

$$\frac{d}{dt} R_0^{(1)}(t) = -(\lambda + \beta_1) R_0^{(1)}(t) \quad (7)$$

$$\frac{d}{dt} R_n^{(1)}(t) = (\lambda + \beta_1) R_n^{(1)}(t) + \lambda \sum_{i=1}^n K_i R_{n-i}^{(1)}(t) + \alpha \int_0^\infty P_{n-1}^{(1)}(x, t) + \alpha \int_0^\infty P_{n-1}^{(2)}(x, t), n \geq 1 \quad (8)$$

$$\frac{d}{dt} R_0^{(2)}(t) = -(\lambda + \beta_2) R_0^{(2)}(t) \quad (9)$$

$$\frac{d}{dt} R_n^{(2)}(t) = -(\lambda + \beta_2) R_n^{(2)}(t) + \lambda \sum_{i=1}^n K_i R_{n-i}^{(2)}(t) + (r \beta_1) R_n^{(1)}(t), n \geq 1 \quad (10)$$

$$\frac{d}{dt} Q(t) = -\lambda(1 - a_1)Q(t) + (1-r) \beta_1 R_0^{(1)}(t) + \beta_2 R_0^{(2)}(t) + (1-p)(1-r) \int_0^\infty P_1^{(1)}(x, t) \mu_1(x) dx + (1-p) \int_0^\infty P_1^{(2)}(x, t) \mu_2(x) dx + \int_0^\infty V_0(x, t) \gamma(x) dx \quad (11)$$

Equations are to be solved subject to the following boundary conditions:

$$P_0^{(1)}(0, t) = \lambda c_1 a_1 Q(t) + (1-r) \beta_1 R_1^{(1)}(t) + \beta_2 R_1^{(2)}(t) + (1-p)(1-r) \int_0^\infty P_1^{(1)}(x, t) \mu_1(x) dx + (1-p) \int_0^\infty P_1^{(2)}(x, t) \mu_2(x) dx + \int_0^\infty V_1(x, t) \gamma(x) dx \quad (12)$$

$$P_n^{(1)}(0, t) = \lambda c_{n+1} a_1 Q(t) + (1-r) \beta_1 R_{n+1}^{(1)}(t) + \beta_2 R_{n+1}^{(2)}(t) + (1-p)(1-r) \int_0^\infty P_{n+1}^{(1)}(x, t) \mu_1(x) dx + (1-p) \int_0^\infty P_{n+1}^{(2)}(x, t) \mu_2(x) dx + \int_0^\infty V_{n+1}(x, t) \gamma(x) dx, \quad (13)$$

$$P_n^{(2)}(0, t) = r \int_0^\infty P_n^{(1)}(x, t) \mu_1(x) dx, n \geq 0 \quad (14)$$

$$V_n(0, t) = (1-r) \int_0^\infty P_n^{(1)}(x, t) \mu_1(x) dx + p \int_0^\infty P_n^{(2)}(x, t) \mu_2(x) dx, n \geq 0 \quad (15)$$

We assume that initially there are no customers in the system and the server is idle. so the initial Conditions are with $Q(0)=1, P_n^{(1)}(0)=P_n^{(2)}(0)$ and $V_n(0)=V_0(0)$

Probability Generating Functions of the Queue Size

We define the probability generating functions,

$$P_q^{(i)}(x, z, t) = \sum_{n=0}^\infty P_n^{(i)}(x, t) Z^n; i=1,2 \quad P_q^{(i)}(z, t) = \sum_{n=0}^\infty P_n^{(i)}(t) Z^n; i=1,2$$

$$V(x, z, t) = \sum_{n=0}^\infty V_n^{(i)}(x, t) Z^n, V(z, t) = \sum_{n=0}^\infty V_n^{(i)}(t) Z^n \quad (*)$$

$$R_q^{(i)}(z, t) = \sum_{n=0}^\infty R_n^{(i)}(t) Z^n \quad i=1,2 \quad K(z) = \sum_{n=1}^\infty K_n Z^n$$

Taking Laplace transforms of equations (1) to (15)

$$\frac{\partial}{\partial x} \bar{P}_n^{(1)}(x, s) + (s + \lambda + \mu_1(x) + \alpha) \bar{P}_n^{(1)}(x, s) = \lambda(1 - a_1) \bar{P}_n^{(1)}(x, s) + \lambda a_1 \sum_{i=1}^{n-1} K_i \bar{P}_{n-i}^{(1)}(x, s), n \geq 1 \quad (16)$$

$$\frac{\partial}{\partial x} \bar{P}_0^{(1)}(x, s) + (s + \lambda + \mu_1(x) + \alpha) \bar{P}_0^{(1)}(x, s) = \lambda(1 - a_1) \bar{P}_0^{(1)}(x, s) \quad (17)$$

$$\frac{\partial}{\partial x} \bar{P}_n^{(2)}(x, s) + (s + \lambda + \mu_1(x) + \alpha) \bar{P}_n^{(2)}(x, s) = \lambda(1 - a_1) \bar{P}_n^{(2)}(x, s) + (x, s) \lambda a_1 \sum_{i=1}^{n-1} K_i \bar{P}_{n-i}^{(2)}(x, s), n \geq 1 \quad (18)$$

$$\frac{\partial}{\partial x} \bar{P}_0^{(2)}(x, s) + (s + \lambda + \mu_1(x) + \alpha) \bar{P}_0^{(2)}(x, s) = \lambda(1 - a_1) \bar{P}_0^{(2)}(x, s) \quad (19)$$

$$\frac{\partial}{\partial x} \bar{V}_n(x, s) + (s + \lambda + \gamma(x)) \bar{V}_n(x, s) = \lambda(1 - a_2) \bar{V}_n(x, s) + \lambda a_2 \sum_{i=1}^{n-1} K_i \bar{V}_{n-i}(x, s), n \geq 1 \quad (20)$$

$$\frac{\partial}{\partial x} \bar{V}_0(x, s) + (s + \lambda + \gamma(x)) \bar{V}_0(x, s) = \lambda(1 - a_2) \bar{V}_0(x, s) \quad (21)$$

$$(s + \lambda + \beta_1) \bar{R}_0^{(1)}(s) = 0 \quad (22)$$

$$(s + \lambda + \beta_1) \bar{R}_n^{(1)}(s) = \lambda \sum_{i=1}^{n-1} K_i \bar{R}_{n-i}^{(1)}(s) + \alpha \int_0^\infty \bar{P}_{n-1}^{(1)}(x, s) + \alpha \int_0^\infty \bar{P}_{n-1}^{(2)}(x, s), n \geq 1 \quad (23)$$

$$(s + \lambda + \beta_2) \bar{R}_0^{(2)}(s) = 0 \quad (24)$$

$$(s + \lambda + \beta_2) \bar{R}_n^{(2)}(s) + \lambda \sum_{i=1}^{n-1} K_i \bar{R}_{n-i}^{(2)}(s) + (r \beta_1) \bar{R}_n^{(1)}(s) \quad (25)$$

$$(s + \lambda) Q(s) = -\lambda(1 - a_1) Q(s) + (1-r) \beta_1 R_0^{(1)}(s) + \beta_2 R_0^{(2)}(s) + (1-p)(1-r) \int_0^\infty P_1^{(1)}(x, s) \mu_1(x) dx + (1-p) \int_0^\infty P_1^{(2)}(x, s) \mu_2(x) dx + \int_0^\infty V_0(x, s) \gamma(x) dx \quad (26)$$

For the boundary conditions

$$\bar{P}_0^{(1)}(0, s) = \lambda c_1 a_1 \bar{Q}(s) + (1-r) \beta_1 \bar{R}_1^{(1)}(s) + \beta_2 \bar{R}_1^{(2)}(s) + (1-p)(1-r) \int_0^\infty \bar{P}_1^{(1)}(x, s) \mu_1(x) dx + (1-p) \int_0^\infty \bar{P}_1^{(2)}(x, s) \mu_2(x) dx + \int_0^\infty \bar{V}_1(x, s) \gamma(x) dx \quad (27)$$

$$\bar{P}_n^{(1)}(0, s) = \lambda c_{n+1} a_1 \bar{Q}(s) + (1-r) \beta_1 \bar{R}_{n+1}^{(1)}(s) + \beta_2 \bar{R}_{n+1}^{(2)}(s) + (1-p) \int_0^\infty \bar{P}_{n+1}^{(1)}(x, s) \mu_1(x) dx + \int_0^\infty \bar{P}_{n+1}^{(2)}(x, s) \mu_2(x) dx + \int_0^\infty \bar{V}_{n+1}(x, s) \gamma(x) dx, \quad (28)$$

$$\bar{P}_n^{(2)}(0, s) = \int_0^\infty \bar{P}_n^{(1)}(x, s) \mu_1(x) dx, n \geq 0 \quad (29)$$

$$\bar{V}_n(0, s) = p(1-r) \int_0^\infty \bar{P}_1^{(1)}(x, s) \mu_2(x) dx + \int_0^\infty \bar{V}_1(x, s) \gamma(x) dx \quad (30)$$

multiplying (16) and (18) by z^n and summing over n from 1 to ∞ , adding to equation (17) and (19) and using the definition of probability generating function, we obtain

$$\frac{\partial}{\partial x} \bar{P}^{(1)}(x, z, s) + [s + a_1 \lambda(1 - K(z)) + \mu_1(x) + \alpha] \bar{P}^{(1)}(x, z, s) = 0 \quad (31)$$

$$\frac{\partial}{\partial x} \bar{P}^{(2)}(x, z, s) + [s + a_1 \lambda(1 - K(z)) + \mu_2(x) + \alpha] \bar{P}^{(2)}(x, z, s) = 0 \quad (32)$$

Similar operations on equations (20),(23),(25) and (26) yields

$$\frac{\partial}{\partial x} \bar{V}_q(x, z, s) + (s + a_2 \lambda(1 - K(z)) + \gamma(x)) \bar{V}_q(x, z, s) = 0 \quad (33)$$

$$[s + \lambda - \lambda K(z) + \beta_1] \bar{R}_q^{(1)}(z, s) = \alpha z \int_0^\infty \bar{P}^{(1)}(x, z, s) dx + \alpha z \int_0^\infty P_2 x, z, s dx \quad (34)$$

$$[s + \lambda - \lambda K(z) + \beta_2] \bar{R}_q^{(2)}(z, s) = (r \beta_1) \bar{R}_q^{(1)}(z, s) \quad (35)$$

Similar operations on equations (27),(28),(29) and (30)

$$z \bar{P}^{(1)}(0, z, s) = (1 - s) \bar{Q}(s) + \lambda a_1 (K(z) - 1) \bar{Q}(s) + (1 -$$

$$r) \beta_1 \bar{R}_1^{(1)}(s) + \beta_2 \bar{R}_1^{(2)}(s) +$$

$$(1 - p)(1 - r) \int_0^\infty \bar{P}_1^{(1)}(x, s) \mu_1(x) dx +$$

$$(1 - p) \int_0^\infty \bar{P}_1^{(2)}(x, s) \mu_2(x) dx + \int_0^\infty \bar{V}_q(x, s) \gamma(x) dx \quad (36)$$

$$\bar{P}^{(2)}(0, z, s) = r \int_0^\infty \bar{P}^{(1)}(x, z, s) \mu_1(x) dx \quad (37)$$

$$\bar{V}_q(0, z, s) =$$

$$p(1-r) \int_0^\infty \bar{P}_q^{(1)}(x, s) \mu_2(x) dx + rp \int_0^\infty \bar{P}_q^{(2)}(x, s) \mu_2(x) dx \quad (38)$$

Integrating equations (31) between 0 and x, we get

$$\bar{P}^{(1)}(x, z, s) = \bar{P}^{(1)}(0, z, s) e^{-(s+a_1\lambda(1-K(z))+\alpha)x - \int_0^x \mu_1(t)dt}$$

(39) Again integrating equation (39) w.r.to x, we have

$$\bar{P}^{(1)}(z, s) = \bar{P}^{(1)}(0, z, s) \left[\frac{1 - \bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)}{(s+a_1\lambda(1-K(z))+\alpha)} \right] \quad (40)$$

where $\bar{B}_1(s + a_1\lambda(1 - C(z)) + \alpha) = \int_0^\infty e^{-(s+a_1\lambda(1-K(z))+\alpha)x} d\bar{B}_1(x)$

is the Laplace transform of service time. Multiply equation (39) by $\mu_1(x)$ and integrate with respect to x

$$\int_0^\infty \bar{P}^{(1)}(x, z, s) \mu_1(x) dx = \bar{P}^{(1)}(0, z, s) \bar{B}_1(s + a_1\lambda(1 - Kz) + \alpha) \quad (41)$$

Multiply equation (32) and (33) after some simplification and using equation (39) , we obtain

$$\bar{P}^{(2)}(x, z, s) = \bar{P}^{(2)}(0, z, s) e^{-(s+a_1\lambda(1-K(z))+\alpha)x - \int_0^x \mu_2(t)dt}$$

$$\bar{V}_q(x, z, s) = \bar{V}_q(0, z, s) e^{-(s+a_2\lambda(1-K(z))+\alpha)x - \int_0^x \gamma(x) dx} \quad (43)$$

Where $\bar{P}^{(2)}(0, z, s)$ and $\bar{V}_q(0, z, s)$ in equation (37) and (38)

Again integrating equation (42) and (43) w.r.to x, we have

$$\bar{P}^{(2)}(z, s) = \bar{P}^{(2)}(0, z, s) \left[\frac{1 - \bar{B}_2(s+a_1\lambda(1-C(z))+\alpha)}{(s+a_1\lambda(1-C(z))+\alpha)} \right] \quad (44)$$

$$\bar{V}_q(z, s) = \bar{V}_q(0, z, s) \left[\frac{1 - \bar{V}(s+a_2\lambda(1-K(z))+\alpha)}{(s+a_2\lambda(1-K(z))+\alpha)} \right] \quad (45)$$

where $\bar{B}_2(s + a_1\lambda(1 - K(z)) + \alpha) = \int_0^\infty e^{-(s+a_1\lambda(1-K(z))+\alpha)x} d\bar{B}_2(x)$

And $\bar{V}(s + a_2\lambda(1 - K(z)) + \alpha) = \int_0^\infty e^{-(s+a_1\lambda(1-K(z)))x} d\bar{V}(x)$

Using equation (42) by $\mu_2(x)$ and integrating with respect to x

$$\bar{R}_q^{(1)}(z, s) = \frac{\alpha z \bar{P}^{(1)}(0, z, s) [1 - (1-r) \bar{B}_1(s+a_1\lambda(1-K(z))+\alpha) - r \bar{B}_1(s+a_1\lambda(1-K(z))+\alpha) \bar{B}_2(s+a_1\lambda(1-K(z))+\alpha)]}{\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha) \bar{B}_1(s+\lambda(1-K(z))+\beta_1)} \quad (52)$$

Equation (35) becomes

$$\bar{R}_q^{(2)}(z, s) = \frac{r \beta_1 \alpha z \bar{P}^{(1)}(0, z, s) [1 - (1-r) \bar{B}_1(s+a_1\lambda(1-K(z))+\alpha) - r \bar{B}_1(s+a_1\lambda(1-K(z))+\alpha) \bar{B}_2(s+a_1\lambda(1-K(z))+\alpha)]}{\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha) \bar{B}_1(s+\lambda(1-K(z))+\beta_1) \bar{B}_1(s+\lambda(1-K(z))+\beta_2)} \quad (53)$$

Solving for $\bar{P}^{(1)}(0, z, s)$ using equation (41),(46),(48),(52),(53) in equation (36)

$$\bar{P}^{(1)}(0, z, s) = \frac{f_1(z)f_2(z)f_3(z)[(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (54)$$

Where

$$Dr = f_1(z)f_2(z)f_3(z)\{z - ((1-p) - p\bar{V}(s + a_2\lambda(1 - K(z)) + \alpha))[(1-r) \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha) - r \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha) \bar{B}_2(s + a_1\lambda(1 - K(z)) + \alpha)] - \alpha\beta z [(1-r) \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha) - r \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha) \bar{B}_2(s + a_1\lambda(1 - K(z)) + \alpha)]\} \quad (55)$$

Where

$$\bar{f}_1(z) = s + a_1\lambda(1 - C(z)) + \alpha, \bar{f}_2(z) = s + a_1\lambda(1 - C(z)) + \beta_1 \text{ and } \bar{f}_3(z) = s + a_1\lambda(1 - C(z)) + \beta_2$$

Substituting the value of $\bar{P}^{(1)}(0, z)$ from equation (40) into equations (49), (51), (52) we get

$$\bar{P}^{(1)}(z, s) = \frac{f_1(z)f_2(z)f_3(z)[1 - \bar{B}_1(\bar{f}_1(z))][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (56)$$

$$\bar{P}^{(2)}(z, s) = \frac{r f_2(z)f_3(z)\bar{B}_1(\bar{f}_1(z))[1 - \bar{B}_2(\bar{f}_1(z))][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (57)$$

$$\bar{V}_q(z, s) = \frac{p f_1(z)f_2(z)f_3(z)[1 - (1-r) \bar{B}_1(\bar{f}_1(z)) + r \bar{B}_1(\bar{f}_1(z)) \bar{B}_2(\bar{f}_1(z))][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)] \left[\frac{1 - \bar{V}(s+a_2\lambda(1-K(z))+\alpha)}{(s+a_2\lambda(1-K(z))+\alpha)} \right]}{Dr} \quad (58)$$

$$\bar{R}_q^{(1)}(z, s) = \frac{\alpha \beta_1 z f_1(z)f_2(z)f_3(z)[1 - (1-r) \bar{B}_1(\bar{f}_1(z)) + r \bar{B}_1(\bar{f}_1(z)) \bar{B}_2(\bar{f}_1(z))][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{DR} \quad (59)$$

$$\bar{R}_q^{(2)}(z, s) = \frac{\alpha z f_1(z)f_2(z)f_3(z)[1 - (1-r) \bar{B}_1(\bar{f}_1(z)) - r \bar{B}_1(\bar{f}_1(z)) \bar{B}_2(\bar{f}_1(z))][(1-s)\bar{Q}(s)+\lambda a_1(K(z)-1)\bar{Q}(s)]}{DR} \quad (60)$$

$$\int_0^\infty \bar{P}^{(2)}(x, z, s) \mu_2(x) dx = \bar{P}^{(2)}(0, z, s) \bar{B}_2(s + a_1\lambda(1 - Kz) + \alpha) \quad (46)$$

Using (46), equation (37) reduces to

$$\int_0^\infty \bar{P}^{(2)}(0, z, s) \mu_2(x) dx = r \bar{P}^{(1)}(0, z, s) \bar{B}_2(s + a_1\lambda(1 - Kz) + \alpha) \quad (47)$$

multiply (43) by $\gamma(x)$ integrating with respect to x

$$\int_0^\infty \bar{V}_q(x, z, s) \gamma(x) dx = \bar{V}_q(0, z, s) \bar{V}(s + a_2\lambda(1 - K(z))) \quad (48)$$

Equation (44) becomes

$$\bar{P}^{(2)}(z, s) = r \bar{P}^{(1)}(0, z, s) \left[\frac{\bar{B}_1(s+a_1\lambda(1-K(z))+\alpha)(1 - \bar{B}_2(s+a_1\lambda(1-K(z))+\alpha))}{s+a_1\lambda(1-K(z))+\alpha} \right] \quad (49)$$

Using (41),(46),(47) in (38)

$$\bar{V}_q(0, z, s) = p(1-r) \bar{P}^{(1)}(0, z, s) \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha +$$

$$r \bar{P}^{(1)}(0, z, s) \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha) \bar{B}_2(s + a_1\lambda(1 - K(z)) + \alpha) \quad (50)$$

Using (50), equation (45) becomes

$$\bar{V}_q(z, s) = \frac{1}{Dr} \{ \bar{P}^{(1)}(0, z, s) p \{ (1-r) \bar{B}_1(s + a_1\lambda(1 - Kz) + \alpha) + r \bar{B}_1(s + a_1\lambda(1 - K(z)) + \alpha) \bar{B}_2(s + a_1\lambda(1 - K(z)) + \alpha) \} \left[\frac{1 - \bar{V}(s+a_2\lambda(1-K(z))+\alpha)}{(s+a_2\lambda(1-K(z))+\alpha)} \right] \} \quad (51)$$

Using (41) and (46) in equations (34) becomes

4. The Steady State Analysis

By using well known Tauberian property,

$$\lim_{s \rightarrow 0} s\bar{f}(s) = \lim_{t \rightarrow \infty} f(t) \quad (61)$$

Multiply both sides of equation (56) to (60) using (61)

$$V_q(z) = \frac{pf_1(z)f_2(z)f_3(z)[1 - (1-r)\bar{B}_1\bar{f}_1(z) + r\bar{B}_1(\bar{f}_1(z))\bar{B}_2(\bar{f}_1(z))][(1-s)\bar{Q}(s) + \lambda a_1(K(z)-1)\bar{Q}(s)]}{Dr} \quad (64)$$

$$\bar{R}_q^{(1)}(z) = \frac{\alpha\beta_1zf_1(z)f_2(z)f_3(z)[1-(1-r)\bar{B}_1(\bar{f}_1(z))+r\bar{B}_1(\bar{f}_1(z))\bar{B}_2(\bar{f}_1(z))][\lambda a_1(K(z)-1)Q]}{DR} \quad (65)$$

$$\bar{R}_q^{(2)}(z) = \frac{r\alpha\beta_1zf_1(z)f_2(z)f_3(z)[1-(1-r)\bar{B}_1(\bar{f}_1(z))+r\bar{B}_1(\bar{f}_1(z))\bar{B}_2(\bar{f}_1(z))][\lambda a_1(K(z)-1)Q]}{DR} \quad (66)$$

$$\text{Let } W_q(z) = P^{(1)}(z) + P^{(2)}(z) + V_q(z) + R_q^{(1)}(z) + R_q^{(2)}(z) \quad (I)$$

The normalizing condition.

$$P^{(1)}(1) + P^{(2)}(1) + V(1) + R_q^{(1)}(1) + R_q^{(2)}(1) = 1 \quad (67)$$

$$P^{(1)}(1) = \frac{\lambda a_1\beta_1Q E(I)(1-\bar{B}_1(\alpha))}{dr} \quad (68)$$

$$P^{(2)}(1) = \frac{r\lambda a_1\beta_1Q E(I)\bar{B}_1(\alpha)[1-\bar{B}_2(\alpha)]}{dr} \quad (69)$$

$$V(1) = \frac{p\lambda a_2\alpha\beta_1\beta_2Q E(I)E(v)[(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]}{dr} \quad (70)$$

$$\bar{R}_q^{(1)}(1) = \frac{\lambda\alpha\beta_2Q E(I)(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)}{dr} \quad (71)$$

$$\bar{R}_q^{(2)}(1) = \frac{r\lambda\alpha\beta_2Q E(I)(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)}{dr} \quad (72)$$

Where $dr = \alpha\beta_1\beta_2[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] - \lambda E(I)[r\alpha\beta_1 + \beta_2\alpha + \beta_1\beta_2] - \alpha\beta_1\beta_2pE(I)E(v)[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]$

$$Q = 1 - \frac{\lambda a_1 E(I) \left[\frac{1}{\alpha[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{r}{\beta_1[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{r}{\beta_2[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} \right]}{\frac{r}{\beta_2} - \frac{r}{\beta_1} - \frac{1}{\alpha} + p E(V)} \quad (73)$$

and the utilization factor ρ of the system is given by

$$\rho = \lambda a_1 E(I) \left[\frac{1}{\alpha[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{1}{\beta_1[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{1}{\beta_2[(1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]} + \frac{r}{\beta_2} - \frac{r}{\beta_1} - \frac{1}{\alpha} + p E(V) \right] \quad (74)$$

Where $\rho < 1$ is the stability condition under which the steady states exists the equation (73) gives the probability that the server is idle. Substitute Q from equation (73) in equation (I). $W_q(z)$ have been completely and explicitly determined which is the probability generating function of the queue size.

$$P^{(1)}(z) = \frac{f_2(z)f_3(z)[1-B_1(\bar{f}_1(z))][\lambda a_1(K(z)-1)Q]}{DR} \quad (62)$$

$$P^{(2)}(z) = \frac{rf_2(z)B_1(\bar{f}_1(z))[1-B_2(\bar{f}_1(z))][\lambda a_1(K(z)-1)Q]}{DR} \quad (63)$$

5. The Average Queue Size

Let L_q denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz} [P_q(z)] \text{ at } z = 1, L_q = \lim_{z \rightarrow 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} \quad (75)$$

Where primes and double primes in equation (75) denote first and second derivation at $z=1$ respectively, carrying out the derivatives at $z=1$.

$$N'(1) = Q\lambda a_1 E(I) \{ [r\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha] + (1-r)B_1\alpha + rB_1\alpha B_2\alpha p\alpha\beta_1\beta_2 E V - r\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha \} \quad (76)$$

$$N''(1) = 2Q[(\lambda a_1 E(I))^2] \{ [(\frac{\alpha}{\lambda E(I)} - 1) + (1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha)(1 - \frac{\alpha}{\lambda E(I)} - p\alpha\beta_1 E(V) - p\beta_1\beta_2 E(V) + \frac{1}{2} p\alpha\beta_1\beta_2 a_2^2 E(V)^2) + (1-r)\bar{B}_1'(\alpha) + r\bar{B}_1'(\alpha)\bar{B}_2'(\alpha)[r\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha - p\alpha\beta_1\beta_2 E V] + Q\lambda a_1 E(I - 1) \{ (r\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha) + (1-r)B_1\alpha + rB_1\alpha B_2\alpha p\alpha\beta_1\beta_2 E V - r\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha$$

$$[p\alpha\beta_1\beta_2 E(V) - (r\alpha\beta_1\beta_2 + \beta_1\beta_2 + \beta_2\alpha)] \} \quad (77)$$

$$D(1) = \lambda a_1 E(I) \{ [r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha] + (1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha) \} \{ \alpha\beta_1\beta_2 + \lambda E(I)[r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha] - p\alpha\beta_1\beta_2 E(V) E(I) \} \quad (78)$$

$$D''(1) = \{ 2\lambda a_1 a_2 \alpha E(I)^2 \{ (1 - \frac{r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha}{\lambda E(I)}) + (1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha) [(-1 - pE(V))(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha - 12p\alpha\beta_1\beta_2 a_2^2 E V^2)] + (1-r)\bar{B}_1'(\alpha) + r\bar{B}_1'(\alpha)\bar{B}_2'(\alpha) [-\frac{\alpha\beta_1\beta_2}{\lambda E(I)} - [r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha + p\alpha\beta_1\beta_2 E V]] \} + \lambda a_2 E(I) (I-1) \{ -r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha + (1-r)\bar{B}_1(\alpha) + r\bar{B}_1(\alpha)\bar{B}_2(\alpha) [(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) - p\alpha\beta_1\beta_2 E(V)] \} \} \quad (79)$$

where $E(V)^2$ is the second moment of the vacation time and Q has been found in equation (73). Then if we substitute the values of $N'(1)$, $N''(1)$, $D'(1)$ and $D''(1)$ from equations (76),(77),(78),(79) in to equation(75). we obtain L_q in a closed form. Mean waiting time of a customer could be found $W_q = \frac{L_q}{\lambda}$ by using Little's formula.

6. Conclusion

In this paper we have studied batch arrival queue with second optional service and repair under optional vacation and balking is considered. The probability generating function of the number of customers in the queue is found using the supplementary variable technique. This model can be utilized in large scale manufacturing industries and communication networks.

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