For α =0 , β >0:

$$E[T^m]_{\alpha=0} = m\int_0^\infty t^{m-1}\overline{G}(t;\beta,\lambda)dt$$

$$= \text{m}e^{-\lambda}\int_0^\infty t^{m-1}e^{-\beta t^2/2}\sum_{n=0}^\infty \frac{(\lambda e^{-\beta t^2/2})^n}{n!}dt \quad (15)$$

$$= \frac{me^{-\lambda}\Gamma(\frac{m}{2})}{2(\beta/2)^{m/2}}\sum_{n=0}^\infty \frac{\lambda^n}{n!(n+1)^{m/2}}$$

The moments of the new extended Rayleigh distribution are given by

3.3. Order Statistics

It will also be useful to derive the pdf of the rth order statistic $T_{(r)}$ of a random sample $T_1,.., T_n$ drawn from the distribution was proposed by equation (5) with parameters α, β and λ. From [13], the pdf of $T_{(r)}$ is given by

$$g_{r:n}(t) = \frac{[G(t)]^{r-1}[\overline{G}(t)]^{n-r}g(t)}{B(r,\text{n - r} + 1)} \quad (16)$$

where B(., .) is the beta function.

Using $h(t) = \alpha + \beta t$, $H(t) = \alpha t + \beta t^2/2$, and substituting (5), (6) and (7) into (16), we get

$$g_{r:n}(t) = n[1+\lambda e^{-H(t)}]h(t)\binom{n-1}{\text{r - 1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{\text{i}}e^{-(n+i+1-r)[\lambda+H(t)-\lambda e^{-H(t)}]} \quad (17)$$

4. Parameter Estimations

In this section, we derive the maximum likelihood estimates of the unknown parameters *α, β* and *λ* of $g(t;\alpha,\beta,\lambda)$ based on a complete sample. Let us assume that we have a simple random sample $T_1, T_2, \ldots, T_n$ from $g(t;\alpha,\beta,\lambda)$ The likelihood function of this sample is

$$L = \prod_{i=1}^n g(t_i;\alpha,\beta,\lambda) \quad (18)$$

Substituting from (7) into (18), we get

$$L = \prod_{i=1}^n \{1+\lambda e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)}\}(\alpha+\beta t_i)\, e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)}e^{-\lambda[1-e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)}]} \quad (19)$$

It can be written as;

$$L = (\alpha^n + \beta^n T_1)e^{-(\alpha T_1+\beta T_2)}e^{-\lambda[n-Y]}(1+\lambda^n e^{-(\alpha T_1+\beta T_2)}) \quad (20)$$

Where $T_j = \frac{1}{j}\sum_{i=1}^n t_i^j$, $j = 1, 2$ and $Y = \sum_{i=1}^n e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)}$

The log-likelihood function becomes

$$\text{lnL} = \ln(\alpha^n + \beta^n T_1) - (\alpha T_1 + \beta T_2) - \lambda(n-Y) + \ln(1+\lambda^n e^{-(\alpha T_1+\beta T_2)}) \quad (21)$$

Setting the first partial derivatives of lnL with respect α, β and λ to zero, the likelihood equations are

$$0= -\frac{\lambda^n T_1 e^{-(\alpha T_1+\beta T_2)}}{1+\lambda^n e^{-(\alpha T_1+\beta T_2)}} + \frac{n\alpha^{n-1}}{(\alpha^n+\beta^n T_1)} - T_1 - \lambda\sum_{i=1}^n t_i e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)} \quad (22)$$

$$0 = \frac{n\beta^{n-1}T_1}{(\alpha^n+\beta^n T_1)} - \frac{\lambda}{2}\sum_{i=1}^n t_i^2 e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)} - \frac{T_2\lambda^n e^{-(\alpha T_1+\beta T_2)}}{1+\lambda^n e^{-(\alpha T_1+\beta T_2)}} - T_2 \quad (23)$$

$$0= -n+\sum_{i=1}^n e^{-(\alpha t_i+\frac{\beta}{2}t_i^2)} + \frac{n\lambda^{(n-1)}e^{-(\alpha T_1+\beta T_2)}}{1+\lambda^n e^{-(\alpha T_1+\beta T_2)}} \quad (24)$$

The maximum likelihood estimates can be obtained by solving the non-linear equations numerically for α, β and λ. This can be done using Mathematica and Maple, among other packages. The relatively large number of parameters can cause problems especially when the sample size is not large. A good set of initial values is essential.

Asymptotic Confidence Bounds: Since the MLEs of the unknown parameters α, β, λ cannot be obtained in closed forms, then it is not easy to derive the exact distributions of the MLE of these parameters. Thus, we derive the approximate confidence intervals of the parameters based on the asymptotic distributions of the MLE of the parameters. It is known that the asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ is given by see [14], $((\hat{\alpha}, \hat{\beta}, \hat{\lambda}) - (\alpha, \beta, \lambda)) \rightarrow N(0, I_0^{-1})$, where I_0^{-1} is the variance covariance matrix of the unknown parameters (α, β, λ), where I_0^{-1} is the inverse of the observed information matrix

$$I_0^{-1} = \begin{pmatrix} \frac{\partial^2 \ln L}{\partial^2 \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial^2 \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ln L}{\partial \lambda \partial \beta} & \frac{\partial^2 \ln L}{\partial^2 \lambda^2} \end{pmatrix}^{-1} \quad (25)$$

thus

$$I_0^{-1} = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}) & Cov(\hat{\alpha}, \hat{\lambda}) \\ Cov(\hat{\beta}, \hat{\alpha}) & Var(\hat{\beta}) & Cov(\hat{\beta}, \hat{\lambda}) \\ Cov(\hat{\lambda}, \hat{\alpha}) & Cov(\hat{\lambda}, \hat{\beta}) & Var(\hat{\lambda}) \end{pmatrix} \quad (26)$$

The derivatives in I_0 are given as follows

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{n(n-1)\alpha^{n-2}}{(\alpha^n + \beta^n T_1)} - \frac{n^2 \alpha^{2n-2}}{(\alpha^n + \beta^n T_1)^2} + \frac{\lambda^n T_1^2 e^{-(\alpha T_1 + \beta T_2)}}{1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)}} - \frac{\lambda^{2n} T_1^2 e^{-2(\alpha T_1 + \beta T_2)}}{(1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)})^2} + \lambda \sum_{i=1}^n t_i^2 e^{-(\alpha t_i + \beta t_i^2/2)} \quad (27)$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = -\frac{T_1 n^2 \alpha^{n-1} \beta^{n-1}}{(\alpha^n + \beta^n T_1)^2} + \frac{\lambda^n T_1 T_2 e^{-(\alpha T_1 + \beta T_2)}}{1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)}} - \frac{\lambda^{2n} T_1 T_2 e^{-2(\alpha T_1 + \beta T_2)}}{(1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)})^2} + \frac{\lambda}{2} \sum_{i=1}^n t_i^3 e^{-(\alpha t_i + \beta t_i^2/2)} \quad (28)$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = -\frac{n \lambda^{n-1} T_1 e^{-(\alpha T_1 + \beta T_2)}}{1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)}} + \frac{n \lambda^{2n-1} T_1 e^{-2(\alpha T_1 + \beta T_2)}}{(1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)})^2} - \sum_{i=1}^n t_i e^{-(\alpha t_i + \beta t_i^2/2)} \quad (29)$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{n(n-1) T_1 \beta^{n-2}}{(\alpha^n + \beta^n T_1)} - \frac{n^2 \beta^{2n-2} T_1^2}{(\alpha^n + \beta^n T_1)^2} + \frac{\lambda^n T_2^2 e^{-(\alpha T_1 + \beta T_2)}}{1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)}} - \frac{\lambda^{2n} T_2^2 e^{-2(\alpha T_1 + \beta T_2)}}{(1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)})^2} + \frac{\lambda}{4} \sum_{i=1}^n t_i^4 e^{-(\alpha t_i + \beta t_i^2/2)} \quad (30)$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = -\frac{n \lambda^{n-1} T_2 e^{-(\alpha T_1 + \beta T_2)}}{1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)}} + \frac{n \lambda^{2n-1} T_2 e^{-2(\alpha T_1 + \beta T_2)}}{(1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)})^2} - \frac{1}{2} \sum_{i=1}^n t_i^2 e^{-(\alpha t_i + \beta t_i^2/2)} \quad (31)$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{n(n-1)\lambda^{n-2} e^{-(\alpha T_1 + \beta T_2)}}{1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)}} - \frac{n^2 \lambda^{2n-2} e^{-2(\alpha T_1 + \beta T_2)}}{(1 + \lambda^n e^{-(\alpha T_1 + \beta T_2)})^2} \quad (32)$$

The above approach is used to derive the $100(1-\alpha)\%$ confidence intervals of the parameters α, β, λ as in the following forms

$$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{Var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\alpha/2}\sqrt{Var(\hat{\beta})}, \quad \hat{\lambda} \pm Z_{\alpha/2}\sqrt{Var(\hat{\lambda})} \quad (33)$$

Here, $Z_{\alpha/2}$ is the upper $(\alpha/2)$ th percentile of the standard normal distribution. It should be mentioned here as it was pointed by a referee that if we do not make the assumption that the true parameter vector (α, β, λ) is an interior point of the parameter space then the asymptotic normality results will not hold. If any of the true parameter value is 0, then the asymptotic distribution of the maximum likelihood estimators is a mixture distribution, see for example [15] in this connection. In that case obtaining the asymptotic confidence intervals becomes quite difficult and it is not pursued here.

5. Application

In this section we use the real data were $(t_1, t_2, \ldots, t_{10})$ = (31, 43, 56, 65, 73, 82, 96, 101, 111, 135), we assume that these data follow the distribution $G(t; \alpha, \beta, \lambda)$. First we compute the maximum likelihood estimator(s) for the parameters (α, β, λ), finally, we compute the asymptotic confidence intervals of the parameters (α, β, λ).

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The MLE of the parameters is $\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\end{bmatrix} = \begin{bmatrix}4.0700\\ 2.1200\\ 0.4025\end{bmatrix}$

and by substituting the MLE of unknown parameters in equation (25), we get estimation of the variance covariance matrix as

$$I_0^{-1} = \begin{bmatrix} 4.307\times10^{-5} & -7.081\times10^{-5} & 9.075\times10^{-3} \\ -8.503\times10^{-4} & 8.725\times10^{-6} & -4.917\times10^{-4} \\ 2.614\times10^{-3} & -9.566\times10^{-4} & 0.0164 \end{bmatrix}$$

The approximate 95% two sided confidence intervals of the parameters (α, β, λ)

Confidence intervals			
Significance level	Parameter Estimation	Upper	lower
a=0.05	α	4.0829	4.0571
a=0.05	β	2.2579	2.1142
a=0.05	λ	0.6535	0.1515

6. Conclusion

The new distribution with three parameters, referred to as the Poisson distribution is used to add a new parameter to the linear exponential distribution given in this study. We discussed some reliability and statistical properties of new distribution. In this paper we have considered the problem of estimation of parameters of new distribution. Procedure for the maximum likelihood estimation has been discussed.

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