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On The Non Homogeneous Heptic Equation with Five Unknowns $(x^2 - y^2)(9x^2 + 9y^2 - 16xy) = 21(X^2 - Y^2)z^5$

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Abstract: ***The non homogeneous Diophantine equation of degree seven with five unknowns represented by $(x^2 - y^2)(9x^2 + 9y^2 - 16xy) = 21(X^2 - Y^2)z^5$ is analyzed for its non - zero distinct integer solutions. Employing suitable linear transformations and applying the method of cross multiplication, four different patterns of non-zero distinct integer solutions to the heptic equation under consideration are obtained. A few interesting relation between the solutions and special numbers namely Polygonal numbers, Pyramidal numbers, Centered Pyramidal numbers, Star numbers and Stella octangular numbers are exhibited.***

Keywords: The non homogeneous Diophantine equation, Heptic equation with five unknowns, integral solutions, special numbers, a few interesting relation

2010 Mathematics Subject Classification: 11D09

Notations: P_n^m = Pyramid number of rank n with size m
$T_{m,n}$ =Polygonal number of rank n with size m
gn_a = Gnomonic number of rank a
So_n = Stella octangular number of rank n
Pr_n = Pronic number of rank n
$CP_{m,n}$ = Centered Pyramidal number of rank n with size m.

1. Introduction

The Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity as can be seen from [1, 9] the problem of finding all integer solutions of a Diophantine equation with three or more variables and degree at least three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables [1-4, 9]. Cubic equations in two variables fall in to the theory of elliptic curves which is a very developed theory but still an important topic of current research [5-7]. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small coefficients. In [1, 8 and 9] a few higher order equations are considered for integral solutions. In this communication a seventh degree non-homogeneous equation with five variables represented by $(x^2 - y^2)(9x^2 + 9y^2 - 16xy) = 21(X^2 - Y^2)z^5$ is considered and in particular a few interesting relations among the solutions among the solutions are presented.

2. Method of Analysis

The Diophantine equation representing the heptic equation with five unknowns under consideration is
$(x^2 - y^2)(9x^2 + 9y^2 - 16xy) = 21(X^2 - Y^2)z^5$ (1)

Introduction of the linear transformation

x = u + v, y = u – v, X = 2u + v, Y = 2u – v (2)

in (1) leads to
$u^2 + 17v^2 = 21z^5$ (3)

Now we solve (3) through different methods and thus obtain different patterns of solutions to (1)

2.1 Pattern: 1

Assume $z = z(a, b) = a^2 + 17b^2$ (4)

Where a and b are non zero distinct integers

Write 21 as
$21 = (2 + i\sqrt{17})(2 - i\sqrt{17})$ (5)
Using (4) and (5) in (3) and applying the method of factorization, define
$(u + i\sqrt{17}\, v) = (2 + i\sqrt{17})(a + i\sqrt{17}\, b)^5$

Equating real and imaginary parts, we get

$u = u(a, b) = 2a^5 - 85a^4b - 340a^3b^2 + 2890a^2b^3 + 2890ab^4 - 4913b^5$
$v = v(a, b) = a^5 + 10a^4b - 170a^3b^2 - 340a^2b^3 + 1445ab^4 + 578b^5$

Hence in view of (2) the corresponding solutions of (1) are given by
$x = x(a, b) = 3a^5 - 75a^4b - 510a^3b^2 + 2550a^2b^3 + 4335ab^4 - 4335b^5$

y = y (a, b) = $a^5 - 95a^4b - 170a^3b^2 + 3230a^2b^3 + 1445ab^4 - 5491b^5$

X = X (a, b) = $5a^5 - 160a^4b - 850a^3b^2 + 5440a^2b^3 + 7225ab^4 - 9248b^5$

Y=Y (a, b) = $3a^5 - 180a^4b - 510a^3b^2 + 612a^2b^3 + 4335ab^4 - 10404b^5$

z = z (a, b) = $a^2 + 17b^2$

Properties:

1. 3X(a,1) – 5Y(a,1) = 420Pr_{a^2} -367$T_{4,2a}$ +24276
2. x (a, 1) – 3y (a,1) - 210Pr_{a^2} ≡ 12138 (mod 94620)
3. 10z (a, a) a Nasty number
4. x (a, 1) - 3y (a, 1) + 10490 $T_{4,3a}$ - 30 g_{na^2} ≡ 168 (mod 210)
5. x (1, b) - 3y(1, b) +47205So_b – 1557 $T_{b,2}$ ≡ 0 (mod 12138)

2.2 Pattern: 2

Rewrite (3) as $u^2 + 17v^2 = 21z^5 * 1$ (6)

Write 1 as 1 = $\frac{1}{81}$ $(8 + i\sqrt{17})\ (8 - i\sqrt{17})$ (7)

Following the procedure similar to Pattern-I, the corresponding non-zero distinct integral solutions of (1) are found to be

x = x (a, b) = $1240029a^5 - 6200145a^4b - 210804930a^3b^2 + 210804930a^2b^3 + 1791841905ab^4 - 358368381b^5$

y = y (a, b) = $964467a^5 - 17222625a^4b - 163959390a^3b^2 + 585569250a^2b^3 + 1393654815ab^4 - 995467725b^5$

X = X (a, b) = $2342277a^5 - 17911530a^4b - 398187090a^3b^2 + 608992020a^2b^3 + 3384590265ab^4 - 1035286434b^5$

Y = Y (a, b) = $2066715a^5 - 28934010a^4b - 351341550a^3b^2 + 983756340a^2b^3 + 2986403175ab^4 - 1672385778b^5$

z = z (a, b) = $81a^2 + 1377b^2$

Properties:

1. z (a, a) – 639gn_{a^2} - 639 Nasty number
2. $\frac{1}{51254532}$ [7x (1, 1) – 9y (1, 1)] a Nasty number
3. 7x (a, 1) –9y (a, b) +105402465 $T_{4,6a}$ ≡ $-22320522 (mod\ 111602610)$
4. 15X (a, 1) – 17Y (a, 1) -55801305gn_{2a^4} ≡ -2276693244(mod7588977480)
5. 15X (1,b) – 17Y(1,b) -77407570296$CP_{b^2,b}$ -2656142118So_b +771196355622 $T_{2,b}$ = 0

2.3: Pattern: 3

Instead of (5) write 21 as

21 = $\frac{1}{81}$ $(1 + i10\sqrt{17})\ (1 - i10\sqrt{17})$ (8)

Following the procedure similar to Pattern-I , and performing a few calculations, the corresponding non-zero distinct integral solutions of (1) are found to be

x = x (a, b) = $72171a^5 - 5544045a^4b - 12269070a^3b^2 + 190728270a^2b^3 + 104287095ab^4 - 320445801b^5$

y = y (a, b) = $-59049a^5 - 5609655a^4b + 10038330a^3b^2 + 188497530a^2b^3 - 85325805ab^4 - 324238059b^5$

X = X (a, b) =$78732a^5 - 11120895a^4b - 13384440a^3b^2 + 380341170a^2b^3 + 113767740ab^4 - 642787731b^5$

Y =Y (a, b) = $-52488a^5 - 11186505a^4b + 8922960a^3b^2 + 378110430a^2b^3 - 75845160ab^4 - 646579989b^5$

z = z (a, b) = $81a^2 + 1377b^2$

Properties:

1. 9x (a,1) +11y (a,1) +31000725 –$T_{4,6a}$ ≡1129423662 mod(37900027260)
2. $\frac{1}{28877148}$ [9x (1,1) – 11y(1,1)] a Nasty number
3. $\frac{1}{27}$ z (a,a) a Nasty number
4. $8X(a,1) + 12Y(a,1) + 1339231320CP_{a,a} - 3678424650gn_{a^2} \equiv 416394254(mod\ 1339231320)$

2.4 Pattern: 4

Instead of (7) write 1 as

1 = $\frac{1}{1089}$ $(1 + i8\sqrt{17})\ (1 - i8\sqrt{17})$ ---- -(9)

Following the procedure similar to Pattern-III, the corresponding non-zero distinct integral solutions of (1) are found to be

x = x (a, b) = $224139069a^5 - 16810430175a^4b - 38103641730a^3b^2 + 571554625950a^2b^3 + 323880954705 - 971642864115b^5$

y = y (a, b) = $-174330387a^5 - 17059473585a^4b + 29636165790a^3b^2 + 580022101890a^2b^3 - 251907409215ab^4 - 986037573213b^5$

X = X (a, b) = $249043410a^5 - 33745382055a^4b - 42337379700a^3b^2 + 1147342989890a^2b^3 + 3598677274 - 1950483082779b^5$

Y = Y (a, b) = $-149426046a^5 - 33994425465a^4b + 25402427820a^3b^2 + 1155810465810a^2b^3 - 215920636470ab^4 - 1964877791877b^5$

z = z (a, b) = $1089a^2 + 18513b^2$

Properties:

1. $7x(a,1) + 9y(a,1) - 542415469807T_{3,a^2-1} \equiv 3308740936578(mod\ 9492289572150)$
2. $\frac{1}{363}$ [z(a, a)] a Nasty number
3. 7x(1, b) + 9y(1, b) ≡ 0 (mod 18080551566)
4. $6X(a,1) + 10Y(a,1) + 7203829673960T_{3,a^2-1} = 7420123880230gn_{a^2} - 23931552535214$

3. Conclusion

In linear transformations (2), the variables X and Y may also be represented by X=2uv +1, Y= 2uv -1. Applying the procedure similar to that of pattern I-IV choices of integral solutions to (1) are obtained. To conclude, one may search for other patterns of solutions and their corresponding properties.

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