

Quasi-Static Thermal Stresses in a Thin Rectangular Plate Due to Heat Generation

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Abstract: This paper is concerned with the determination of displacement and thermal stresses in a thin rectangular plate due to internal heat generation within it. Initially the plate is at arbitrary temperature $f(x, y)$, while the boundary at $y = 0$ is kept at temperature $\phi(x, t)$ and the remaining boundaries are kept at zero temperature. The governing heat conduction equation has been solved by the method of double integral transform technique. The results are obtained in series form in term of circular functions. The results for displacement and thermal stresses have been computed numerically and illustrated graphically.

Keywords: Heat conduction problem, Thermal stresses, Heat generation, Non-homogeneous, Rectangular plate.

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1. Introduction

Rectangular plates are one of the most widely used structural elements in various engineering applications such as the pavements of highways and airports, building walls and bridge decks and so on. In most cases, the plates have to carry various loads. Therefore, a thorough understanding of their mechanics Characteristics is essential for designers.

Ootao et al. [1] studied theoretical analysis of a three dimensional transient thermal stress problem for a non-homogeneous hollow circular cylinder due to a moving heat source in the axial direction from the inner and outer surfaces. Tanigawa et al. [2] have studied theoretical analysis of two dimensional thermoplastic bending deformation of plate subjected to partially distributed heat supply. Tanigawa et al. [3] have discussed stress analysis of a rectangular plate and its thermal stress intensity factor for compressive field. Vihak et al. [4] have investigated the solution of the plane thermoelasticity problem for a rectangular domain. Adam et al. [5] have determined thermoelastic vibration of a laminated rectangular plate subjected to a thermal shock. Ghadle et al. [6] have studied the study of an inverse steady-state thermoelastic problem of a thin rectangular plate. Gaikwad et al. [7] have studied the quasi-static thermal stresses in a thick rectangular plate subjected to constant heat supply on extreme edges where as the initial edges are thermally insulated. Recently, Salve et al. [8] studied an inverse transient quasi-static thermal stresses problem in a thin rectangular plate.

In this article, we analyzed a non-homogeneous heat conduction problem due to internal heat generation in a thin rectangular plate and determined the expressions for temperature, displacement and thermal stresses. Initially, the plate is at arbitrary temperature $f(x, y)$, while the boundary at $y=0$ is kept at temperature $\phi(x, t)$ and the remaining boundaries are kept at zero temperature. The governing heat conduction equation has been solved by the method of double integral transform technique. The results are obtained in series form in term of circular functions. The results for

thermal displacement and stress components have been computed numerically and illustrated graphically.

To the author knowledge, no literature on quasi-static thermal stresses in a thin rectangular plate due to heat generation has been published. The results presented here will be more useful in engineering problem particularly, in the determination of the state of strain in thin rectangular plate constituting foundations of containers for hot gases or liquids, in the foundations for furnaces etc.

2. Formulation of the Problem

Consider a thin rectangular plate occupying the space $D: 0 \leq x \leq a, 0 \leq y \leq b$. Initially the rectangular plate is at arbitrary temperature $f(x, y)$. For time $t > 0$, heat is generated within the solid at a rate of $g(x, y, t)$ Btu/hr ft³, while the boundary at $y = 0$ is kept at temperature $\phi(x, t)$ and the remaining boundaries are kept at zero temperature. Under these realistic prescribed conditions, the displacement and thermal stresses in a thin rectangular plate due to internal heat generation are required to be determined.

The temperature $T(x, y, t)$ of the thin rectangular plate satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{g(x, y, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1)$$

with the boundary conditions,

$$T = 0 \text{ at } x = 0, 0 \leq y \leq b, t > 0 \quad (2)$$

$$T = 0 \text{ at } x = a, 0 \leq y \leq b, t > 0 \quad (3)$$

$$T = \phi(x, t) \text{ at } y = 0, 0 \leq x \leq a, t > 0 \quad (4)$$

$$T = 0 \text{ at } y = b, 0 \leq x \leq a, t > 0 \quad (5)$$

and the initial condition

$$T(x, y, t) = f(x, y) \text{ at } t = 0, 0 \leq x \leq a, 0 \leq y \leq b \quad (6)$$

where k and α are thermal conductivity and thermal diffusivity of the material of the plate.

Here the plate is assumed sufficiently thin and considered free from traction. Since the plate is in a plane stress state without bending. Airy stress function method is applicable to the analytical development of the thermoelastic field. Airy stress function $U(x, y, t)$ which satisfy the following relation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 U = -\lambda E \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T \quad (7)$$

Where λ and E are linear coefficient of the thermal expansion, Young's modulus elasticity of the material of the plate.

The displacement components u_x and u_y in the X and Y direction are represented in the integral form as

$$u_x = \int \left\{ \frac{1}{E} \left(\frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \lambda T \right\} dx \quad (8)$$

$$u_y = \int \left\{ \frac{1}{E} \left(\frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \lambda T \right\} dy \quad (9)$$

where ν is the Poisson's ratio of the material of the plate.

The stress components in terms of U are given by

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} \quad (10)$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} \quad (11)$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial^2 xy} \quad (12)$$

Equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

3. Solution of the Heat Conduction Problem

To find the temperature function $T(x, y, t)$ we introduce the "double-integral transform" and its corresponding "double-inversion formula" as defined in Ozisik [9] respectively as

$$\bar{T}(\beta_m, \nu_n, t) = \int_{x'=0}^a \int_{y'=0}^b K(\beta_m, x') \cdot K(\nu_n, y') \cdot T(x', y', t) dx' dy' \quad (13)$$

$$T(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} K(\beta_m, x) \cdot K(\nu_n, y) \cdot \bar{T}(\beta_m, \nu_n, t) \quad (14)$$

where the kernels

$$K(\beta_m, x) = \sqrt{\frac{2}{a}} \cdot \sin(\beta_m x) \quad (15)$$

$$K(\nu_n, y) = \sqrt{\frac{2}{b}} \cdot \sin(\nu_n y) \quad (16)$$

and eigenvalues are

$$\beta_m \text{ is } m^{\text{th}} \text{ root of transcendental equation } \sin(\beta_m a) = 0$$

$$\text{i.e. } \beta_m = \frac{m\pi}{a}, \quad m = 1, 2, 3, \dots \quad (17)$$

$$\nu_n \text{ is } n^{\text{th}} \text{ root of transcendental equation } \sin(\nu_n b) = 0$$

$$\text{i.e. } \nu_n = \frac{n\pi}{b}, \quad n = 1, 2, 3, \dots \quad (18)$$

On applying double-integral transform defined in equation (13) to Eqs. (1) - (3) and then using their inversions defined in equation (14), one obtains the expressions of the temperature as

$$T(x, y, t) = \frac{4}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-(\beta_m^2 + \nu_n^2)t} \sin(\beta_m x) \cdot \sin(\nu_n y) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} \cdot A(\beta_m, \nu_n, t') dt' \right] \quad (19)$$

where

$$A(\beta_m, \nu_n, t') = \frac{\alpha}{k} \bar{g}(\beta_m, \nu_n, t') + \alpha \cdot \nu_n \int_{x'=0}^x \phi(x', t') \sin(\beta_m x') dx'$$

$$\bar{g}(\beta_m, \nu_n, t') = \int_{x'=0}^x \int_{y'=0}^y \sin(\beta_m x') \sin(\nu_n y') dx' dy'$$

$$\bar{f}(\beta_m, \nu_n) = \int_{x'=0}^x \int_{y'=0}^y \sin(\beta_m x') \sin(\nu_n y') f(x', y') dx' dy'$$

4. Airy's Stress Function

Using equation (19) in (7), one obtains

$$U = \frac{4\lambda E}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{e^{-(\beta_m^2 + \nu_n^2)t}}{\beta_m^2 + \nu_n^2} \right] \sin(\beta_m x) \cdot \sin(\nu_n y) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} \cdot A(\beta_m, \nu_n, t') dt' \right] \quad (20)$$

5. Displacement Components

Now using equations (19) and (20) in equations (8) to (12), one obtains

$$u_x = \frac{4\lambda}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-(\beta_m^2 + \nu_n^2)t} \left[\frac{(\nu-1)\beta_m^2 - 2\nu\nu_n^2}{\beta_m^2 + \nu_n^2} \right] \left(\frac{\cos(\beta_m x)}{\beta_m} \right) \sin(\nu_n y) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} \cdot A(\beta_m, \nu_n, t') dt' \right] \quad (21)$$

$$u_y = \frac{4\lambda}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-(\beta_m^2 + \nu_n^2)t} \left[\frac{(\nu-1)\nu_n^2 - 2\beta_m^2}{\beta_m^2 + \nu_n^2} \right] \sin(\beta_m x) \left(\frac{\cos(\nu_n y)}{\nu_n} \right) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} \cdot A(\beta_m, \nu_n, t') dt' \right] \quad (22)$$

6. Thermal Stresses

$$\sigma_{xx} = \frac{-4\lambda E}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{e^{-(\beta_m^2 + \nu_n^2)t}}{\beta_m^2 + \nu_n^2} \right] \nu_n^2 \sin(\beta_m x) \sin(\nu_n y) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} \cdot A(\beta_m, \nu_n, t') dt' \right] \quad (23)$$

$$\sigma_{yy} = \frac{-4\lambda E}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{e^{-(\beta_m^2 + \nu_n^2)t}}{\beta_m^2 + \nu_n^2} \right] \beta_m^2 \sin(\beta_m x) \sin(\nu_n y) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} A(\beta_m, \nu_n, t') dt' \right] \quad (24)$$

$$\sigma_{xy} = \frac{-4\lambda E}{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{e^{-(\beta_m^2 + \nu_n^2)t}}{\beta_m^2 + \nu_n^2} \right] \beta_m \nu_n \cos(\beta_m x) \cos(\nu_n y) \times \left[\bar{f}(\beta_m, \nu_n) + \int_{t'=0}^t e^{-(\beta_m^2 + \nu_n^2)t'} A(\beta_m, \nu_n, t') dt' \right] \quad (25)$$

7. Special case and Numerical calculations

Setting

$f(x, y) = (a - x)(1 - e^x)(b - y)$,
 $g(x, y, t) = g_i \delta(x - x_0) \delta(y - y_0) \delta(t - \tau)$ Btu/hr.ft³ and
 $\phi(x, t) = (a - x)(1 - e^x)(1 - e^t)$
 where δ is the Dirac-delta function, $A > 0$.

The heat source $g(x, y, t)$ is an instantaneous line heat source of strength $g_i=50$ Btu/hr.ft, situated at the center of the rectangular plate and releases it's instantaneously at the time $t = \tau = 2$ hr.

Dimension

Length of rectangular plate $a = 2$ ft
 Breadth of rectangular plate $b = 1$ ft
 Central length of rectangular plate $x_0 = 1$ ft
 Central breadth of rectangular plate $y_0 = 0.5$ ft.

Material Properties

The aluminum (pure) rectangular plate is chosen for purpose of numerical evaluations.

The associated constants are taken as,
 Density $\rho = 169$ lb/ft³,
 Specific heat $c_p = 0.208$ Btu/lb⁰F,
 Thermal conductivity $k = 117$ Btu/(hr.ft. ⁰F),
 Thermal diffusivity $\alpha = 3.33$ ft²/hr,
 Coefficients of thermal expansion $\lambda = 12.84 \times 10^{-6}$ 1/F
 Young's modulus elasticity of the material of the plate $E = 70$ GPa,
 Poisson ratio $\nu = 0.35$.

For convenience setting $A = \left(\frac{-4\lambda E}{ab} \right)$

Considering

$$\lim_{m \rightarrow \infty} \beta_m = \lim_{n \rightarrow \infty} \nu_n = \infty$$

$$\lim_{m \rightarrow \infty} \left(e^{-k\beta_m^2 t} \right) = \lim_{n \rightarrow \infty} \left(e^{-k\nu_n^2 t} \right) = 0$$

Also the term $\sin(\beta_m x)$ and $\cos(\beta_m x)$ are bounded.

Thus necessary condition for convergence is satisfied, by applying D-Alembert's ratio test it can be easily verify that

all the series in (19) to (25) are convergent. Also the term in the expression for displacements and stresses are negligible for large value of m and n it converges to zero at infinity. The numerical calculation has been carried out with help of computational mathematical software Mathcad-2007, and the graphs are plotted with the help of Excel (MS Office-2007).

8. Discussion

In this study, we analyzed a non-homogeneous heat conduction problem due to internal heat generation in a thin rectangular plate. As an illustration, we carried out numerical calculations for the aluminum (pure) rectangular plate. The heat source $g(x,y,t)$ is an instantaneous line, heat source of strength g_i , is situated at center of the rectangular plate in X and Y direction and releases instantaneously at the time $t = \tau = 2$ hr. The thermoelastic behavior is examined such as temperature, displacement and stress components with the help of temperature.

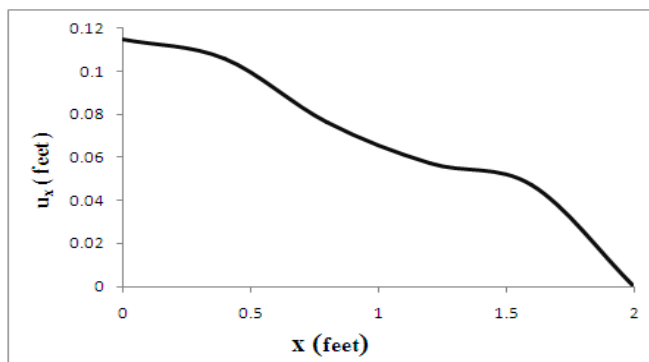


Figure 1: The displacement function u_x / A in X direction

From Fig. 1 and 2, it is observe that the displacement function u_x decreases from the inner boundary surface to the outer boundary surface and it becomes zero at $x = 2$ in X direction and $y = 0.8$ in Y direction.

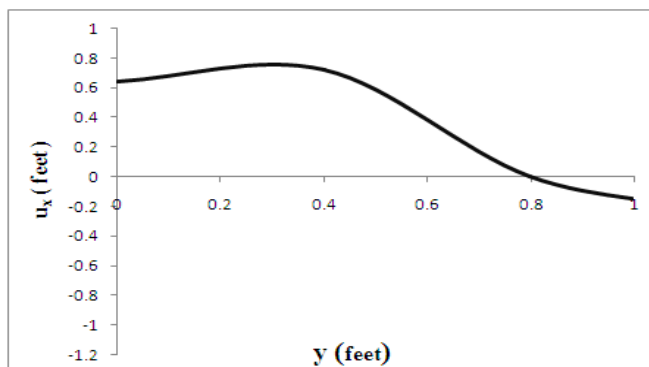


Figure 2: The displacement function u_x / A in Y direction.

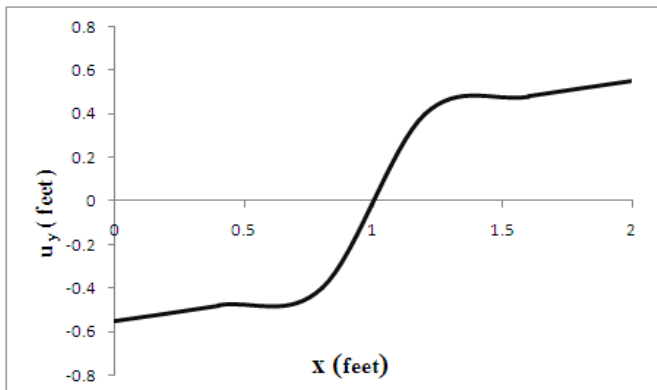


Figure 3: The displacement function u_y/A in X direction.

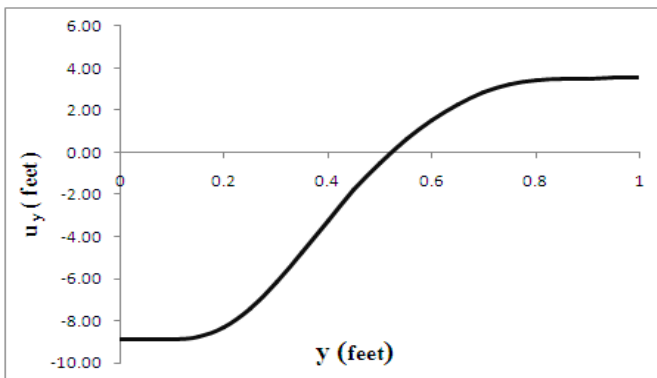


Figure 4: The displacement function u_y/A in Y direction.

From Fig. 3 and 4, it is clear that the displacement function u_y increases from the inner boundary surface to the outer boundary surface and it becomes zero at $x = 0.5$ in X direction and $y = 1$ in Y direction.

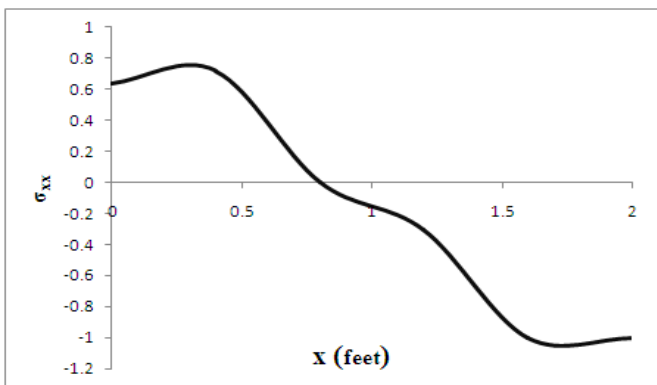


Figure 5: The stress function σ_{xx}/A in X direction

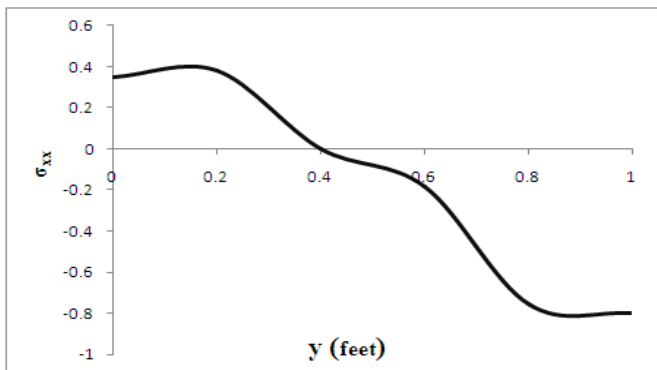


Figure 6: The stress function σ_{xx}/A in Y direction.

From Fig. 5 and 6, it is clear that the stress function σ_{xx} decreases non-uniformly from the inner boundary surface to the outer boundary surface and it becomes zero at $x = 0.8$ in X direction and $y = 0.4$ in Y direction.

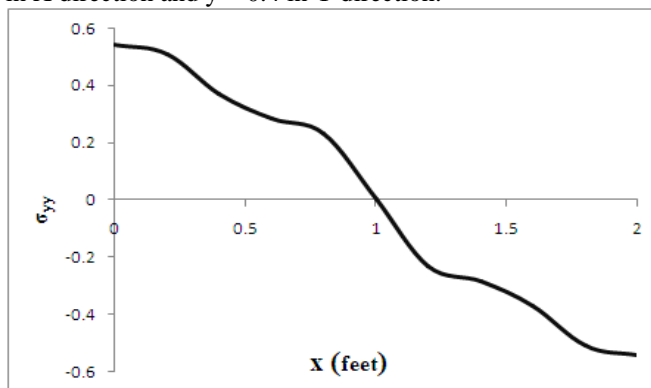


Figure 7: The stress function σ_{yy}/A in X direction.

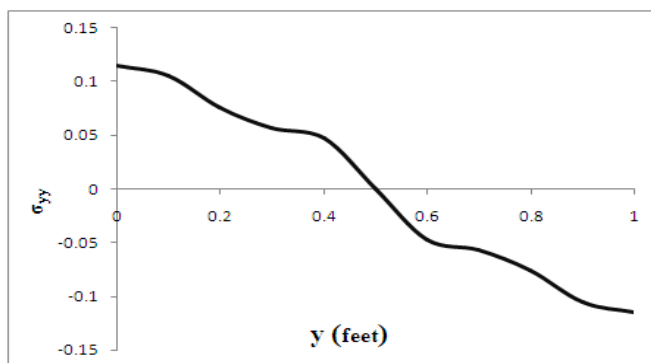


Figure 8: The stress function σ_{yy}/A in Y direction.

From Fig. 7 and 8, it is clear that the stress function σ_{yy} decreases from the inner boundary surface to the outer boundary. It is zero at the center $x = 1$ in X direction and $y = 0.5$ in Y direction.

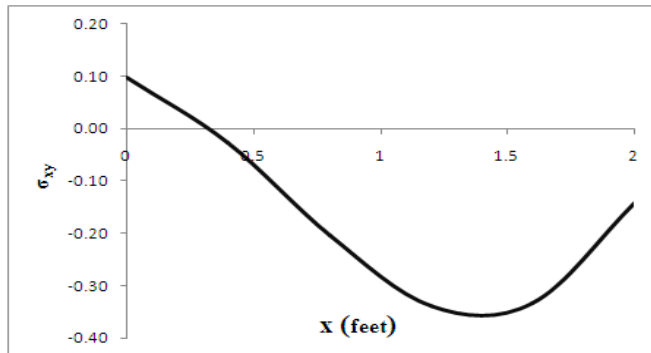


Figure 9: The stress function σ_{xy}/A in X direction.

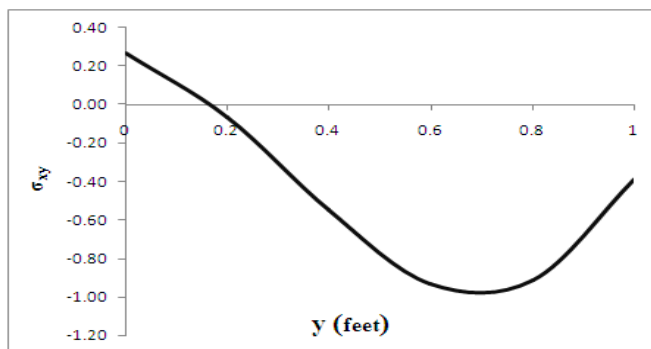


Figure 9: The stress function σ_{xy}/A in X direction.

From Fig. 9 and 10, it is clear that the stress function σ_{xy} decreases from the inner boundary surface to the outer boundary. It is zero at the center $x = 1$ in X direction and $y = 0.5$ in Y direction and displaces upward in both X and Y direction.

9. Concluding Remarks

In this article, we extend the problem studied by Salve et al. [8], the thermal stress problem in a thin rectangular slab without heat generation. We have considered the thermal problem with heat generation. The displacement and thermal stresses of a non-homogeneous heat conduction problem in a thin rectangular under unsteady-state temperature field due to internal heat generation is presented. The present method is based on the direct method, using the double integral transform technique and their inversion. We observe that a displacement and stress component occurs near heat source. Due to internal heat generation within the thin rectangular plate, from the figure of displacement, the direction of heat flow in X and Y are opposite to each other and they are inversely proportional. Also, it can be observe that, the stress function σ_{xx} develops the tensile stress and σ_{yy} develops the compressive stress in both X and Y direction.

The results, obtained here mainly applicable in engineering problems, particularly for industrial machines subjected to the heating such as the main shaft of a lathe, turbines, the roll of rolling mill and practical applications in air-craft structures. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (19)–(25).

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