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Integral Solutions of the Homogeneous Biquadratic Diophantine Equation with 6 Unknowns $(x^3 - y^3)Z = (W^2 - P^2)R^2$

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Abstract: ***The homogeneous biquadratic equation with 6 unknowns represented by $(x^3 - y^3)Z = (W^2 - P^2)R^2$ is analyzed for its non - zero distinct integer solutions. Employing the transformations and applying the method of factorization, five different patterns of non-zero distinct integer solutions to the above biquadratic equation are obtained. A few interesting relations between the solutions and special numbers namely Polygonal numbers and Pyramidal numbers are presented.***

Keywords: Biquadratic equation with 6 unknowns, Homogeneous Biquadratic equation, integral solutions, Special numbers, A few interesting relations

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1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations homogeneous and non homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. In this context one may refer [5-8] for various problems on the biquadratic Diophantine equations come with four variables. However, often we across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. It is towards this end this paper concerns with the problem of determining non-trivial integral solutions of the biquadratic equation with six unknowns given by

$(x^3 - y^3)Z = (W^2 - P^2)R^2$.

2. Method of Analysis

The equation under consideration is

$$(x^3 - y^3)Z = (W^2 - P^2)R^2 \quad (1)$$

Assuming the transformations

x = 4u + v, y = 4 u – v, z = 2u, w = u v +1,
p = u v - 1 in (1) we get (2)

$$v^2 + 48u^2 = R^2 \quad (3)$$

2.1 Pattern: 1

$$\text{Let } R(a, b) = a^2 + 48b^2 \quad (4)$$

Using (4) in (3) we get

$v^2 + 48u^2 = (a^2 + 48b^2)^2$

By method of factorization, we get

$$(v + i\sqrt{48}\, u)(v - i\sqrt{48}\, u) = (a + i\sqrt{48}\, b)^2 (a - i\sqrt{48}\, b)^2 \quad (5)$$

Equating positive and negative factors, we get

$$(v + i\sqrt{48}\, u) = (a + i\sqrt{48}\, b)^2 \quad (6)$$

$$(v - i\sqrt{48}\, u) = (a - i\sqrt{48}\, b)^2 \quad (7)$$

Equating real and imaginary parts, we get

$$v = a^2 - 48b^2 \quad (8)$$
$$u = 2ab \quad (9)$$

Substitute (8) & (9) in (2) the integer solutions of (1) are found to be

x (a, b) = $8ab + a^2 - 48b^2$
y (a, b) = $8ab - a^2 + 48b^2$
z (a, b) = 4ab
w (a, b)= $2a^3b - 96ab^3 + 1$
P (a, b) = $2a^3b - 96ab^3 - 1$
R (a, b) = $a^2 + 48b^2$

Properties:

1.3[x (a, 1) + y (a, 1) + z(a,1) $-6gn_a$ + 4] Nasty number
2. P (1, b) + y (1, b) $-18Pr_b - 8gn_b$ Nasty number
3. x (a, 2) - 2 $T_{3,a}$ ≡ 3 (mod 15)
4. x(a, 1) – y (a, 1) + z(a, 1)$- 2gn_a$ ≡ 0 (mod 2)
5. w (a, 1) – y(a, 1) $-SO_a - 2T_{3,a}$ ≡ -3 (mod 44)
6. $\frac{1}{3}$ [x (a, b) – y (a, b) + R (a, b)] = Difference between two perfect squares
7. x (n(n+1), n+2) + y (n(n+1), n+2) = $96P_n^3$

2.2 Pattern: 2

$$\text{Rewrite (3) as } v^2 + 48u2 = R^{2\,*}\, 1 \quad (10)$$

$$\text{Write 1 as } 1 = \frac{(1+i\sqrt{48})\,(1-i\sqrt{48})}{49} \quad (11)$$

Substitute (4) & (11) in (10) and by method of factorization, we get

$$\left(v + i\sqrt{48}\, u\right) = \left(a + i\sqrt{48}\, b\right)^2 \frac{(1+i\sqrt{48})}{7} \quad (12)$$

Equating real and imaginary parts

$$v = \frac{1}{7}(a^2 - 96ab - 48b^2) \quad (13)$$
$$u = \frac{1}{7}(a^2 + 2ab - 48b^2) \quad (14)$$

As our interest is on finding integer solutions, it is seen that the values of u and v are integers for suitable choice of the parameters a and b.

Putting a = 7A and b = 7B in (13) & (14), we get

$$v = (7A^2 - 672AB - 336B^2) \quad (15)$$
$$u = (7A^2 + 14AB - 336B^2) \quad (16)$$

Substitute (15) & (16) in (2) the integer solutions of (1) are found to be

x (A, B) = $35A^2 - 616AB - 1680B^2$
y (A, B) = $21A^2 + 728AB - 1008B^2$
z (A, B) = $14A^2 + 28AB - 672B^2$
w (A, B) = $49A^4 - 4606A^3B - 14112A^2B^2 + 221088AB^3 + 112896B^4 + 1$
P (A, B) = $49A^4 - 4606A^3B - 14112A^2B^2 + 221088AB^3 + 112896B^4 - 1$
R (A, B) = $49A^2 + 2352B^2$

Properties:
1. x (A, 1) – $T_{72,A} \equiv 66 (mod\ 582)$
2. 5z (n (n + 1), n + 2) – 2x (n (n + 1), n + 2) = $8232P_n^3$
3. z (A, 1) + R (A, 1) + x (A,1) + gn_{246A} +1 $-2T_{4,A}$ Nasty number
4. $\frac{1}{14}[y(A,1) + z(A,1) - x(A,1) - 1]$ Nasty number
5. 3[w (A, 1) – P (A,1)] Nasty number.

2.3: Pattern: 3

In (10) write 1 as

$$1 = \frac{(1+i14\sqrt{48})\,(1-i14\sqrt{48})}{9409} \quad (17)$$

Substitute (4) &(17) in (10) by employing the method of factorization and equating the positive and negative factors, we get

$$\left(v + i\sqrt{48}\, u\right) = \left(a + i\sqrt{48}\, b\right)^2 \frac{(1+i14\sqrt{48})}{97} \quad (18)$$

Equating real and imaginary parts, we get

$$v = \frac{1}{97}(a^2 - 1344ab - 48b^2) \quad (19)$$
$$u = \frac{1}{97}(14a^2 + 2ab - 672b^2) \quad (20)$$

As our interest is on finding integer solutions, it is seen that the values of u and v are integers for suitable choice of the parameters a and b.

Putting a = 97A and b = 97B in (19) & (20) we get the corresponding integer solutions of (1) are found to be

x (A, B) = $5529A^2 - 129592AB - 133680B^2$
y (A, B) = $5335A^2 + 131144AB - 124368B^2$
z (A, B) = $2716A^2 + 388AB - 64512B^2$
w (A, B)
= $131726A^4 - 177020926A^3B - 34743072A^2B^2 + 4204246944AB^2 + 150183936B^4 + 1$
P (A, B) =
$131726A^4 - 177020926A^3B - 34743072A^2B^2 4204246944AB^3 + 150183936B^4 - 1$
R (A, B) = $9409A^2 + 451632B^2$

Properties:

1. x (A, 1) – $T_{11060,A} \equiv -9616 (mod\ 124064)$
2. x (1, B) +267360 $\equiv 5529 (mod\ 137768)$
3. y (1, B) +2591 $T_{98,B} \equiv 4784 (mod\ 551)$
4. z(A, 1) –194 $gn_A \equiv 866 (mod\ 2716)$
5. x(A, 1) + R(A, 1) $-679T_{46,A} + 64796gn_A \equiv 73027 (mod\ 129592)$
6. x (A, 1) - y(A, 1) -14Pr_A + 56$T_{A+2,97}$ + 4060 Nasty number.

2.4: Pattern: 4

$$\text{Rewrite (10) as } R^2 - 48u^2 = v^2 * 1 \quad (21)$$

$$\text{Take } v = a^2 - 48b^2 \quad (22)$$

$$\text{Write 1 as } 1 = (7 + \sqrt{48})(7 - \sqrt{48}) \quad (23)$$

Substitute (22) & (23) in (21), we get

$R^2 - 48u^2 = (a^2 - 48b^2)^2 (7 + \sqrt{48}) (7 - \sqrt{48})$

By method of factorization and by equating positive and negative factors, we get

$$R + \sqrt{48}\, u = (a + \sqrt{48}\, b)^2 (7 + \sqrt{48}) \quad (24)$$

Equating rational and irrational parts, we get

$$R = 7a^2 + 96ab + 336b^2 \quad (25)$$
$$u = a^2 + 14ab + 48b^2 \quad (26)$$

Substitute (22) & (26) in (2) the integer solutions of (1) are found to be

x (a, b) = $5a^2 + 56ab + 144b^2$
y (a, b) = $3a^2 + 56ab + 240b^2$
z (a, b) = $2a^2 + 28ab + 96b^2$
w (a, b) = $a^4 + 14a^3b - 672ab^3 - 2304b^4 + 1$
P (a, b) = $a^4 + 14a^3b - 672ab^3 - 2304b^4 - 1$
R (a, b) = $7a^2 + 96ab + 336b^2$

Properties:
1. 12x (a, 1) -12 y (a, 1) + 912 Nasty number
2. 3 [x (a, a) – y (a, a) + z (a, a)] Nasty number
3. $Pr_a^2 + 21\, OH_a - w(a,1) - x(a,1) + 8T_{3,a} \equiv 278$ (mod 627)
4. 3[w (a, 1) – P (a, 1)] Nasty number
5. R (a, 1)$-14T_{3,a} \equiv$ -20 (mod 89)
6. 7z (n^2, n + 1) – 2R (n^2, n + 1) ≡ 0 (mod 4)

2.5: Pattern: 5

In (21) rewrite 1 as

$$1 = (97 + 14\sqrt{48})(97 - 14\sqrt{48}) \qquad (27)$$

Substitute (27) in (21), we get

$$R^2 - 48u^2 = (a^2 - 48b^2)^2 (97 + 14\sqrt{48})(97 + 14\sqrt{48})$$

By method of factorization and equating rational and irrational parts, we get

$$R = 97a^2 + 1344ab + 4656b^2$$

$$u = 14a^2 + 194ab + 672b^2$$

and therefore the integer solutions of (1) becomes

$x(a, b) = 57a^2 + 776ab + 2640b^2$
$y(a, b) = 55a^2 + 776ab + 2736b^2$
$z(a, b) = 28a^2 + 388ab + 1344b^2$
$w(a, b) = 14a^4 + 194a^3b - 9312ab^3 - 32256b^4 + 1$
$P(a, b) = 14a^4 + 194a^3b - 9312ab^3 - 32256b^4 - 1$
$R(a, b) = 97a^2 + 1344ab + 4656b^2$

Properties:

1. $x(a,1) - y(a,1) + 4T_{4,a}$ Nasty number
2. $\frac{10}{103}[x(a,a) - y(a,a) - z(a,a)]$ Nasty number
3. $R(a,1) - 194T_{3,a} \equiv 915 \pmod{1247}$
4. $3[w(a,1) - P(a,1)]$ Nasty number

3. Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

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