Table 5: Simulation of price and delta

	1st Euler scheme		2nd Euler scheme	
	Price	Delta	Price	Delta
MC FX values at time 0	0.123	0.611	0.120	0.612
1st order Black-Scholes approx.	0.126	0.634	0.126	0.634
MC relative deviations	3.1%	3.6%	4.8%	3.5%
3rd order Black-Scholes approx.	0.129	0.628	0.129	0.628
MC relative deviations	5.0%	2.6%	6.8%	2.5%

The option prices from the 1st order Black-Scholes approximation are closer to the MC values than those from the 3rd order Black-Scholes approximation. The opposite is true for the delta values. Moreover, the first Euler scheme yields closer approximations of the option prices to the MC values than the second one. But, no significant difference between the Euler schemes is observed concerning delta.

7. Conclusions and Outlook

Although a quite general LIBOR market model has been presented, a lot of issues remain open for investigation. Detailed implementation issues, further applications and careful testing are left to the practitioner. In the following, some more theoretical concerns are raised.

Clearly, the Gaussian setting of the general LIBOR market model can be questioned and there is an increased demand to go beyond it. However, the construction of multivariate non-Gaussian processes for financial purposes is a complex topic. For market consistent valuation one needs to deflate these processes with state price deflators. A possible approach to circumvent complexity can be based on normal variance-mean mixtures, as presented in author [30]. Here, we would like to emphasize modeling within the framework of copulas that are increasingly used in all areas of mathematical sciences (e.g. Cherubini et al. [8] for financial applications). Two main issues are discussed.

Compatibility of Correlation Matrices

The inaccurate approximation $\hat{C}$ of Example 2.2 suffers from another drawback within the copula setting. It is a simple example of a positive semi-definite correlation matrix which is not the rank correlation matrix of a multivariate normal copula. In fact, general compatibility conditions under which a joint normal distribution with a specified rank correlation matrix can be realized have only been derived quite recently in author [26], Corollary 4.2. One can ask for non-normal trivariate distributions for which $\hat{C}$ is compatible. From [26], Section 2, it is known that $\hat{C}$ is not compatible with a Bernoulli mixture trivariate reduction model. Note that the compatibility of three bivariate distributions in Fréchet spaces has been studied since Dall'Aglio [11], [12] (see also Rüschendorf [56], [57], Joe [33], Section 3, and Durante et al. [15]). The compatibility problem leads to our next issue.

Existence and Construction of Universal Copulas

In general, one says that a n-dimensional copula is n-universal if every n-dimensional valid correlation matrix can be realized as a rank correlation matrix, i.e. there exists a n-variate uniform distribution with this rank correlation structure. Clearly, there exist quite a lot of 2-universal copulas, the so-called comprehensive or inclusive copulas (see e.g. Nelsen [43]). Although the existence of 3-universal copulas has been settled by several authors (e.g. Joe [33], Exercise 4.17, pp. 137-138, Kurowicka and Cooke [38], Section 4.4.6, p.102, Devroye and Letac [13]), the effective construction of 3-universal copulas is more difficult. The author obtains in [31] an analytical 3-universal copula that is based on the bivariate linear circular copula in Perlman and Wellner [46]. The latter copula seems to have been independently obtained by Kurowicka et al. [39] under the naming „elliptical copula". According to Letac [40], the linear circular copula is a special case of probability distributions studied by Gasper [19]. It is remarkable that this copula can be used to settle the existence question for rank two extremal correlation matrices. By Theorem 7.1 below it generalizes to the copula framework the two-factor LMM of Rebonato [53] (see the Remarks 3.1). The construction, which mimics essentially Letac [40], is elementary.

Let $B_2 \subset R^2$ be the unit disk and $C_2 = [-1,1]^2$ the centred square. Consider the linear circular copula density with uniform [-1,1] margins U, V defined by

$$p_{(U,V)}(u,v) = \begin{cases} \dfrac{1}{2\pi\sqrt{1-u^2-v^2}}, & (u,v) \in B_2, \\ 0, & (u,v) \in C_2 - B_2. \end{cases} \qquad (7.1)$$

Lemma 7.1 (*n-universal rank two extreme linear circular copula*) Given is the extreme correlation matrix of rank two of the form

$$r = (r_{ij}) = (\cos(\alpha_i - \alpha_j)), \quad \alpha_i \in [0,2\pi], 1 \le i, j \le n.$$

Then, there exist a random vector $(X_1, X_2, ..., X_n)$ with uniform [-1,1] margins $X_i, i = 1,...,n$, and rank two correlation matrix $r = (r_{ij})$.

Proof. Consider the random vector $(X_1, X_2, ..., X_n)$ defined by

$$X_i = \cos(\alpha_i) \cdot U + \sin(\alpha_i) \cdot V, \quad i = 1,...,n,$$

where the random pair (U, V) has the linear circular copula density (7.1). Clearly, the variables $X_i, i = 1,...,n$, are uniform [-1,1] random variables. Moreover, through application of the Jacobian transformation method, one sees that the probability density of $(X_i, X_j), 1 \le i \le j \le n$, is given by

$$p_{(X_i,X_j)}(x,y)=\begin{cases}\dfrac{1}{2\pi\sqrt{(1-r_{ij}^2)(1-x^2)-(y-r_{ij}x)^2}}, & (x,y)\in E_{r_{ij}},\\ 0, & (u,v)\in C_2-E_{r_{ij}},\end{cases} \quad (7.2)$$

where the support $E_{r_{ij}}=\{(x,y)\mid x^2+y^2-2r_{ij}xy<1-r_{ij}^2\}$ is the inner of an ellipse, and $r_{ij}=\cos(\alpha_i-\alpha_j)$ coincides with the correlation coefficient of the pair (X_i,X_j) (e.g. Kurowicka et al. [39], Perlman and Wellner [46], author [31], Section 3). ◊

Theorem 7.1 (*n-universal rank two copula*) Given a rank two correlation matrix $r=(r_{ij}), 1\le i,j\le n$, there exist a random vector $(X_1,X_2,...,X_n)$ with uniform [-1,1] margins $X_i, i=1,...,n$, and rank two correlation matrix $r=(r_{ij})$.

Proof. This follows through application of the theorem of Carathéodory [7] and Steinitz [61]. Any valid correlation matrix (of rank two) is a finite convex combination of extreme correlation matrices (of rank two). Since the result holds for the extreme correlation matrices of rank two by Lemma 7.1, the result follows. ◊

Besides the existence of a single currency two-factor LMM of arbitrary dimension with arbitrary margins, Theorem 7.1 also settles the existence question for n-universal copulas, $n=3,4,5$. This follows because correlation matrices of dimensions $n=3,4,5$ have maximum rank two (e.g. Ycart (1985), Theorem 2).

Clearly, for simulation and analytical purposes, there is a need for alternative more explicit results. For the trivariate case, one can use the analytical results in [31], Theorem 5.3 and Theorem 6.1. Finally, a computational approach to (almost) n-universal copulas, which is based on doubly stochastic matrices and the checkerboard copula, has been proposed in the Sections 4 of Piantadosi et al. [47], [48]. Future researchers should appreciate the challenge to extend classical LIBOR market models to a copula setting.

References

[1] H. Abdi, L.J. Williams, "Principal component analysis," WIREs Computational Statistics 2(4), pp. 433-459, 2010.
[2] C. Alexander, D. Lvov, "Statistical properties of LIBOR forward rates," ISMA Discussion Papers, 2003. [Online]. Available: http://www.icmacentre.ac.uk/pdf/discussion/DP2003-03.pdf. [Accessed: Nov. 11, 2014].
[3] L. Andersen, J. Andreasen, "Factor dependence of Bermudan swaptions: fact or fiction?" Journal of Financial Economics 62(1), pp. 3-37, 2001.
[4] F. Black, "The pricing of commodity contracts," J. of Financial Economics 3, pp. 167-179, 1976.
[5] D. Brigo, "A note on correlation and rank reduction," 2002. [Online]. Available: www.damianobrigo.it/correl.pdf. [Accessed: Nov. 11, 2014].
[6] M. Budden, P. Hadavas, L. Hoffman, "On the generation of correlation matrices," Applied Mathematics E-Notes 8, pp. 279-282, 2008.
[7] C. Carathéodory, "Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen," Rend. del Circolo Matem. Palermo 32, pp. 193-217, 1911.
[8] U. Cherubini, E. Luciano, W. Vecchiato, Copula Methods in Finance, J. Wiley Finance Series, New York, 2004.
[9] M. Chesney, L. Scott, "Pricing European currency options: a comparison of the modified Black-Scholes model and a random variance model," J. Financial and Quantitative Analysis 24, pp. 267-284, 1989.
[10] Q. Dai, K. Singleton "Term structure dynamics in theory and reality," Review of Financial Studies 16(3), pp. 631-678, 2003.
[11] G. Dall'Aglio "Sulla compatibiltà delle funzioni di repartizione doppia," Rend. Mat. 18, pp. 385-413, 1959.
[12] G. Dall'Aglio, "Fréchet classes and compatibility of distribution functions," In: Symposia Mathematica, vol. IX, pp. 131-150, 1972, Academic Press, London.
[13] L. Devroye, G. Letac, "Copulas in three dimensions with prescribed correlations," 2010. [Online]. Available: http://arxiv.org/abs/1004.3146v1 [math.ST]. [Accessed: Nov. 11, 2014].
[14] J. Driessen, P. Klaassen, B. Melenberg, "The performance of multi-factor term stucture models for pricing and hedging caps and swaptions," J. of Financial and Quantitative Analysis 38(3), pp. 635-672, 2003.
[15] F. Durante, E.P. Klement, J.J. Quesada-Molina, "Copulas: compatibility and Fréchet classes," Preprint, 2007. [Online]. Available: arXiv:0711.2409v1 [math.ST]. [Accessed: Nov. 11, 2014].
[16] R. Fan, A. Gupta, P. Ritchken, "Hedging in the possible presence of unspanned stochastic volatility: evidence from swaption markets," J. of Finance 58(5), pp. 2219-2248, 2003.
[17] Ch. Fries, Mathematical Finance - Theory, Modeling and Implementation, J. Wiley, New York, 2007.
[18] B. Flury, Common principal components and related multivariate models, J. Wiley, 1988.
[19] G. Gasper, "Banach algebra for Jacobi series and positivity of a kernel," Annals of Mathematics 95, pp. 261-280, 1971.
[20] P. Glasserman, X. Zhao, "Fast Greeks by simulation in forward Libor models,"Journal of Computational Finance 3, pp. 5–39, 1999.
[21] I. Grubisic, R. Pietersz, "Efficient rank reduction of correlation matrices," Linear Algebra and its Applications 422, pp. 629-653, 2007.
[22] A. Gupta, M. Subrahmanyam, "Pricing and hedging interest rate options: evidence from cap-floor markets," J. of Banking and Finance 29(3), pp. 701-733, 2005.
[23] S.L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and

currency options," Review of Financial Studies 6, pp. 327-343, 1993.

[24] N.J. Higham, "Computing the nearest correlation matrix – a problem from finance," IMA Journal of Numerical Analysis 22, pp. 329-344, 2002.

[25] W. Hürlimann, "An extension of the Black-Scholes and Margrabe formulas to a multiple risk economy," Applied Mathematics 2(4), pp. 137-142, 2011.

[26] W. Hürlimann, "Compatibility conditions for the multivariate normal copula with given rank correlation matrix," Pioneer Journal of Theoretical and Applied Statistics 3(2) (2012), pp. 71-86, 2012.

[27] W. Hürlimann, "Option pricing in the multidimensional Black-Scholes market with Vasicek interest rates. Math. Finance Lett. 2(1) (2013), pp. 1-18, 2013.

[28] W. Hürlimann, "Option pricing in the multidimensional Black-Scholes-Merton market with Gaussian Heath-Jarrow-Morton interest rates: the parsimonious and consistent Hull-White models of Vasicek and Nelson-Siegel type," Math. Finance Lett., Vol. 2014, Article ID 3, 2014.

[29] W. Hürlimann, "Cartesian and polar coodinates for the n-dimensional elliptope," Theoretical Mathematics & Applications 4(3), 1-17, 2014.

[30] W. Hürlimann, "Normal variance-mean mixtures (III) Option pricing through state-price deflators," J. of Mathematical Sciences: Advances and Applications 25(1), pp. 13-41, 2014.

[31] W. Hürlimann, "A closed-form universal trivariate pair-copula," Journal of Statistical Distributions and Applications 1(7), 25p., 2014.

[32] P. Jäckel, C. Kahl, "Positive semi-definite correlation marix completion," 2009. [Online]. Available: http://www.pjaeckel.webspace.virginmedia.com/PositiveSemidefiniteCorrelationMatrixCompletion.pdf. [Accessed: Nov. 11, 2014].

[33] H. Joe, Multivariate models and dependence concepts, Monographs on Statistics and Applied Probability, vol. 73, Chapman & Hall, London, 1997.

[34] I.T. Joliffe, Principal Component Analysis (2nd ed.), Springer-Verlag, New York, 2002.

[35] C. Kahl, "Modeling and simulation of stochastic volatility in finance," PhD Thesis, Bergische Universität Wuppertal and ABN Amro, 2007.

[36] C. Kahl, M. Günther "Complete the correlation matrix," 2005. [Online]. Available: http://www2.math.uni-wuppertal.de/~kahl/publications/CompleteTheCorrelationMatrix.pdf. [Accessed: Nov. 11, 2014].

[37] A. Kawai, P. Jäckel, "An asymptotic FX option formula in the cross currency LIBOR market model," Willmott, March 2007, 74-84, March, 2007.

[38] D. Kurowicka, M.R. Cooke, Uncertainty analysis with high dimensional dependence modelling, J. Wiley, Chichester, England, 2006.

[39] D. Kurowicka, D., M.R.Cooke, J. Misiewicz, "Elliptical copulae," In: GI Schuëller & PD Spanos (Eds.), Proceedings Monte Carlo Simulation, pp. 209-214, Lisse: Balkema, 2000.

[40] G. Letac, "The convex set of correlation matrices and Jacobi polynomials," Conference Presentation, Orthogonal Polynomials and Hypergroups, Toulouse, June 18, 2014.

[41] M. Lim, "Calibrating a 3-factor LIBOR market model," Ecole Polytechnique, 2004. [Online]. Available: http://homepages.nyu.edu/~mrl306/fichiers/rapporte.pdf [Accessed: Nov. 11, 2014].

[42] F. Longstaff, P. Santa-Clara, E. Schwartz, "Throwing away a billion dollars: the cost of suboptimal exercise strategies in the swaptions market," Journal of Financial Economics 62(1), pp. 39-66, 2001.

[43] R.B. Nelsen, An introduction to copulas, Lecture Notes in Statistics, vol. 139 (2nd edition). Springer-Verlag, New York, 2006.

[44] P.J. Olver, "Orthogonal bases and the QR algorithm" University of Minnesota, 2008. [Online]. Available: http://www.math.umn.edu/~olver/aims_/qr.pdf. [Accessed: Nov. 11, 2014].

[45] N. Packham, "Correlation parameterization and calibration for the LIBOR market Model," MsC Thesis, BS Finance & Management, Frankfurt am Main, 2005.

[46] M.D. Perlman, J.A. Wellner, "Squaring the circle and cubing the sphere: circular and spherical copulas," Symmetry 3, pp. 574-599, 2011.

[47] J. Piantadosi, P.G. Howlett, J.W. Boland, "Matching the grade correlation coefficient using a copula with maximum disorder," J. Indust. Man. Opt. 3(2), pp. 305-312, 2007.

[48] J. Piantadosi, P.G. Howlett, J.W. Borwein, "Copulas with maximum entropy," Optim. Lett. 6, pp. 99-125, 2012.

[49] R. Pietersz, P. Groenen, "Rank reduction of correlation matrices by majorization", Quantitative Finance 4(6), pp. 649-662, 2004.

[50] R. Pietersz, A. Pelsser, "A comparison of single Markov-functional and multifactor market models," Review of Derivatives Research 13, pp. 245-272, 2010.

[51] F. Rapisarda, D. Brigo, F. Mercurio, "Parameterizing correlations: a geometric interpretation," IMA Journal of Management Mathematics 18(1), pp. 55-73, 2007.

[52] R. Rebonato, Interest-rate Option Models, J. Wiley, New York, 1996.

[53] R. Rebonato, "On the simultaneous calibration of multifactor lognormal interest rate models to Black volatilities and to the correlation matrix," J. Comp. Finance 2, pp. 5-27, 1999.

[54] R. Rebonato, Volatility and Correlation (2nd ed.), J. Wiley, New York, 2004.

[55] R. Rebonato, P. Jäckel, "The most general methodology to create a valid correlation matrix for risk management and option pricing purposes," Journal of Risk 2(2), pp. 17-27, 2000.

[56] L. Rüschendorf, "Bounds for distribution with multivariate marginal," In: Stochastic Orders and Decision under Risk, IMS Lecture Notes Monograph Series, vol. 19, Institute Math. Statist., Hayward, 1991.

[57] L. Rüschendorf, "Fréchet bounds and their applications," In: G. Dall'Aglio, S. Kotz, G. Salinetti (Eds.). Advances in Probability Distributions with Given Marginals, Kluwer, Dordrecht, 1991.

[58] R. Schöbel, J. Zhu, "Stochastic volatility with an Ornstein Uhlenbeck process: an Extension," European Finance Review 3, pp. 23-46, 1999.

[59] J. Schoenmakers, C. Coffey, "LIBOR rate models, related derivatives and calibration," WIAS Preprint No. 480, Weierstrass Institute Berlin, 1999.
[60] J. Schoenmakers, C. Coffey, "Systematic generation of parametric correlation structures for the LIBOR market model," Int. J. Theor. Applied Finance 6(5), pp. 507-519, 2003.
[61] E. Steinitz, "Bedingt konvergente Reihen und konvexe Systeme," Journal Reine Angew. Math. 143, pp. 128–175, 1914.
[62] L. Tamminen, "Pricing of constant maturity spread options in the deterministic LIBOR market model framework," MsC Thesis, Tampere University of Technology, Finnland, 2008.
[63] B. Ycart, "Extreme points in convex sets of symmetric matrices," Proceedings of the American Mathematical Society 95(4), pp. 607-612, 1985.
[64] G. Yule, M. Kendall, An Introduction to the Theory of Statistics (14th ed.), Charles Griffin & Company, Belmont, California, 1965.
[65] Z. Zhang, L. Wu, "Optimal low-rank approximation to a correlation matrix," Linear Algebra and its Applications 364, pp. 161-187, 2003.

Author Profile

Werner Hürlimann has studied mathematics and physics at ETHZ, where he obtained his PhD in 1980 with a thesis in higher algebra. After postdoctoral fellowships at Yale University and at the Max Planck Institute in Bonn, he became 1984 an actuary at Winterthur Life and Pensions, a senior actuary at Aon Re and IRMG Switzerland 2003-06, a senior consultant, business expert and product manager at IRIS, FRSGlobal Switzerland and Wolters Kluwer Financial Services Switzerland in Zürich between 2006-2014. He has been visiting associate professor in actuarial science at the University of Toronto during the academic year 1988-89. After early retirement in April 2014 he works as an Independent Scholar in various fields of interest, mainly in Algebra and Number Theory, Probability and Statistics.