Analysis of Batch Arrival Queue with Second Optional and Repair Under Optional Vacation, Balking

R. Vimaladevi

Arignar Anna Govt.Arts College, Villupuram

Abstract: ***This paper analyze a batch arrival queue with second optional service and repair under optional vacation and balking is considered. All coming customers demand the first essential service where in only some of them demand the second optional service. After completion of service, the server will go for vacation with probability p or remain staying back in the system for providing the service to the next customer with probability 1-p if any. The service times of both essential and optional services and vacation time follow arbitrary (general) distribution. The system may breakdown at random time and the breakdowns occur according to poisson stream. Once the server breakdown, it must be spend to repair process immediately. The most realistic aspect in modeling of a unreliable server, multi optional repair is required. Both essential and optional repair times follow exponential distribution. In addition to this , due to the annoyance of seeing long queue in the system the customer may decide not to join the queue, called balking. Such a customer behavior is considered in both busy time and vacation time of the system. The time dependent probability generating functions has been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly.***

Keywords: Batch arrival, Optional service, Optional repair, Balking

Mathematics Subject Classification: 60K25, 60K30

1. Introduction

Due to the improvement and advancement of science and technology , performance in modeling is one of the vital parts that the design, configuration , and implement of any real time system. Batch queueing models have been analyzed in the past by several authors. Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Application of vacation models can be found in production line systems, designing local area networks and data communication systems.

The research study of queueing models with second optional service has a prominent place in queueing theory. The server performs first essential service to all arriving customers and after completing the first essential service, second optional service wii be provided to some customers those who demand a second optional service. Madan (2000) has first introduced the concept of second optional service of an M/G/1 queueing system in whuch he has analyzed the time – dependent as well as the steady state behavior of the model by using supplementary variable technique. Madhi (2001) proposed an M/G/1 queueing model with second optional channel who developed the explicit expressions for the mean queue length and mean waiting time. Later Madan (2002) studied second optional service by incorporating Bernoulli schedule server vacations. Gaudham choudhury (2003) analyzed some aspects of M/G/1 queueing system with second optional service and obtained the steady state queue size distribution at the stationary point of time for general second optional service.

Whenever the server encounters a breakdown, it would not be able to serve unless it should be repaired. Therefore the server should undergo a repair process, but sometimes the repair process will not be started immediately due to the non availability of the repairing equipment or repairmen.Such situations can be also be modeled as queueing model .Burke studied delays in single – server queues with batch input. Madan studied queueing system with random failures and delayed repairs. Choudhury et al. discussed a batch arrival, single server queue with two phase of service subject to the breakdown and delay time.

The arriving customers may be discourged due to long queue ,and decide not to join the queue and leave the system at once. This behavior of customers is referred as balking. Sometimes customers get impatient after joing the queue and leave the sytem without getting service. This behavior of customer recognizes as reneging.A queue with balking was initially studied by Haight(1957). Queue with balking has been studied by authours like Altlan and Yechiali(2006).Ancker et al.(1963), Choudhry and Medhi (2011) in the last few years.

In this paper, we analyze a queueing system such that the customers are arriving in batches according to poisson stream. The server provides a first essential service to all incoming customers and second optional service will be provide to some of then those who demand it. Both essential and optional service times are assumed to follow general distribution. After the service completion the server may go for a vacation with probability p or continue staying in the system to next customer, if any, with probability 1-p. On account of, the system amy subject to breakdowns during busy time, the breakdown occur according to poisson process with mean breakdown rate α (>0). Once the breakdown, it is immediately sent for repair where in the repairman or repairing apparatus provides the essential repair (FER). After the completion of FER, the server may opt for second optional repair (SOR) with probability r or may join the system with complementary probability 1-r to render the service to the customers. After completion of requires repair, the server provides service with the same efficiency as before failure according to FCFS discipline. Both first essential and second optional repair follow exponential with mean $\frac{1}{\beta_1}$and$\frac{1}{\beta_2}$respectively.After the repair process complete, the server resumes its work immediately. Also whenever the system meet a breakdown, the customer whose service is interrupted goes back to the head of the queue. Also assuming that the batch arrival units may decide not to join the system (balks) by estimating the duration of waiting time for a service to get completed or by witnessing the long length of the queue.

2. Mathematical Description of the Model

We assume the following to describe the queueing model of our study. Let λk_i dt i=1,2,3… be the first order probability of arrival of 'I' customers in batches in the system during a short period of time ((*t*, *t*+ d*t*] where $0\leq k_i\leq 1$; $\sum_{i=}^{\infty} k_i=1$, $\lambda>0$ is the mean arrival rate of batches.

The single server provides the first essential service to all arriving customers. Let B_1 (*v*))and b_1 (*v*) be the distribution function and the density function of the first service times respectively.

As soon as the first service of a customer is complete , the customer may opt for the second service with probability r, in which case his second service will immediately commence or else with probability 1-r he may opt to leave the system, in which case another customer at the head of the queue taken up for his essential service. The second service is assumed to be general with the distribution function B_2 (*v*) and the density function b_2 (*v*).

Let μ_j (*x*)d*x* be the conditional probability of completion of the j*th* stage of service during the interval (*x*, *x*+ d*x*] given that elapsed time is *x*, so that

$$\mu_j(x)=\frac{bj\,(x)}{1-B_j\,(x)},\ j=1,2$$

and therefore,

$$bj(x)=\mu_j\,(x)\,e^{-\int_0^{\infty}\mu_j(x)dx}\ ,j=1,2$$

As soon as a service is completed, the server may take a vacation of random length with probability p or he may stay in the system providing service with probability (1-p). The vacation periods follow general (arbitrary) distribution with distribution function V(s) and the density function v(s). Let γ (x)dx be the conditional probability of a completion of a vacation during the interval (*x*, *x*+ d*x*] given that elapsed vacation time is x , so that

$$\gamma\,(x)=\frac{v(x)}{1-V(x)}$$

and therefore, v(s) = γ (s) $e^{-\int_0^{\infty}\gamma(x)dx}$

The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha>0$. as soon as the server is broken down, it is immediately sent for repair where in the repairman or repairing apparatus provides the essential repair (FER). After the completion of FER, the server may opt for second optional repair (SOR) with probability r or may join the system with complementary probability 1-r to render the service to the customers.

The repair process provides two types of repair in which the first type of repair is essential and the second type of repairs is optional. Both exponentially distributed with mean $\frac{1}{\beta_1}$and$\frac{1}{\beta_2}$. After completion of the required repair, the server provides service with the same efficiency as before failure according to FCFS discipline.

Also we assume that $(1-a_1)$ ($0\leq a_1\leq 1$) is the probability that an arriving customers balks during the period when the server is busy and $(1-a_2)$ ($0\leq a_2\leq 1$) is the probability that an arriving customer balks during the period when the server is on vacation.

3. Definitions and Equations Governing the System

We define

$P_n^{(i)}(x,t)$ = Probability that at time *t*, the server is active providing i^th service and there are *n* ($n\geq 1$) customers in the queue including the one being served and the elapsed service time for this customer is *x*. Consequently $P_n^{(i)}(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in i ^th service irrespective of the value of *x*.

$V_n(x,t)$ = Probability that at time *t*, the server is on vacation with elapsed vacation time x, and there are *n* ($n\geq 1$) customers waiting in the queue for service. Consequently $V_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

$R_n^{(1)}(x,t)$ = Probability that at time *t*, the server is inactive due to breakdown and the system is under first essential repair while there are in *n* ($n\geq 0$) customers in the queue .

$R_n^{(2)}(x,t)$ = Probability that at time *t*, the server is inactive due to breakdown and the system is under second optional repair while there are in *n* ($n\geq 0$) customers in the queue

Q(t) = Probability that at time *t*, there are no customers in the system and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+\left(\lambda+\mu_1(x)+\alpha\right)P_n^{(1)}(x,t)=\lambda\,(1-a_1)P_n^{(1)}(x,t)+\lambda a_1\sum_{i=1}^{\infty}K_iP_{n-i}^{(1)}(x,t)\ ,n\geq 1\ (1)$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+\left(\lambda+\mu_1(x)+\alpha\right)P_0^{(1)}(x,t)=\lambda\,(1-a_1)P_0^{(1)}(x,t)\ (2)$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+\left(\lambda+\mu_2(x)+\alpha\right)P_n^{(2)}(x,t)=\lambda\,(1-a_1)P_n^{(2)}(x,t)+\lambda a_1\sum_{i=1}^{\infty}K_iP_{n-i}^{(2)}(x,t)\ ,n\geq 1\ (3)$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+\left(\lambda+\mu_2(x)+\alpha\right)P_0^{(2)}(x,t)=\lambda\,(1-a_1)P_0^{(2)}(x,t)\ (4)$$

$$\frac{d}{dt}V_n(x,t)+\frac{\partial}{\partial x}V_n(x,t)+(\lambda+\gamma(x))V_n(x,t)=\lambda(1-a_2)V_n(x,t)+\lambda a_2\sum_{i=1}^{\infty}K_iV_{n-i}(x,t)\ ,n\geq 1\ (5)$$

$$\frac{d}{dt}V_0(t)+\frac{\partial}{\partial x}V_0(x,t)+(\lambda+\gamma(x))V_0(x,t)=\lambda(1-a2V0x,t\ (6)$$

$\frac{d}{dt}R_0^{(1)}(t) = -(\lambda+\beta_1)R_0^{(1)}(t)$ (7)

$\frac{d}{dt}R_n^{(1)}(t) = (\lambda+\beta_1)R_n^{(1)}(t) + \lambda\sum_{i=1}^{n} K_i R_{n-i}^{(1)}(t) + \alpha\int_0^\infty P_{n-1}^{(1)}(x,t) + \alpha\int_0^\infty P_{n-1}^{(2)}(x,t),\ n\ge 1$ (8)

$\frac{d}{dt}R_0^{(2)}(t) = -(\lambda+\beta_2)R_0^{(2)}(t)$ (9)

$\frac{d}{dt}R_n^{(2)}(t) = -(\lambda+\beta_2)R_n^{(2)}(t) + \lambda\sum_{i=1}^{n} K_i R_{n-i}^{(2)}(t) + (r\,\beta_1)R_n^{(1)}(t), n\ge 1$ (10)

$\frac{d}{dt}Q(t) = -\lambda(1-a_1)Q(t) + (1-r)\,\beta_1 R_0^{(1)}(t) + \beta_2 R_0^{(2)}(t) + (1-p)\,(1-r)\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)dx + (1-p)\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx$ (11)

Equations are to be solved subject to the following boundary conditions:

$P_0^{(1)}(0,t) = \lambda c_1\, a_1 Q(t) + (1-r)\,\beta_1 R_1^{(1)}(t) + \beta_2 R_1^{(2)}(t) + (1-p)(1-r)\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)dx + (1-p)\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_1(x,t)\gamma(x)dx$ (12)

$P_n^{(1)}(0,t) = \lambda c_{n+1}\, a_1 Q(t) + (1-r)\,\beta_1 R_{n+1}^{(1)}(t) + \beta_2 R_{n+1}^{(2)}(t) + (1-p)(1-r)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx + (1-p)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_{n+1}(x,t)\gamma(x)dx,$ (13)

$P_n^{(2)}(0,t) = r\int_0^\infty P_n^{(1)}(x,t)\,\mu_1(x)dx,\ n\ge 0$ (14)

$V_n(0,t) = (1-r)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_n^{(2)}(x,t)\,\mu_2(x)dx,\ n\ge 0$ (15)

We assume that initially there are no customers in the system and the server is idle. so the initial Conditions are with Q(0)=1, $P_n^{(1)}(0)= P_n^{(2)}=0$ and $V_n(0)= V_0(0)$

Probability Generating Functions of the Queue Size

We define the probability generating functions,

$P_q^{(i)}(x,z,t) = \sum_{n=0}^\infty P_n^{(i)}(x,t)\, Z^n$; i=1,2 $P_q^{(i)}(z,t) = \sum_{n=0}^\infty P_n^{(i)}(t)\, Z^n$; i=1,2

$V(x,z,t) = \sum_{n=0}^\infty V_n^{(i)}(x,t)\, Z^n$, $V(z,t) = \sum_{n=0}^\infty V_n^{(i)}(t)\, Z^n$ (*)

$R_q^{(i)}(z,t) = \sum_{n=0}^\infty R_q^{(i)}(t) Z^n$ i=1,2 $K(z) = \sum_{n=1}^\infty K_n\, Z^n$

Taking Laplace transforms of equations (1) to (15)

$\frac{\partial}{\partial x}\bar P_n^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar P_n^{(1)}(x,s) = \lambda(1-a_1)\bar P_n^{(1)}(x,s) + \lambda a_1\sum_{i=1}^{n-1}K_i\bar P_{n-i}^{(1)}(x,s),\ n\ge 1$ (16)

$\frac{\partial}{\partial x}\bar P_0^{(1)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar P_0^{(1)}(x,s) = \lambda(1-a_1)\bar P_0^{(1)}(x,s)$ (17)

$\frac{\partial}{\partial x}\bar P_n^{(2)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar P_n^{(2)}(x,s) = \lambda(1-a_1)\bar P_n^{(2)}(x,s) + (x,s)\lambda a_1\sum_{i=1}^{n-1}K_i\bar P_{n-i}^{(2)}(x,s),\ n\ge 1$ (18)

$\frac{\partial}{\partial x}\bar P_0^{(2)}(x,s) + (s+\lambda+\mu_1(x)+\alpha)\bar P_0^{(2)}(x,s) = \lambda(1-a_1)\bar P_0^{(2)}(x,s)$ (19)

$\frac{\partial}{\partial x}\bar V_n(x,s) + (s+\lambda+\gamma(x))\bar V_n(x,s) = \lambda(1-a_2)V_n(x,s) + \lambda a_2\sum_{i=1}^{n-1}K_iV_{n-i}(x,s),\ n\ge 1$ (20)

$\frac{\partial}{\partial x}\bar V_0(x,s) + (s+\lambda+\gamma(x))\bar V_0(x,s) = \lambda(1-a_2)\bar V_0(x,s)$ (21)

$(s+\lambda+\beta_1)\bar R_0^{(1)}(s) = 0$ (22)

$(s+\lambda+\beta_1)\bar R_n^{(1)}(s) = \lambda\sum_{i=1}^{n-1}K_i\bar R_{n-i}^{(1)}(s) + \alpha\int_0^\infty \bar P_{n-1}^{(1)}(x,s) + \alpha\int_0^\infty \bar P_{n-1}^{(2)}(x,s),\ n\ge 1$ (23)

$(s+\lambda+\beta_2)\bar R_0^{(2)}(s) = 0$ (24)

$(s+\lambda+\beta_2)\bar R_n^{(2)}(s) + \lambda\sum_{i=1}^{n-1}K_i\bar R_{n-i}^{(2)}(s) + (r\,\beta_1)\bar R_n^{(1)}(s)$ (25)

$(s+\lambda)Q(s) = -\lambda(1-a_1)Q(s) + (1-r)\,\beta_1 R_0^{(1)}(s) + \beta_2 R_0^{(2)}(s) + (1-p)\,(1-r)\int_0^\infty P_1^{(1)}(x,s)\mu_1(x)dx + (1-p)\int_0^\infty P_1^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty V_0(x,s)\gamma(x)dx$ (26)

For the boundary conditions

$\bar P_0^{(1)}(0,s) = \lambda c_1\,\overline{a_1 Q}(s) + (1-r)\,\beta_1\bar R_1^{(1)}(s) + \beta_2\bar R_1^{(2)}(s) + (1-p)(1-r)\int_0^\infty \bar P_1^{(1)}(x,s)\mu_1(x)dx + (1-p)\int_0^\infty \bar P_1^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty \bar V_1(x,s)\gamma(x)dx$ (27)

$\bar P_n^{(1)}(0,s) = \lambda c_{n+1}\, a_1\bar Q(s) + (1-r)\,\beta_1\bar R_{n+1}^{(1)}(s) + \beta_2\bar R_{n+1}^{(2)}(s) + (1-p)\int_0^\infty \bar P_{n+1}^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty V_{n+1}(x,s)\gamma(x)dx,$ (28)

$\bar P_n^{(2)}(0,s) = \int_0^\infty \bar P_n^{(1)}(x,s)\,\mu_1(x)dx,\ n\ge 0$ (29)

$\bar V_n(0,s) = p\,(1-r)\int_0^\infty \bar P_1^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty \bar V_1(x,s)\gamma(x)dx$ (30)

multiplying (16) and (18) by z^n and summing over n from 1 to ∞ , adding to equation (17) and (19) and using the definition of probability generating function , we obtain

$\frac{\partial}{\partial x}\bar P^{(1)}(x,z,s) + [s + a_1\lambda(1-K(z)) + \mu_1(x) + \alpha]\bar P^{(1)}(x,z,s) = 0$ (31)

$\frac{\partial}{\partial x}\bar P^{(2)}(x,z,s) + [s + a_1\lambda(1-K(z)) + \mu_2(x) + \alpha]\bar P^{(2)}(x,z,s) = 0$ (32)

Similar operations on equations (20),(23),(25) and (26) yields

$\frac{\partial}{\partial x}\bar V_q(x,z,s) + (s + a_2\lambda(1-K(z)+\gamma(x))\bar V_q(x,z,s) = 0$ (33)

$[s+\lambda-\lambda K(z)+\beta_1]\,\bar R_q^{(1)}(z,s) = \alpha z\int_0^\infty \bar P^{(1)}(x,z,s)dx + \alpha z\int_0^\infty P^{(2)}(x,z,s)dx$ (34)

$[s+\lambda-\lambda K(z)+\beta_2]\bar R_q^{(2)}(z,s) = (r\,\beta_1)\bar R_q^{(1)}(z,s)$ (35)

Similar operations on equations (27),(28),(29) and (30)

$z\,\bar P^{(1)}(0,z,s) = (1-s)\bar Q(s) + \lambda a_1(K(z)-1)\bar Q(s) + (1-r)\,\beta_1\bar R_1^{(1)}(s) + \beta_2\bar R_1^{(2)}(s) + (1-p)(1-r)\int_0^\infty \bar P_1^{(1)}(x,s)\mu_1(x)dx + (1-p)\int_0^\infty \bar P_1^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty \bar V_q(x,s)\gamma(x)dx$ (36)

$\bar P^{(2)}(0,z,s) = r\int_0^\infty \bar P^{(1)}(x,z,s)\mu_1(x)dx$ (37)

$\bar V_q(0,z,s) = p\,(1-r)\int_0^\infty \bar P_q^{(1)}(x,s)\mu_2(x)dx + rp\int_0^\infty \bar P_q^{(2)}(x,s)\mu_2(x)dx$ (38)

Integrating equations (31) between 0 and x, we get

$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)\, e^{-(s+a_1\lambda(1-K(z))+\alpha)x-\int_0^\infty \mu_1(t)dt}$
(39)
Again integrating equation (39) w.r.to x, we have

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)}{(s+a_1\lambda(1-K(z))+\alpha)}\right] \quad (40)$$

$where\ \bar{B}_1(s+a_1\lambda(1-C(z))+\alpha) = \int_0^\infty e^{-(s+a_1\lambda(1-K(z))+\alpha)x} d\bar{B}_1(x)$
is the Laplace transform of service time. Multiply equation (39) by $\mu_1(x)$ and integrate with respect to x
$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1(s+a_1\lambda(1-$
$Kz)+\alpha$ (41)
Multiply equation (32) and (33) after some simplification and using equation (39) , we obtain

$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)\, e^{-(s+a_1\lambda(1-K(z))+\alpha)x-\int_0^\infty \mu_2(t)dt}$
(42)
$\bar{V}_q(x,z,s) = \bar{V}_q(0,z,s)e^{-(s+a_2\lambda(1-K(z))+\alpha)x-\int_0^\infty \gamma(x)\,dx}$ (43)
Where $\bar{P}^{(2)}(0,z,s)$ and $\bar{V}_q(0,z,s)$ in equation (37) and (38)
Again integrating equation (42) and (43) w.r.to x, we have

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+a_1\lambda(1-C(z))+\alpha)}{(s+a_1\lambda(1-C(z))+\alpha)}\right] \quad (44)$$
$$\bar{V}_q(z,s) = \bar{V}_q(0,z,s)\left[\frac{1-\bar{V}(s+a_2\lambda(1-K(z))+\alpha)}{(s+a_2\lambda(1-K(z))+\alpha)}\right] \quad (45)$$
$where\ \bar{B}_2(s+a_1\lambda(1-K(z))+\alpha) = \int_0^\infty e^{-(s+a_1\lambda(1-K(z))+\alpha)x} d\bar{B}_2(x)$
And
$\bar{V}(s+a_2\lambda(1-K(z))+\alpha) == \int_0^\infty e^{-(s+a_1\lambda(1-K(z)))x} d\bar{V}(x)$
Using equation (42) by $\mu_2(x)$ and integrating with respect to x

$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2\,(s+a_1\lambda(1-$
$Kz)+\alpha$ (46)

Using (46), equation (37) reduces to
$\int_0^\infty \bar{P}^{(2)}(0,z,s)\mu_2(x)dx = r\bar{P}^{(1)}(0,z,s)\bar{B}_2\,(s+a_1\lambda(1-$
$Kz)+\alpha$ (47)
multiply (43) by $\gamma(x)$ integrating with respect to x

$\int_0^\infty \bar{V}_q(x,z,s)(x)\gamma(x)\,dx = \bar{V}_q(0,z,s)\bar{V}\left(s+a_2\lambda(1-K(z)\right)$
(48)
Equation (44) becomes
$$\bar{P}^{(2)}(z,s) = r\bar{P}^{(1)}(0,z,s)\left[\frac{\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)(1-\bar{B}_2(s+a_1\lambda(1-K(z))+\alpha)}{s+a_1\lambda(1-K(z))+\alpha}\right]$$
(49)
Using (41),(46),(47) in (38)

$\bar{V}_q(0,z,s) = p(1-r)\bar{P}^{(1)}(0,z,s)\,\bar{B}_1\big(s+a_1\lambda(1-K(z)\big)+$
$\alpha+$
r $\quad \bar{P}^{(1)}(0,z,s)\,\bar{B}_1\big(s+a_1\lambda(1-K(z))+\alpha\big)\bar{B}_2(s+$
$a1\lambda 1-Kz+\alpha$ (50)
Using (50), equation (45) becomes

$\bar{V}_q(z,s) = \frac{1}{Dr}\{\bar{P}^{(1)}(0,z,s)\,p\,\{(1-r)\ \bar{B}_1\big(s+a_1\lambda(1-$
$Kz+\alpha+$
r $\quad \bar{B}_1\big(s+a_1\lambda(1-K(z))+\alpha\big)\bar{B}_2\big(s+a_1\lambda(1-K(z))+$
$\alpha)\,\}\left[\frac{1-\bar{V}(s+a_2\lambda(1-K(z))+\alpha)}{(s+a_2\lambda(1-K(z))+\alpha)}\right]$ (51)
Using (41)and (46) in equations (34) becomes

$$\bar{R}_q^{(1)}(z,s) = \frac{\alpha z\bar{P}^{(1)}(0,z,s)[1-(1-r)\,\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)-r\,\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)\,\bar{B}_2(s+a_1\lambda(1-K(z))+\alpha)]}{\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)\,\bar{B}_1(s+\lambda(1-K(z))+\beta_1)} \quad (52)$$
Equation (35)becomes

$$\bar{R}_q^{(2)}(z,s) = \frac{r\beta_1\alpha z\bar{P}^{(1)}(0,z,s)[1-(1-r)\,\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)-r\,\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)\,\bar{B}_2(s+a_1\lambda(1-K(z))+\alpha)]}{\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)\,\bar{B}_1(s+\lambda(1-K(z))+\beta_1)\,\bar{B}_1(s+\lambda(1-K(z))+\beta_2)} \quad (53)$$

Solving for $\bar{P}^{(1)}(0,z,s)$ using equation (41),(46),(48),(52),(53) in equation (36)

$$\bar{P}^{(1)}(0,z,s) = \frac{f_1(z)f_2(z)f_3(z)[(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (54)$$
Where
$Dr = f_1(z)f_2(z)f_3(z)\{z-((1-p)-p\bar{V}\big(s+a_2\lambda(1-K(z))+\alpha\big))[(1-r)\,\bar{B}_1\big(s+a_1\lambda(1-K(z))+\alpha\big)-$
$r\,\bar{B}_1\big(s+a_1\lambda(1-K(z))+\alpha\big)\bar{B}_2\big(s+a_1\lambda(1-K(z))+\alpha\big)]-\alpha\beta z\,[(1-r)\,\bar{B}_1\big(s+a_1\lambda(1-K(z))+\alpha\big)-\ r\,\bar{B}_1(s+$
$a1\lambda 1-Kz+\alpha\ B2s+a1\lambda 1-Kz+\alpha\}$ (55)
Where
$\bar{f}_1(z) = s+a_1\lambda\big(1-C(z)\big)+\ \alpha\,,\bar{f}_2\,(z) = s+a_1\lambda(1-C(z))\ +\beta_1\ and\ \bar{f}_3(z) = s+a_1\lambda(1-C(z))\ +\ \beta_2$

Substituting the value of $P^{(1)}(0,z)$ from equation (40) into equations (49), (51) ,(52) we get

$$\bar{P}^{(1)}(z,s) = \frac{f_1(z)f_2(z)f_3(z)[1-\bar{B}_1\bar{f}_1(z)[(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (56)$$
$$\bar{P}^{(2)}(z,s) = \frac{rf_2(z)f_3(z)\bar{B}_1\bar{f}_1(z)[1-\bar{B}_2\bar{f}_1(z)][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (57)$$
$$\bar{V}_q(z,s) = \frac{pf_1(z)f_2(z)f_3(z)[1-(1-r)\,\bar{B}_1\big(\bar{f}_1(z)\big)+r\,\bar{B}_1\big(\bar{f}_1(z)\big)\,\bar{B}_2\big(\bar{f}_1(z)\big)][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr}\left[\frac{1-\bar{V}(s+a_2\lambda(1-K(z))+\alpha)}{(s+a_2\lambda(1-K(z))+\alpha)}\right] \quad (58)$$
$$\bar{R}_q^{(1)}(z,s) = \frac{\alpha\beta_1 zf_1(z)f_2(z)f_3(z)[1-(1-r)\,\bar{B}_1\big(\bar{f}_1(z)\big)+r\,\bar{B}_1\big(\bar{f}_1(z)\big)\,\bar{B}_2\big(\bar{f}_1(z)\big)][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{DR} \quad (59)$$
$$\bar{R}_q^{(2)}(z,s) = \frac{\alpha zf_1(z)f_2(z)f_3(z)[1-(1-r)\,\bar{B}_1\bar{f}_1(z)-r\,\bar{B}_1\big(\bar{f}_1(z)\big)\,\bar{B}_2\big(\bar{f}_1(z)\big)][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{DR} \quad (60)$$

4. The Steady State Analysis

By using well known Tauberian property,

$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$ (61)

Multiply both sides of equation (56) to (60) using (61)

$$V_q(\mathrm{z}) = \frac{\mathrm{p} f_1(z) f_2(z) f_3(z)[1-(1-r)\,\bar{B}_1 \bar{f}_1(\mathrm{z}) + r\,\bar{B}_1\left(\bar{f}_1(\mathrm{z})\right)\bar{B}_2\left(\bar{f}_1(\mathrm{z})\right)[(1-s)\bar{Q}(\mathrm{s}) + \lambda a_1(\mathrm{K}(\mathrm{z})-1)\bar{Q}(\mathrm{s})]}{\mathrm{Dr}}$$

(64)

$$\bar{R}_q^{(1)}(z) = \frac{\alpha\beta_1 z f_1(z) f_2(z) f_3(z)[1-(1-r)\,\bar{B}_1\left(\bar{f}_1(\mathrm{z})\right) + r\,\bar{B}_1\left(\bar{f}_1(\mathrm{z})\right)\bar{B}_2\left(\bar{f}_1(\mathrm{z})\right)][\lambda a_1(\mathrm{K}(\mathrm{z})-1)\mathrm{Q}]}{DR} \quad (65)$$

$$\bar{R}_q^{(2)}(z) = \frac{r\alpha\beta_1 z f_1(z) f_2(z) f_3(z)[1-(1-r)\,\bar{B}_1\left(\bar{f}_1(\mathrm{z})\right) + r\,\bar{B}_1\left(\bar{f}_1(\mathrm{z})\right)\bar{B}_2\left(\bar{f}_1(\mathrm{z})\right)][\lambda a_1(\mathrm{K}(\mathrm{z})-1)Q]}{DR} \quad (66)$$

Let W_q (z)= $P^{(1)}(z) + P^{(2)}(z) + V_q(z) + R_q^{(1)}(z) + R_q^{(2)}(z)$ (I)

$$P^{(1)}(z) = \frac{f_2(z) f_3(z)\left[1-B_1\left(\bar{f}_1(\mathrm{z})\right)\right][\lambda a_1(\mathrm{K}(\mathrm{z})-1)\mathrm{Q}]}{DR} \quad (62)$$

$$P^{(2)}(z) = \frac{r f_2(z) B_1\left(\bar{f}_1(\mathrm{z})\right)\left[1-B_2\left(\bar{f}_1(\mathrm{z})\right)\right][\lambda a_1(\mathrm{K}(\mathrm{z})-1)\mathrm{Q}]}{DR} \quad (63)$$

The normalizing condition.

$P^{(1)}(1) + P^{(2)}(1) + V(1) + R_q^{(1)}(1) + R_q^{(2)}(1) = 1$ (67)

$$P^{(1)}(1) = \frac{\lambda a_1\beta_1 \mathrm{Q}\ \mathrm{E(I)}(1-\bar{B}_1(\alpha))}{dr} \quad (68)$$

$$P^{(2)}(1) = \frac{r\lambda a_1\beta_1 \mathrm{Q}\ \mathrm{E(I)}\bar{B}_1(\alpha)[1-\bar{B}_2(\alpha)}{dr} \quad (69)$$

$$V(1) = \frac{p\lambda a_2\alpha\beta_1\beta_2 \mathrm{Q}\ \mathrm{E(I)E(v)}[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]}{dr} \quad (70)$$

$$\bar{R}_q^{(1)}(1) = \frac{\lambda\alpha\beta_2 \mathrm{Q}\ \mathrm{E(I)}(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]}{dr} \quad (71)$$

$$\bar{R}_q^{(2)}(1) = \frac{r\lambda\alpha\beta_2 \mathrm{Q}\ \mathrm{E(I)}(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]}{dr} \quad (72)$$

Where $dr = \alpha\beta_1\beta_2[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] - \lambda E(I)[r\alpha\beta_1 + \beta_2\alpha + \beta_1\beta_2] - \alpha\beta_1\beta_2 \mathrm{pE(I)E(v)}[(1-r)B_1(\alpha) + rB_1(\alpha)B_2(\alpha)]$

$$\mathrm{Q}=1-\lambda a_1 E(I)\Big[\frac{1}{\alpha[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{1}{\beta_1[[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{r}{\beta_2[[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)} + \frac{r}{\beta_2} - \frac{r}{\beta_1} - \frac{1}{\alpha} + p\,E(V)\Big] \quad (73)$$

and the utilization factor ρ of the system is given by

$$\rho = \lambda a_1 E(I)\left[\frac{1}{\alpha[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{1}{\beta_1[[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{r}{\beta_2[[(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)} + \frac{r}{\beta_2} - \frac{r}{\beta_1} - \frac{1}{\alpha} + p\,E(V\right]$$

(74)

Where $\rho < 1$ is the stability condition under which the steady states exits the equation (73)gives the probability that the server is idle. Substitute Q from equation (73) in equation (I). W_q (z) have been completely and explicitly determined which is the probability generating function of the queue size.

5. The Average Queue Size

Let L_q denote the mean number of customers in the queue under the steady state. Then we have

$L_q = \frac{d}{dz}[P_q(z)]$ at z = 1, $L_q = \lim_{z\to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2}$

(75)

Where primes and double primes in equation (75) denote first and second derivation at z=1 respectively, carrying out the derivatives at z=1.

$N'^{(1)} = \mathrm{Q}\lambda a_1 \mathrm{E}(I)\{[\mathrm{r}\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha] + (1 - r\,B1\alpha+rB1\alpha B2\alpha \mathrm{p}\alpha\beta1\beta2\mathrm{EV}-\mathrm{r}\alpha\beta1\beta2+\beta1\beta2+\beta2\alpha\}$ (76)

$N''(1) = 2\mathrm{Q}[(\lambda\alpha a_1\mathrm{E(I)})^2]\{[(\frac{\alpha}{\lambda\mathrm{E}(I)} - 1) + (1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)(1 - \frac{\alpha}{\lambda\mathrm{E}(I)} - \mathrm{pr}\alpha\beta_1\mathrm{E(V)} - \mathrm{p}\beta_1\beta_2\mathrm{E(V)} - \mathrm{p}\alpha\beta_2\mathrm{E(V)} + \frac{1}{2}\mathrm{p}\alpha\beta_1\beta_2 a_2^2\mathrm{E}(V)^2] + (1-r)\,\bar{B}_1' + r\bar{B}_1'(\alpha)\bar{B}_2'(\alpha)[\mathrm{r}\alpha\beta_1\beta_2 + \beta1\beta2+\beta2\alpha-\mathrm{p}\alpha\beta1\beta2\mathrm{EV}\}$
$+ \mathrm{Q}\lambda a_1\mathrm{E}(I-1)\{(\mathrm{r}\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha) + (1 - r\,B1\alpha+rB1\alpha B2\alpha\mathrm{p}\alpha\beta1\beta2\mathrm{EV}-\mathrm{r}\alpha\beta1\beta2+\beta1\beta2+\beta2\alpha$

$[\mathrm{p}\alpha\beta_1\beta_2\mathrm{E(V)} - (\mathrm{r}\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha)]\}$ (77)

$\mathrm{D}'(1) = \lambda a_1\mathrm{E}(I)\{[\mathrm{r}\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha] + (1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)\{\alpha\beta_1\beta_2 + \lambda\mathrm{E}(I)[\mathrm{r}\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha] - \mathrm{p}\alpha\beta_1\beta_2\mathrm{E(V)E(I)}$ (78)

$\mathrm{D}''(1)$= $\{2\lambda a_1 a_2\alpha\mathrm{E}(I)^2\{(1 - \frac{\mathrm{r}\alpha\beta_1+\beta_1\beta_2+\beta_2\alpha}{\lambda\mathrm{E}(I)}) + (1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)[(-1-\mathrm{pE(V)})(\mathrm{r}\alpha\beta_1 + \beta1\beta2+\beta2\alpha - 12\,\mathrm{p}\alpha\beta1\beta2 a22\mathrm{E}V2]$

$+ (1-r)\,\bar{B}_1' + r\bar{B}_1'(\alpha)\bar{B}_2'(\alpha)[-\frac{\alpha\beta_1\beta_2}{\lambda\mathrm{E}(I)} - [\mathrm{r}\alpha\beta_1 + \beta_1\beta_2 + \beta2\alpha + \mathrm{p}\alpha\beta1\beta2\mathrm{EV}]\} + \lambda a2\mathrm{E}(I)(I-1)\{-\mathrm{r}\alpha\beta1+\beta1\beta2+\beta2\alpha+$

$(1-r)\,\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)[(\mathrm{r}\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) - \mathrm{p}\alpha\beta_1\beta_2\mathrm{E(V)}]$ (79)

where $\mathrm{E}(V)^2$is the second moment of the vacation time and Q has been found in equation (73). Then if we substitute the values of N'(1), N"(1), D'(1) and D"(1) from equations (76),(77),(78),(79) in to equation(75). we obtain L_q in a closed form. Mean waiting time of a customer could be found $W_q = \frac{Lq}{\lambda}$ by using Little's formula.

6. Conclusion

In this paper we have studied batch arrival queue with second optional service and repair under optional vacation and balking is considered. The probability generating function of the number of customers in the queue is found using the supplementary variable technique. This model can be utilized in large scale manufacturing industries and communication networks.

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