respectively be the distribution function and the density function of the first service times respectively.

- As soon as the first service of a customer is completed, then he may demand for a certain second optional service from m kind of different service with probability r_k $(1 \leq k \leq m$, or else he may decide to leave the system with probability $r_0 = 1 - \sum_{k=1}^{m} r_k$, he may opt to leave the system.
- The second service times as assumed to be general with the distribution function $B_k(v)$ and the density function $b_k(v)$. Further, Let $\mu_k(x)dx$ be the conditional probability density function of k^{th} service completion during the interval (x, x+dx] given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, i = 0,1,\dots,m$$

and therefore,

$$b_i(s) = \mu_i(s)e^{-\int_0^s \mu_i(x)dx}, i = 0,1,\dots,m$$

- As soon as the second service completed and if the customer is dissatisfied with his service for certain reason or if he received unsuccessful service, the customer may immediately join the tail of the original queue with probability p $(0 \leq p < 1)$. Otherwise the customer may depart forever from the system with probability q (= 1 − p. The customers are served according to First come First service rule.
- If there is no customer waiting in the queue, then the server goes for a vacation. The vacation periods are exponentially distributed with mean vacation time $\frac{1}{\gamma}$. On returning from vacation if the server again founds no customer in the queue, then it goes for another vacation. So the server takes multiple vacations
- The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$
- Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.
- Various stochastic processes involved in the system are independent of each others.

3. Definitions and Equations Governing the System

- $P_n^0(x,t) =$ Probability that at time 't', there are 'n' customers in the queue including the one being provided the first essential service, with the elapsed service time for this customer is x. Consequently $P_n^0(t) =$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one being provided the first essential service irrespective of the value of x,
- $P_n^k(x,t) =$ Probability that at time't', there are 'n' customers in the queue including the one being provided the k^{th} optional service, with the elapsed service time for this customer is x. Consequently $P_n^k(t) =$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one being provided the k^{th} optional service irrespective of the value of x.
- $V_n(t)$ = the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.
- $R_n(t) =$ Probability that at time t, the server is inactive due to break down and the system is under repair while there are 'n' customers in the queue.

In steady state condition, we have

$P_n^i(x)\,dx = \lim_{t\to\infty} P_n^i$ (x, t), i=1, 2 ; $x > 0$; $n \geq 0$

$$V_n = \lim_{t\to\infty} V_n(t) \ ; \ n \geq 0$$

$$R_n = \lim_{t\to\infty} R_n(t) ; \ n \geq 0$$

Assume that

$V_0(0) = 1, V_n(0) = 0$ and $P_n^i(0) = 0$ for $n \geq 0$ and i=1,2,…,m

and for i=0,1,…,m

$B_i(0), B_i(\infty) = 1$

Also V(x) and $B_i(x)$ are continuous at $x = 0$.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{(0)}(x,t) + \frac{\partial}{\partial t}P_n^{(0)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_n^{(0)}(x,t) = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{(0)}(x,t)\,, n \geq 1 \quad (3.1)$$

$$\frac{\partial}{\partial x}P_0^{(0)}(x,t) + \frac{\partial}{\partial t}P_n^{(0)}(x,t) + \big(\lambda + \mu_1(x) + \alpha\big)P_0^{(0)}(x,t) = 0 \quad (3.2)$$

$$\frac{\partial}{\partial x}P_n^{(k)}(x,t) + \frac{\partial}{\partial t}P_n^{(k)}(x,t) + \big(\lambda + \mu_2(x) + \alpha\big)P_n^{(k)}(x,t) = \lambda \sum_{k=1}^{\infty} C_k P_{n-k}^{(k)}(x,t)\,; \quad n \geq 1, k = 1,2,\dots,m \quad (3.3)$$

$$\frac{\partial}{\partial x}P_0^{(k)}(x,t) + \frac{\partial}{\partial t}P_n^{(k)}(x,t) + \big(\lambda + \mu_2(x) + \alpha\big)P_0^{(k)}(x,t) = 0; k = 1,2,\dots,m \quad (3.4)$$

$$\frac{d}{dt}V_n(t) + (\lambda + \gamma)V_n(t) = \lambda \sum_{k=1}^{\infty} C_k V_{n-k}(t)\,, n \geq 1 \quad (3.5)$$

$$\frac{d}{dt}V_0(t) + (\lambda + \gamma)V_0(t) = \gamma V_0(t) + r_0 \int_0^{\infty} P_0^0(x,t)\mu_1(x)d + q\sum_{k=1}^{m} \int_0^{\infty} P_0^{(k)}(x,t)\mu_k(x)dx \quad (3.6)$$

$$\frac{d}{dt}R_0(t) + (\lambda + \beta)\,R_0(t) = 0 \quad (3.7)$$

$$\frac{d}{dt}R_n(t) + (\lambda + \beta)R_n(t) = \lambda \sum_{k=1}^{\infty} C_k R_{n-k}(t) + \alpha \int_0^{\infty} P_{n-1}^{(0)}(x,t)dx + \alpha \sum_{k=1}^{m} \int_0^{\infty} P_{n-1}^{(k)}(x,t)dx\,; \ n \geq 1 \quad (3.8)$$

Equations are to be solved subject to the following boundary conditions:

$$P_n^{(0)}(0) = \gamma V_{n+1}(t) + \beta R_{n+1}(t) + r_0 \int_0^{\infty} P_{n+1}^{(0)}(x,t)\mu_0(x)dx +$$

$q\sum_{k=1}^{m}\int_0^\infty P_{n+1}^{(k)}(x,t)\mu_k(x)dx\ +$
$(1-q)\sum_{k=1}^{m}\int_0^\infty P_n^{(k)}(x,t)\mu_k(x)dx\ ;\ n\geq 1$ (3.9)

$P_n^{(k)}(0)=r_k\int_0^\infty P_n^{(0)}(x)\,\mu_0(x)dx\ ,\ n\geq 0$ (3.10)

4. Time Dependent Solution

Generating functions of the queue length

Now we define the probability generating function as follows
$P^{(0)}(x,z,t)=\sum_0^\infty P_n^{(0)}(x,t)z^n;\ P^{(0)}(z,t)=\sum_0^\infty P_n^{(0)}(t)z^n\ ,|z|\leq 1\ ,x>0$

$P^{(k)}(x,z,t)=\sum_0^\infty P_n^{(k)}(x,t)z^n;$
$P^{(k)}(z,t)=\sum_0^\infty P_n^{(k)}(t)z^n\ ,|z|\leq 1\ ,x>0$

$V(z,t)=\sum_0^\infty z^nV_n(t)\ ;\ R(z,t)=\sum_0^\infty z^nR_n(t)\ ;\ C(z)=$ *0∞Cnzn, |z|≤1* (4.1)

Taking Laplace transforms of equations (3.1) to (3.11)

$\frac{\partial}{\partial x}\bar{P}_n^{(0)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\bar{P}_n^{(0)}(x,s)=\lambda\sum_{k=1}^{\infty}C_k\bar{P}_{n-k}^{(0)}(x,s)\ ,n\geq 1$ (4.2)

$\frac{\partial}{\partial x}\bar{P}_0^{(0)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\bar{P}_0^{(0)}(x,s)=0$ (4.3)

$\frac{\partial}{\partial x}\bar{P}_n^{(k)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\bar{P}_n^{(k)}(x,s)=\lambda\sum_{k=1}^{\infty}C_k\bar{P}_{n-k}^{(2)}(x,s)\ ;\ n\geq 1,k=1,2,\dots,m$ (4.4)

$\frac{\partial}{\partial x}\bar{P}_0^{(k)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\bar{P}_0^{(k)}(x,s)=0;k=1,2,\dots,m$ (4.5)

$(s+\lambda+\gamma)\bar{V}_0(s)=1+r_0\int_0^\infty\bar{P}_0^{(0)}(x,s)\mu_0(x)dx+q\sum_{k=1}^{m}\int_0^\infty\bar{P}_0^{(k)}(x,s)\mu_k(x)dx+\gamma\bar{V}_0(s)$ (4.6)
$(s+\lambda+\gamma)\bar{V}_n(s)=\lambda\sum_{k=1}^{\infty}C_k\bar{V}_{n-1}(s);\ n\geq 1$ (4.7)

$(s+\lambda+\beta)\bar{R}_0(s)=0$ (4.8)

$(s+\lambda+\beta)\bar{R}_n(s)=\lambda\bar{R}_{n-1}(s)+\alpha\int_0^\infty\bar{P}_{n-1}^{(1)}(x,s)dx+\alpha\sum_{k=1}^{m}\int_0^\infty\bar{P}_{n-1}^{(k)}(x,s)dx\ ;\ n\geq 1$
(4.9)
$\bar{P}_n^{(0)}(o,s)=$
$r_0\int_0^\infty\bar{P}_{n+1}^{(0)}(x,s)\mu_0(x)dx+q\sum_{k=1}^{m}\int_0^\infty\bar{P}_{n+1}^{(k)}(x,s)\mu_k(x)dx$
$+(1-q)\sum_{k=1}^{m}\int_0^\infty\bar{P}_n^{(k)}(x,s)\mu_k(x)dx+\gamma\bar{V}_{n+1}(s)+$
$\beta\bar{R}_{n+1}(s);\ n\geq 1$ (4.10)
$\bar{P}_0^{(0)}(o,s)=$
$r_0\int_0^\infty\bar{P}_1^{(0)}(x,s)\mu_0(x)dx+q\sum_{k=1}^{m}\int_0^\infty\bar{P}_1^{(k)}(x,s)\mu_k(x)dx$
$+(1-q)\sum_{k=1}^{m}\int_0^\infty\bar{P}_0^{(k)}(x,s)\mu_k(x)dx+\gamma\bar{V}_1(s)+\beta\bar{R}_1(s)$
(4.11)
$\bar{P}_n^{(k)}(o,s)=r_k\int_0^\infty\bar{P}_n^{(0)}(x,s)\mu_0(x)dx\ ,\ n\geq 0$ (4.12)
We multiply both sides of equations (4.2) and (4.3) by suitable powers of z, sum over n and use (4.1) and simplify. We thus have after algebraic simplifications

$\frac{\partial}{\partial x}\bar{P}^{(0)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_0(x)+\alpha]\bar{P}^{(0)}(x,z,s)=0$ (4.13)
Performing similar operations on equations (4.4) and (4.5) and using (4.1), we have
$\frac{\partial}{\partial x}\bar{P}^{(k)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_k(x)+\alpha]\bar{P}^{(k)}(x,z,s)=0$ (4.14)
Similar operations on equations (4.6),(4.7),(4.8) and (4.9) yields
$[s+\lambda-\lambda C(z)+\gamma]\bar{V}(z,s)=$
$1+r_0\int_0^\infty\bar{P}_0^{(0)}(x,s)\mu_0(x)dx+$
$q\sum_{k=1}^{m}\int_0^\infty\bar{P}_{n+1}^{(k)}(x,s)\mu_k(x)dx+\gamma\bar{V}_0(s)$ (4.15)
$[s+\lambda-\lambda C(z)+\beta]\bar{R}(z,s)=\alpha z\int_0^\infty\bar{P}^{(0)}(x,z,s)dx+$
αzk=1m0∞Pkx,z,sdx (4.16)
Now, We multiply both sides of equation (4.11) by z, multiply both sides of equation (4.10) by z^{n+1}, sum over n from 1 to ∞ , add the two results and use (4.1)&(4.6).Thus we obtain after mathematical adjustments
$z\,\bar{P}^{(1)}(0,z,s)=(q+pz)\sum_{k=1}^{m}\int_0^\infty\bar{P}^{(k)}(x,z,s)\mu_k(x)dx-$
$q\sum_{k=1}^{m}\int_0^\infty\bar{P}_0^{(k)}(x,s)\mu_k(x)dx$
$+\int_0^\infty\bar{P}^{(0)}(x,z,s)\mu_0(x)dx-\int_0^\infty\bar{P}_0^{(0)}(x,s)\mu_0(x)dx+$
*γ*Vz,s+β*Rz,s* (4.17)

$\bar{P}^{(k)}(0,z,s)=r_k\int_0^\infty\bar{P}^{(0)}(x,z,s)\mu_0(x)dx$ (4.18)
Using (4.15) in (4.17), we get
$z\,\bar{P}^{(1)}(0,z,s)=(q+pz)\sum_{k=1}^{m}\int_0^\infty\bar{P}^{(k)}(x,z,s)\mu_k(x)dx+\int_0^\infty\bar{P}^{(0)}(x,z,s)\mu_0(x)dx+1$
$-[s+\lambda-\lambda C(z)]\bar{V}(z,s)+\beta\bar{R}(z,s)$ (4.19)
Integrating equations (4.2), (4.3) and (4.4) between 0 and x, we get
$\bar{P}^{(0)}(x,z,s)=\bar{P}^{(0)}(0,z,s)\,e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x\mu_{0(t)dt}}$ (4.20)
$\bar{P}^{(k)}(x,z,s)=\bar{P}^{(k)}(0,z,s)\,e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x\mu_{k(t)dt}}$ (4.21)
Again integrating equation (4.10) w.r.to x, we have
$\bar{P}^{(0)}(z,s)=\bar{P}^{(0)}(0,z,s)\left[\frac{1-\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right]$ (4.22)
where $\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}d\bar{B}_0(x)$ (4.23)
is the Laplace transform of first essential service time.
Now from equation (4.10) after some simplification and using equation (1.1) , we obtain
$\int_0^\infty\bar{P}^{(k)}(x,z,s)\mu_k(x)dx=\bar{P}^{(k)}(0,z,s)\bar{B}_k\,(s+\lambda-\lambda C(z)+$ *α;k=1,2,…m* (4.24)
Again integrating equation (4.11) w.r.to x, we have
$\bar{P}^{(k)}(z,s)=\bar{P}^{(k)}(0,z,s)\left[\frac{1-\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right];k=1,2,\dots m$ (4.25)
where $\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}d\bar{B}_k(x)\ ;k=1,2,\dots m$ (4.26)
is the Laplace transform of second optional service time.
Now from equation (4.11) after some simplification and using equation (1.1) , we obtain
$\int_0^\infty\bar{P}^{(k)}(x,z,s)\mu_k(x)dx=\bar{P}^{(k)}(0,z,s)\bar{B}_k\,(s+\lambda-\lambda C(z)+$ *α;k=1,2,…m* (4.27)
Using (4.24)& (4.27) in (4.16) we get,
$\bar{R}(z,s)=$

$\alpha z\bar{P}^{(0)}(0,z,s)\frac{[1-\sum_{k=1}^{m} r_k \bar{B}_k(s+\lambda-\lambda C(z)+\alpha)\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)-r_0\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)]}{(s+\lambda-\lambda C(z)+\alpha)(s+\lambda-\lambda C(z)+\beta)}$ (4.28)

Now using equations (4.18) (4.21), (4.23),(4.24),(4.26) and (4.27) in equation (4.19) and solving for $\bar{P}^{(1)}(0,z)$ we get

$\bar{P}^{(0)}(0,z,s) = \frac{f_1(z)f_2(z)[1-(s+\lambda-\lambda C(z))\bar{V}(z,s)]}{DR}$ (4.29)

Where

$DR = f_1(z)f_2(z)\{z-(q+pz)\sum_{k=1}^{m} r_k\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)\bar{B}_0(s+\lambda-\lambda C(z)+\alpha) - r_0\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)\} - \alpha\beta z[1-\sum_{k=1}^{m} r_k\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)\bar{B}_0(s+\lambda-\lambda C(z)+\alpha) - r_0\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)]$ *(4.30)*

$f_1(z) = s+\lambda-\lambda C(z)+\alpha$ and $f_2(z) = s+\lambda-\lambda C(z)+\beta$

Substituting the value of $P^{(1)}(0,z)$ from equation (4.22) into equations (4.13), (4.16) & (4.18) we get

$\bar{P}^{(0)}(z,s) = \frac{f_2(z)[1-\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)]}{DR}[1-(s+\lambda-\lambda C(z))\bar{V}(z,s)]$ (4.31)

$\bar{P}^{(k)}(z,s) = \frac{f_2(z)r_0\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)[1-\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)]}{DR}[1-(s+\lambda-\lambda C(z))\bar{V}(z,s)];$
$k = 1,2,\ldots,m$ (4.32)

$R(z,s) = [1-(s+\lambda-\lambda C(z))\bar{V}(z,s)]$
$\frac{\alpha z[1-\sum_{k=1}^{m} r_k\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)-r_0\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)]}{DR}$

(4.33)

In this section we shall derive the steady state probability distribution for our Queuing model. To define the steady state probabilities, suppress the arguments where ever it appears in the time dependent analysis. By using well known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

$P^{(0)}(z) = \frac{f_2(z)[1-B_0(\lambda-\lambda C(z)+\alpha)]}{DR}\lambda(C(z)-1)V(z)$ (4.34)

$P^{(k)}(z) = \frac{f_2(z)r_k B_0(\lambda-\lambda C(z)+\alpha)[1-B_k(\lambda-\lambda C(z)+\alpha)]}{DR}\lambda(C(z)-1)V(z); k = 1,2,\ldots,m$ (4.35)

$R(z) = \frac{\alpha z[1-\sum_{k=1}^{m} r_k\bar{B}_k(s+\lambda-\lambda C(z)+\alpha)\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)-r_0\bar{B}_0(s+\lambda-\lambda C(z)+\alpha)]}{DR}\lambda(C(z)-1)V(z)$ (4.36)

In order to determine $P^{(1)}(z), P^{(2)}(z)$, R(z) completely, we have yet to determine the unknown $V(z)$ which appears in the numerator of the right sides of equations (4.34), (4.35) and (4.36). For that purpose, we shall use the normalizing condition.

$P^{(1)}(1) + P^{(2)}(1) + V(1) + R(1) = 1$ (4.37)

$P^{(0)}(1) = \frac{\lambda\beta C'(1)(1-B_0(\alpha))}{dr}V(1)$ (4.38)

$P^{(k)}(1) = \frac{\lambda\beta C'(1)r_k B_0(\alpha)(1-B_k(\alpha))}{dr}V(1); k = 1,2,\ldots,m$ (4.39)

$R(1) = \frac{\lambda\alpha C'(1)[1-\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha)-r_0B_0(\alpha)]}{dr}V(1)$ (4.40)

where

$$dr = \alpha\beta\left[(1-p)\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha) + r_0B_0(\alpha)\right] - \left[1-\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha) - r_0B_0(\alpha)\right]\lambda C'(1)(\alpha+\beta)$$

$P^{(0)}(1), P^{(k)}(1)$ *and* $R(1)$ denote the steady state probabilities that the server is providing first essential service, second optional service and server under repair without regard to the number of customers in the queue. Now using equations (4.38), (4.39), (4.40) into the normalizing condition (4.37) and simplifying, we obtain

$V(1) = 1 - \frac{[1-\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha)-r_0B_0(\alpha)]\lambda C'(1)(\alpha+\beta)}{\alpha\beta[(1-p)\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha)+r_0B_0(\alpha)]}$ (4.41)

and hence, the utilization factor ρ of the system is given by

$\rho = \frac{\lambda C'(1)(\alpha+\beta)[1-\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha)-r_0B_0(\alpha)]}{\alpha\beta[(1-p)\sum_{k=1}^{m} r_k B_k(\alpha)B_0(\alpha)+r_0B_0(\alpha)]}$ (4.42)

where $\rho < 1$ is the stability condition under which the steady states exits.

5. The Mean queue size and the mean system size

Let $P_q(z)$ denote the probability generating function of the queue size irrespective of the server state. Then adding equation (4.27), (4.28) and (4.29) we obtain

$$P_q(z) = P^{(0)}(z) + \sum_{k=1}^{m} P^{(k)}(z) + R(z)$$

$$P_q(z) = \frac{N(z)}{D(z)} \quad (5.1)$$

$D(z) = f_1(z)f_2(z)\{z - (q + pz)\sum_{k=1}^{m} r_k \bar{B}_k(\lambda - \lambda C(z) + \alpha)\bar{B}_0(\lambda - \lambda C(z) + \alpha) - r_0\bar{B}_0(\lambda - \lambda C(z) + \alpha)\} - \alpha\beta z[1 - \sum_{k=1}^{m} r_k \bar{B}_k(\lambda - \lambda C(z) + \alpha)\bar{B}_0(\lambda - \lambda C(z) + \alpha) - r_0\bar{B}_0(\lambda - \lambda C(z) + \alpha)]$

$N(z) = (\lambda C(z) - 1)[1 - \sum_{k=1}^{m} r_k \bar{B}_k(\lambda - \lambda C(z) + \alpha B0\lambda-\lambda Cz+\alpha-r0B0\lambda-\lambda Cz+\alpha$

$$(\alpha z + f_2(z))V(z)$$

Let L_q denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz}[P_q(z)] \text{ at } z = 1$$

$$L_q = \lim_{z\to 1}\frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} \quad (5.2)$$

where primes and double primes in (4.36) denote first and second derivative at z = 1, respectively. Carrying out the derivative at z = 1 we have

$N'(1) =$

$$\lambda C'(1)(\alpha + \beta)V(1)[1 - \sum_{k=1}^{m} r_k \bar{B}_k(\alpha)\bar{B}_0(\alpha) - r_0\bar{B}_0(\alpha)] \quad (5.3)$$

$$N''(1) = \left[1 - \sum_{k=1}^{m} r_k \bar{B}_k(\alpha)\bar{B}_0(\alpha) - r_0\bar{B}_0(\alpha)\right]\{\lambda C''(1)(\alpha + \beta)V(1) - 2\left(\lambda C'(1)\right)^2 V(1) + 2\lambda C'(1)\alpha V(1)$$

$$+2\lambda C'(1)(\alpha + \beta)V'(1)\} - 2\lambda^2\left(C'(1)\right)^2(\alpha + \beta V(1)[k=1mrkB0\alpha Bk'\alpha+Bk\alpha B0'\alpha+r0B0'\alpha]$$

(5.4)

$$D'(1) = \alpha\beta\left[(1-p)\sum_{k=1}^{m} r_k \bar{B}_k(\alpha)\bar{B}_0(\alpha) + r_0\bar{B}_0(\alpha)\right] - \left(1 - \sum_{k=1}^{m} r_k \bar{B}_k(\alpha)\bar{B}_0(\alpha) - r_0\bar{B}_0(\alpha)\right)[(\alpha + \beta)\lambda C'(1)] \quad (5.5)$$

$$D''(1) = 2\alpha\beta(p-1)\left[\sum_{k=1}^{m} r_k\left(\bar{B}_0(\alpha)\bar{B}_k'(\alpha) + Bk\alpha B0'\alpha-r0B0'\alpha\right.\right.$$

$$-(\alpha + \beta)\lambda C''(1)\left(1 - \sum_{k=1}^{m} r_k \bar{B}_k(\alpha)\bar{B}_0(\alpha) - r_0\bar{B}_0(\alpha)\right)$$

$$-2(\alpha + \beta)\lambda C'(1)\left[1 - p\sum_{k=1}^{m} r_k \bar{B}_k(\alpha)\bar{B}_0(\alpha) + \lambda C'1k=1mrkB0\alpha Bk'\alpha+Bk\alpha B0'\alpha+\lambda C'1r0B0'\alpha \right. \quad (5.6)$$

Then if we substitute the values from (5.3), (5.4), (5.5) and (5.6) into (5.2) we obtain L_q in the closed form. Further we find the mean system size L using Little's formula. Thus we have

$$L = Lq + \rho$$

where L_q has been found by equation (5.2) and ρ is obtained from equation (4.35).

6. Conclusion

In this paper we have studied a batch arrival feedback queue with Second m-optional Service, multiple vacation, breakdown and repair. The probability generating function of the number of customers in the queue is found using the supplementary variable technique. This model can be utilized in large scale manufacturing industries and communication networks.

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