out in the practical real world for scientific and mathematical processing without this acceptance!

One of the earliest successful attempts in this directions is the development of the Theories of Probability and Statistics. However, both of them have their own natural limitations. Another successful attempt again in the same direction is the so called Fuzzy Set Theory, introduced by Zadeh.

According to Zadeh, a fuzzy subset of a set X is any function f from the set X itself to the closed interval $[0,1]$ of real numbers. An element x belonging to the set X belongs to the fuzzy subset f with the degree of membership fx, a *real number* between 0 and 1. These fuzzy sub sets of a set X are also called type 1 fuzzy sub sets of x.

Observing that fuzzy subsets themselves require a *specific* real number between/ including 0 and 1 to be associated with *each* element of X while crisp subsets require the *specific* real numbers 0 or 1 to be associated with *each* element of X, which in itself was *not* quite capturing the necessary mathematical abstraction for subsets being fuzzy, Zadeh[18]-[20] himself introduced the so called interval valued fuzzy subsets and type-2 fuzzy sub sets of a set X as means to handle even *more inexact/uncertain* information Murthy-Lakshmi Prasanna[17].

Thus, a type-2 fuzzy sub set of a set X is any function f from the set X itself to the complete lattice of all type 1 fuzzy subsets of the closed interval $[0,1]$ of real numbers. An element x belonging to the set X belongs to the fuzzy subset f with the degree of membership fx, a type 1 fuzzy sub set of the closed interval $[0,1]$.

The same Zadeh[18]-[20] also defines 1. type-2 fuzzy union, type-2 fuzzy intersection for any pair of type-2 fuzzy sub sets and both of these are easily extendable to arbitrary collections of type-2 fuzzy sub sets and 2. the type-2 fuzzy compliment for a type-2 fuzzy subset.

However, Mizumoto and Tanaka [13],]14] have studied the set-theoretic operations of type-2 sets, properties of membership grades of such sets, and examined the operations of algebraic product and algebraic sum for them. Nieminen[15] has provided more details about algebraic structure of type-2 sets.

Dubois and Prade[1]-[3] discussed composition of type-2 relations as an extension of the type-1 sup-star composition, but this formula is only for minimum t-norm. Karnik and Mendel [11] have provided a general formula for the extended sup-star composition of type-2 relations.

For more details one can refer to Hamrawi[4], which offers an excellent panaromic view of type-2 fuzzy sets along with comparisons and contrasts with interval valued fuzzy subsets. In fact, Hamrawi and coupland[5]-[7], studied non-specificity measures for type-2 fuzzy subsets, type-2 fuzzy arithmetic using α-planes and measures of un-certainty for type-2 fuzzy subsets. Hamrawi, Coupland and John[8],[9], studied extension of operations on type-2 fuzzy subsets and α-cut representations for type-2 fuzzy subsets.

Interestingly, the (lattice) algebraic properties of type-2 fuzzy images and type-2 fuzzy inverse images of type-2 fuzzy sub sets under a crisp map which are not only important in the study of both type-2 fuzzy Algebra and type-2 fuzzy Topology but also are necessary for the individual/exclusive development of Type-2 Fuzzy Set Theory are *not* yet investigated.

However, Mendel and John[12] introduced the notion of type-2 fuzzy image of a type-2 Cartesian product of n type-2 fuzzy subsets of n universes as follows (Cf. Theorem 2.4.1. on Page 36-37 of Hamrawi[4]):

Let, $X = X_1 \times \ldots \times X_n$, be the Cartesian product of universes, and $\tilde{A}_1, \ldots, \tilde{A}_n$ be type-2 fuzzy subsets in each respective universe. Also let Y be another universe and $\tilde{B}$ be a type-2 fuzzy subset of Y such that $\tilde{B} = f(\tilde{A}_1, \ldots \tilde{A}_n)$, where $f : X \to Y$ is any mapping. Then applying the Extension Principle to type-2 fuzzy subsets (Type-2 Extension Principle) leads to the following:

$\tilde{B} \Leftrightarrow \tilde{B}(y) = sup_{(x_1,\ldots,x_n) \in f^{-1}(y)} min(\tilde{A}_1(x_1), \ldots, \tilde{A}_n(x_n))$, where $y = f(x_1, \ldots, x_n)$.

Consequently, in this Paper (a) we *re*introduce the notion of type-2 fuzzy image of a type-2 fuzzy subset under a crisp map (b) introduce the notion of type-2 fuzzy inverse image of a type-2 fuzzy subset under a crisp map (c) make an exclusive Lattice theoretic study of both type-2 fuzzy images and type-2 fuzzy inverse images of type-2 fuzzy subsets under a crisp map in lines similar to the one in various other Set Theories, like Crisp Set Theory, (L-) Zadeh's (Interval Valued) Fuzzy Set Theory, Goguen's L-Fuzzy Set Theory, Atanassov's (L-) Intuitionistic Fuzzy Set Theory etc.. Further, as in the crisp set up, injectivity and surjectivity of maps in terms of some lattice algebraic properties of type-2 fuzzy images and type-2 fuzzy inverse images are also characterized.

Now coming back to the developments in this side of these Uncertainty Theories, Goguen unified some of them mathematically. However, one must observe here that, as mentioned earlier, when it comes to practical applications, the fuzzy subsets and the type-2 fuzzy subsets are quite different because fuzzy sets or L-fuzzy sets require a specific real number between 0 and 1 of [0,1] or a specific lattice element of L to be associated with *each* of its elements while type-2 fuzzy sets require a *reasonable set of ordered pairs in [0,1]* to be associated with *each* of its elements. For an excellent treatment of various applications of type 2 fuzzy logic, one can refer to Type 2 Fuzzy Logic: Theory and Applications by Oscar Castillo and Patricia Melin, Studies in Fuzziness and Soft Computing, Vol. 223, Springer-Verlag, 2008.

A preliminary version of this paper was *orally* presented in a conference and the conference details are available in Murthy-Sujatha[16].

2. Preliminaries

We assume the following notions from Lattice Theory: (sub)poset, order preserving map between posets, (least) upper bound, (greatest) lower bound, least element, greatest element in a poset, (complete) (semi) lattice, (complete) sub (semi) lattice, (complete) homomorphism of (semi) lattices, ideal, filter and Galois connection etc.. One can refer to any standard text book on Lattice Theory for them. Observe that by a complete lattice we mean a poset in which every *nonempty* subset has both infimum(denoted by $\wedge$) and supremum(denoted by $\vee$), a subset of a complete lattice is a complete sub lattice if and only if it is closed under infimums and supremums for its *nonempty* subsets.

For any set X, the set of all fuzzy subsets of X be denoted by $Z(X)$. By defining, for any pair of fuzzy subsets $\bar{A}$ and $\bar{B}$ of X, $\bar{A} \leq \bar{B}$ iff $\bar{A}x \leq \bar{B}x$ for all $x \in X$, $Z(X)$ becomes a complete infinitely distributive lattice.

In this case for any family $(\bar{A}_i)_{i \in I}$ of fuzzy subsets of X, the *fuzzy union* denoted by $\vee_{i \in I} \bar{A}_i$ is defined by $(\vee_{i \in I} \bar{A}_i)x = \vee_{i \in I} \bar{A}_i x$ for each $x \in X$. The *fuzzy intersection* of $(\bar{A}_i)_{i \in I}$, denoted by $\wedge_{i \in I} \bar{A}_i$, is defined by, $(\wedge_{i \in I} \bar{A}_i)x = \wedge_{i \in I} \bar{A}_i x$ for each $x \in X$.

For any set X, one can naturally associate, with X, the fuzzy subset $\bar{X}$, where $\bar{X}x$ takes the value 1 of $[0,1]$ for each $x \in X$, which is the whole fuzzy subset of X and turns out to be the largest element in $Z(X)$. Observe that then, the fuzzy subset $\bar{\phi}$ of X such that $\bar{\phi}x$ takes the value 0 of $[0,1]$ for each $x \in X$, which is the empty fuzzy subset of X and turns out to be the least element in $Z(X)$.

Sometimes the whole fuzzy subset is denoted by $\bar{1}$ and the empty fuzzy subset, by $\bar{0}$.

For any fuzzy subset $\bar{A}$ the fuzzy complement of $\bar{A}$, denoted by $\bar{A}^c$ is defined by $\bar{A}x = 1 - \bar{A}x$ for each $x \in X$. Further for any pair $\bar{A}$, $\bar{B}$ of fuzzy subsets of X, we define $\bar{B} - \bar{A}$ to be $\bar{B} \wedge (\bar{A})^c$.

Since for any subset $(\bar{A}_i)_{i \in I}$ of $Z(X)$, $(\vee_{i \in I} \bar{A}_i)^c = \wedge_{i \in I} \bar{A}_i^c$ and $(\wedge_{i \in I} \bar{A}_i)^c = \vee_{i \in I} \bar{A}_i^c$, it follows that $Z(X)$ is a complete infinite distributive complete de Morgan lattice.

In particular, $I^I = Z(I)$ is a complete infinite distributive complete de Morgan lattice.

The following results from lattice theory will be used in this paper later:

(1) In any chain C, for any $\alpha \in C$ and for any subset $\{\beta_i \mid i \in I\}$ of C, the following are true:

(a) $\alpha \wedge (\vee_{i\in I} \beta_i)$ = $\vee_{i\in I} (\alpha \wedge \beta_i)$ (the infinite meet distributive law).

(b) $\alpha \vee (\wedge_{i\in I} \beta_i)$ = $\wedge_{i\in I} (\alpha \vee \beta_i)$ (the infinite join distributive law).

(2) In a $\vee$-complete poset $(P,\leq)$, $\vee_{j\in J} \vee_{i\in I} \alpha_{ij}$ = $\vee_{i\in I} \vee_{j\in J} \alpha_{ij}$, for any subset $\{\alpha_{ij} \mid i \in I, j \in J\} \subseteq P$.

(3) In any $\vee$-complete poset $(P,\leq)$, for any family $(P_i)_{i\in I}$ of subsets of P, $\vee_{i\in I} (\vee P_i) = \vee(\cup_{i\in I} P_i)$.

(4) For any subset $(t_i)_{i\in I}$ of I = $[0,1]$ of real numbers,

(a) 1 - $\vee_{i\in I} t_i$ $\leq$ $1-t_i$ $\leq$ $\vee_{i\in I} (1-t_i)$ (b) $\wedge_{i\in I} (1-t_i)$ $\leq$ $1-t_i$ $\leq$ 1 - $\wedge_{i\in I} t_i$

(c) 1 - $\wedge_{i\in I} t_i$ = $\vee_{i\in I} (1-t_i)$ (d) 1 - $\vee_{i\in I} t_i$ = $\wedge_{i\in I} (1-t_i)$.

(5) For any subset $(\overline{\alpha}_i)_{i\in I}$ of I^I,

(a) $\overline{1}$ - $\vee_{i\in I} \overline{\alpha}_i$ $\leq$ $\overline{1}-\overline{\alpha}_i$ $\leq$ $\vee_{i\in I} (\overline{1}-\overline{\alpha}_i)$ (b) $\wedge_{i\in I} (\overline{1}-\overline{\alpha}_i)$ $\leq$ $\overline{1}-\overline{\alpha}_i$ $\leq$ $\overline{1}$ - $\wedge_{i\in I} \overline{\alpha}_i$.

(c) $\overline{1}$ - $\wedge_{i\in I} \overline{\alpha}_i$ = $\vee_{i\in I} (\overline{1}-\overline{\alpha}_i)$ (d) $\overline{1}$ - $\vee_{i\in I} \overline{\alpha}_i$ = $\wedge_{i\in I} (\overline{1}-\overline{\alpha}_i)$.

Throughout this paper the capital letters $X, Y Z$ stand for arbitrary but fixed (crisp) sets, the small letters f, g stand for arbitrary but fixed (crisp) maps $f: X \to Y$ and $g: Y \to Z$, the barred italic capital letters $\overline{A},\overline{B},\overline{C},\overline{D},\overline{E},\overline{F}$ together with their suffixes stand for fuzzy subsets, the ltalic capital letters A, B, C, D, E, F and their suffixes stand for the type 2 fuzzy subsets and the capital letters I and J stand for the index sets. Note that the capital letter I is also used for the set of all real numbers between 0 and 1 and whenever there is an ambiguity we make explicit mentions of the same. Also we frequently use the standard convention that $\vee \phi$ = 0 and $\wedge \phi$ = 1.

Later on we show that for any set X, the set $Z_2(X)$ of all type-2 fuzzy subsets of X is a complete infinite distributive complete DeMorgan lattice. Let us recall that, a complete DeMorgan Lattice is a complete lattice with a unary complement operation that satisfies the complete DeMorgan identities, namely, for any subset $(\alpha_i)_{i\in I}$ of L, $(\vee_{i\in I} \alpha_i)^c$ = $\wedge_{i\in I} \alpha_i^c$ and $(\wedge_{i\in I} \alpha_i)^c$ = $\vee_{i\in I} \alpha_i^c$.

A complete infinite distributive DeMorgan lattice is a complete DeMorgan lattice which is also a complete infinite distributive lattice.

3. Type 2 Fuzzy Subsets

Definitions and Statements 3.1

(a) A *type 2 fuzzy subset*, A of a set X is any map $A: X \to I^I$, where I^I is the complete infinite distributive complete de Morgan lattice of the Zadeh fuzzy subsets of I, the interval of all real numbers between 0 and 1.

(b) For any set X, the type 2 fuzzy subset of X denoted by X, defined by Xx = $\overline{1}$ for each $x \in X$, is the *type 2 whole fuzzy subset* of X, where $\overline{1}$ is the constant map on I assuming the value 1 of I and the type 2 fuzzy subset of X denoted by ϕ, defined by ϕ x = $\overline{0}$ for each $x \in X$, is the *type 2 empty fuzzy subset* of X, where $\overline{0}$ is the constant map on I assuming the value 0 of I.

Sometimes the whole type 2 fuzzy subset is denoted by 1 and the empty type 2 fuzzy subset, by 0.

(c) The collection of all type 2 fuzzy subsets of a set X is denoted by $Z_2(X)$.

(d) For any pair of type 2 fuzzy subsets A, B of X, define $A \leq B$ if and only if $Ax \leq Bx$ in I^I for each $x \in X$ or equivalently $Ax\alpha \leq Bx\alpha$ in I for each $\alpha \in I$ and for each $x \in X$.

(e) For any type 2 fuzzy subset A of a set X, the *compliment* of A, denoted by A^c or $1-A$, is defined by $A^c x$ = $\overline{1}$ - Ax, for each $x \in X$, where $(\overline{1}-Ax)\alpha$ = $1-Ax\alpha$ for each $\alpha \in I$.

For any pair of type 2 fuzzy subsets A and B of a set X, the *compliment* of A in B, denoted by $B-A$, is defined by $(B-A)x$ = $Bx \wedge A^c x$ = $Bx \wedge (\overline{1}-Ax)$, for each $x \in X$, where $(Bx \wedge (\overline{1}-Ax))\alpha$ = $Bx\alpha \wedge (\overline{1}-Ax)\alpha$ = $Bx\alpha \wedge (1-Ax\alpha)$ for each $\alpha \in I$.

Let $(A_i)_{i\in I}$ be a family of type 2 fuzzy subsets of a set X. Then

(a) The *type 2 fuzzy union* of $(A_i)_{i\in I}$, denoted by $\vee_{i\in I} A_i$, is defined by, $(\vee_{i\in I} A_i)x$ = $\vee_{i\in I} A_i x$ for each $x \in X$.

(b) The *type 2 fuzzy intersection* of $(A_i)_{i\in I}$, denoted by $\wedge_{i\in I} A_i$, is defined by, $(\wedge_{i\in I} A_i)x$ = $\wedge_{i\in I} A_i x$ for each $x \in X$.

4. DeMorgan Algebra Of Type 2 Fuzzy Subsets

Theorem 4.1 *For any set X, the set $Z_2(X)$ of all type 2 fuzzy subsets of a set X is a complete infinite distributive complete DeMorgan lattice, since I^I is so.*

Proof: (1): $Z_2(X)$ with $\leq$ defined by $A \leq B$ if and only if $Ax \leq Bx$ in I^I for each $x \in X$, is a poset with the largest element, the type 2 fuzzy subset X and the least element, the type 2 fuzzy subset ϕ.

(2): $Z_2(X)$ is a complete lattice with the join and meet defined for any family $F = (A_i)_{i \in I}$ of type 2 fuzzy subsets of X, by $\vee F = \vee_{i \in I} A_i$ where $\vee_{i \in I} A_i$ is the type 2 fuzzy union of $(A_i)_{i \in I}$ and $\wedge F = \wedge_{i \in I} A_i$ where $\wedge_{i \in I} A_i$ is the type 2 fuzzy intersection of $(A_i)_{i \in I}$.

(3): Since $I^I = Z(I)$ is a complete infinite distributive complete deMorgan lattice, it follows that for any type 2 fuzzy subset A of X and for any family $(B_i)_{i \in I}$ of type 2 fuzzy subsets of X, $A \wedge (\vee_{i \in I} B_i) = \vee_{i \in I} (A \wedge B_i)$ and $A \vee (\wedge_{i \in I} B_i) = \wedge_{i \in I} (A \vee B_i)$. Hence $Z_2(X)$ is a complete infinite distributive lattice.

(4): Since (a) for any subset $(\overline{\alpha}_i)_{i \in I}$ of I^I, $\overline{1}$ - $\wedge_{i \in I} \overline{\alpha}_i$ = $\vee_{i \in I} (\overline{1} - \overline{\alpha}_i)$ and $\overline{1}$ - $\vee_{i \in I} \overline{\alpha}_i$ = $\wedge_{i \in I} (\overline{1} - \overline{\alpha}_i)$ and (b) A^c = 1 - A, it follows that $Z_2(X)$ is a complete DeMorgan lattice.

5. Images and Inverse Images of type-2 fuzzy sub sets

In this section the well known notions of fuzzy image and fuzzy inverse image for a fuzzy subset of a set under a crisp map of Zadeh are extended to type 2 fuzzy image and type 2 fuzzy inverse image for a type 2 fuzzy subset of a set under a crisp map in lines similar to L-fuzzy image and L-fuzzy inverse image for an L-fuzzy subset of a set under a crisp map in Goguen's L-Fuzzy Set Theory.

Definitions and Statements 5.1 *Let X, Y be a pair of sets and let $f : X \to Y$ be an arbitrary but fixed map. Let $A : X \to I^I$ and $B : Y \to I^I$ be a pair of type 2 fuzzy subsets of X, Y respectively, where I^I is the complete infinite distributive complete de Morgan lattice of the Zadeh fuzzy subsets of I, the interval of all real numbers between 0 and 1.*

(a) The type 2 fuzzy image of A, denoted by fA, is defined by $fA : Y \to I^I$ such that fAy = $\vee Af^{-1}y$ for each $y \in Y$. Observe that

(1) whenever $y \in Y$ is such that $f^{-1}y = \phi$, fAy = $\vee Af^{-1}y = \vee \phi = \overline{0}$

(2) Whenever $y \in fX$, $(fX)^c y$ = $(Y - fX)y$ = $(Y \wedge (fX)^c)y$ = $Yy \wedge (fX)^c y$ = $Yy \wedge (\overline{1} - fXy)$, where for each $\alpha \in I$, $(Yy \wedge (\overline{1} - fXy))\alpha$ = $Yy\alpha \wedge (\overline{1} - fXy)\alpha$ = $\overline{1}\alpha \wedge (1 - fXy\alpha)$ = $1 \wedge (1 - fXy\alpha)$ = $1 \wedge (1 - (\vee Xf^{-1}y)\alpha)$ = $1 \wedge (1 - (\vee_{x \in f^{-1}y} Xx)\alpha)$ = $1 \wedge (1 - \overline{1}\alpha)$ = $1 \wedge (1-1)$ = $1 \wedge 0 = 0$ for each $\alpha \in I$ implying $Yy \wedge (\overline{1} - fXy) = \overline{0}$ or $(fX)^c y = \overline{0}$ and whenever $y \notin fX$, $(fX)^c y = \overline{1}$.

(b) The type 2 fuzzy inverse image of B, denoted by $f^{-1}B$, is defined by $f^{-1}B : X \to I^I$ such that $f^{-1}Bx$ = Bfx for each $x \in X$.

Lemma 5.2 *For any pair of type 2 fuzzy subsets A and C of X and for any subset E of X such that for each $e \in E$, $Ae \leq Ce$, we have $\vee AE \leq \vee CE$.*

Proof: $e \in E$ implies $Ae \leq Ce \leq \vee CE$. Therefore $\vee_{e \in E} Ae \leq \vee CE$ or $\vee AE \leq \vee CE$.

Lemma 5.3 *For any map $A : X \to I^I$ and for any pair of subsets P and Q of X, such that $P \subseteq Q$, we have $\vee AP \leq \vee AQ$.*

Proof: If $a \in P$ then $a \in Q$. Therefore $Aa \in AQ$ which implies $Aa \leq \vee AQ$, for each $a \in P$. Hence $\vee_{a \in P} Aa \leq \vee AQ$ or $\vee AP \leq \vee AQ$.

6. Main Results

In what follows, we show that several of the (lattice) algebraic properties that hold good for images and inverse images of crisp sets are also held good for type 2 fuzzy subsets.

Since, the proofs of almost all the statements in the following theorems are straight forward, follow from the definitions and use the standard properties of the complete lattice I^I and the results stated in this paper, we do not explicitly prove them and in stead, choose to state all of them in three relevant Theorems.

Theorem 6.1 *For any map $f : X \to Y$, for any type 2 fuzzy subsets A, C and $(A_i)_{i \in I}$ of X and B, D and $(B_i)_{i \in I}$ of Y, the following are true:*

(1) $A \le C$ implies $fA \le fC$.

(2) $B \le D$ implies $f^{-1}B \le f^{-1}D$.

(3) $A \le f^{-1}fA$. In particular, $X = f^{-1}fX$.

(4) $ff^{-1}B \le B$.

(5) $\vee_{i\in I} fA_i = f(\vee_{i\in I} A_i)$.

(6) $f(\wedge_{i\in I} A_i) \le \wedge_{i\in I}(fA_i)$. The equality is true whenever f is 1-1 and strict inequality is possible otherwise.

(7) $f^{-1}(\vee_{i\in I} B_i) = \vee_{i\in I} f^{-1}B_i$.

(8) $f^{-1}(\wedge_{i\in I} B_i) = \wedge_{i\in I} f^{-1}B_i$.

(9) $fA = \phi$ iff $A = \phi$. In particular $f\ \phi = \phi$, and when f is 1-1, $fA = fX$ iff $A = X$.

(10)(1) $f^{-1}B = X$ iff $B \ge fX$. In particular, $f^{-1}Y = f^{-1}fX = X$.

(2) $f^{-1}B = \phi$ iff $B \le Y - fX$. In particular, $f^{-1}\ \phi = \phi$.

(11) $fX - fA \le f(X - A)$ and the equality holds whenever f is one-one.

(12) $f^{-1}(B^c) = (f^{-1}B)^c$ or $f^{-1}(Y - B) = X - f^{-1}B$.

(13) $ff^{-1}\ B = B \wedge fX$ and hence always $ff^{-1}B \le B$. In particular, f is onto implies $ff^{-1}B = B$.

(14) $f^{-1}B = f^{-1}(B \wedge fX)$.

(15) $fA \le B$ iff $A \le f^{-1}(B)$.

(16)(1) $ff^{-1}f(A) = f(A)$ (2) $f^{-1}ff^{-1}(B) = f^{-1}(B)$.

(17)(1) $\wedge_{i\in I} fA_i = \phi$ implies $\wedge_{i\in I} A_i = \phi$. However the converse is true whenever f is one-one and it may be false otherwise.

(2) $\vee_{i\in I} fA_i = \phi$ iff $\vee_{i\in I} A_i = \phi$.

(18)(1) $\wedge_{i\in I} B_i = \phi$ implies $\wedge_{i\in I} f^{-1}B_i = \phi$. However the converse is true whenever f is onto.

(2) $\vee_{i\in I} B_i = \phi$ implies $\vee_{i\in I} f^{-1}B_i = \phi$. However the converse is true whenever f is onto.

In what follows we show that the identities

1. $(g \circ f)A = g(f(A))$ for all fuzzy subsets A of X
2. $(g \circ f)^{-1}C = f^{-1}(g^{-1}C)$, for all fuzzy subsets C of Z

remain valid even for type 2 fuzzy subsets whenever $f : X \to Y$ and $g : Y \to Z$ are crisp maps. But first we make the following simple observations which will be used in the proofs of the above:

Let X, Y, Z and f, g be as above. Then

- $f^{-1}g^{-1}z \ne \phi$ iff $g^{-1}z \ne \phi$ and $f^{-1}y \ne \phi$ for some $y \in g^{-1}z$ or equivalently
- $f^{-1}g^{-1}z = \phi$ iff $g^{-1}z = \phi$ or $g^{-1}z \ne \phi$ and $f^{-1}y = \phi$ for all $y \in g^{-1}z$.

Theorem 6.2 *For any pair of maps $f : X \to Y$ and $g : Y \to Z$ and for any pair of type 2 fuzzy subset A of X and C of Z, the following are true:*

(1) $(g \circ f)(A) = g(f(A))$.

(2) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

In what follows, as in the crisp set up, we characterize injectivity and surjectivity of maps in terms of some lattice algebraic properties of type 2 fuzzy images and type 2 fuzzy inverse images.

Theorem 6.3 *For any map $f : X \to Y$, the followimg are true:*

(1) f is injective iff for any type 2 fuzzy subset A of X, $f^{-1}fA = A$.

(2) f is injective iff for any pair of type 2 fuzzy subsets A_1 and A_2 of X, $A_1 < A_2$ implies $fA_1 < fA_2$.

(3) f is injective iff for any pair of type 2 fuzzy subsets A_1 and A_2 of X, $fA_1 \le fA_2$ implies $A_1 \le A_2$.

(4) f is injective iff for any family of type 2 fuzzy subsets $(A_i)_{i\in I}$ of X, $f(\wedge_{i\in I} A_i) = \wedge_{i\in I} fA_i$.

(5) f is surjective iff for any type 2 fuzzy subset B of Y, $B = ff^{-1}B$.

(6) f is surjective iff for any pair of type 2 fuzzy subsets B_1 and B_2 of Y, $B_1 < B_2$ implies $f^{-1}B_1 < f^{-1}B_2$.

(7) f is surjective iff for any type 2 fuzzy subset B of Y, $f^{-1}B_1 \le f^{-1}B_2$ implies $B_1 \le B_2$.

(8) f is surjective iff for any type 2 fuzzy subset B of Y, $f^{-1}B = 0$ implies $B = 0$.

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