International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Lower Bounds on Variance of Estimators Past, Present and Future

A. Shanubhogue[1], D. B. Jadhav[2]

[1]Department of Statistics, Sardar Patel University, Vallabh Vidyanagar-388120, Gujrat, India

[2]Department of Statistics, D. P. Bhosale College, Koregaon (Satara)-415501, Maharashtra, India

Abstract: ***The approach to obtain uniformly minimum variance unbiased estimator (UMVUE) through the lower bound on the variance of an unbiased estimator involves two problems: 1. To provide a lower bound and 2. To identify the classes of (a) probability distributions (b) estimators and (c) parametric functions for which a particular bound is attained by the variance of the UMVUE. Many authors have provided various solutions to these problems starting from Fréchet (1943), Cramér(1946) and Rao (1945). In this paper we are trying to provide a brief history of lower bounds on the variance of unbiased estimators. It is tried to take an account of little bit of present and possible future work in the field of non-regular families of distributions. Sometimes different inequalities provide different bounds for the variance of estimators of the same parametric function. In such situations, it is natural to compare the bounds. We have made a brief mention of these efforts also. A few important references are provided.***

Keywords: Cramér- Rao type inequalities, history of variance bounds, information inequalities, lower bounds on variance, minimum variance unbiased estimation.

1. Introduction

The approach through the variance bound, to obtain UMVUE, leads to two important problems. The first, which we call, 'the problem of construction', is to construct a lower bound for the variance of all the unbiased estimators in a certain situation. The second, whom we call, 'the problem of attainment', is to investigate whether the variance bound is attainable, and to find, if possible, an explicit expression for an estimator with minimum variance, in a specified situation. Different lower bounds on the variance are provided by different inequalities. Therefore, the lower bounds on the variance of the same estimator provided by different inequalities can be different. Therefore, a natural and third problem which we call, 'the problem of comparison', comes up.

We intend to discuss the past, present and future of the solutions to these three problems.

2. The Past

The history of lower bounds on the variance of estimators is long and has many contributors. The widely known bound and the basis of this theory is the so called Cramér-Rao bound (Cramér (1946),Rao(1945)). It is equal to the inverted value of Fisher's information quantity (Fisher (1922), (1925)). The earliest expression involving 'Fisher information' is given by Pearson and Filon(1898) in a different context. Doob (1936) and Dugué (1937)also used Fisher information in their expressions. There intension is not to obtain UMVU estimator or to provide a bound on the variance of the estimator. In fact, what Doob has done is as follows- he considered a class $\{T_n, n\geq 1\}$ of maximum likelihood estimators of θ which follows asymptotic normal distribution and proved that the asymptotic variance of $\sqrt{n}$ $(T_n - \theta)$ is the inverted value of Fisher information quantity. He has also obtained least upper bound on the Fisher information quantity. The actual problem of obtaining minimum variance unbiased estimator is considered by Aitken and Silverstone (1942). Under certain restrictions they proved that there exists a minimum variance unbiased estimator T(X) of θ, if the derivative of the log f(x, θ) can be written as

$$\frac{d}{d\theta} log f(x,\theta) = \frac{[T(x) - \theta]}{\lambda(\theta)} \quad (2.1)$$

where $E[T(X)] = \theta$ and $Var[T(X)] = \lambda(\theta)$. They also showed that $E\{\frac{d}{d\theta} \log f(x, \theta)\} = 0$ (a result already proved by Pearson(1898)).If T(X) satisfies (2.1), they showed that

$$Var(T(X)) = [E\{\frac{-d^2 \log f(x,\theta)}{d\theta^2}\}]^{-1}$$
$$= [E\{\frac{d}{d\theta} \log f(x,\theta)\}^2]^{-1}$$

Thus, it is clear that Aitken and Silverstone have calculated Cramér-Rao bound as the variance of the estimator T(X) when (1.1) holds. But they as well as others mentioned above did not obtain the bounds for the variances.

It seems that Fréchet (1943) has given the inequality which is now known as the Cramér-Rao inequality in the statistical literature, after its explicit and independent publication by Cramér(1946) and Rao(1945).Bhattacharyya(1946) generalized Rao's results, under some additional conditions, to give a sequence of sharper bounds. Darmois(1945) extended Fréchet's inequality to n- dimensions.

Cramér-Rao inequality and Bhattacharyya inequality hold under certain regularity conditions. Therefore, many other authors tried to provide the lower bounds on the variance of estimators, by dropping the regularity assumptions or by giving sharper bounds.

Barankin (1949), with a complementary remark in the (1951) paper, starts with the goal to obtain (locally) attainable variance bounds. His results are very general, but unfortunately difficult to apply. He demonstrates that the Cramér-Rao and Bhattacharyya bounds are special cases of bound obtained by him. The variance bounds "without regularity conditions" appeared around 1950. Hammersley (1950)and Chapman and Robbins (1951) use the same idea to give bounds without regularity conditions use the same idea to give bounds without regularity conditions. Fraser and Guttman (1952) used the idea applied by Chapman and Robbins to give Bhattacharyya bounds without regularity assumptions. In (1952), Kiefer gave a modified form of Barankin's bound, and showed at the same time that it is a generalization of Hammersley-Chapman and Robbins bound. Blischke (1969), Polfeldt (1967),(1970) and many others including Mitra (1954), Vincze (1979), Khatri (1980), Chatterji (1982) etc., have provided some bounds.

Fr$\acute{e}$chet (1943) as well as many others mentioned that Cram$\acute{e}$r-Rao lower bound is attained only if the family of distributions of X is one parameter exponential. In this way, together with providing a lower bound on the variance of unbiased estimators, Fr$\acute{e}$chet has studied its attainment. Fend (1959) gives proof of the above result but his proof is not rigorous. He provided the situation where Bhattacharyya bound is attained by considering a larger family of distributions. His result for the larger family seems to be vacuous. Revealing this Jadhav and Prasad modified Fend's results on attainment of Bhattacharyya bound. Their results are similar to those by Zacks (1971) and are presented in Jadhav's M. Phil. dissertation. A rigorous account on attainment of Cram$\acute{e}$r-Rao lower bound is provided by Wijsman (1973) and Joshi (1976). Sen and Ghosh (1976) provided results about attainment of Chapman-Robbin's bound. The bound constructed by Kiefer (1952) is the most general variance bound including non-regular situations. It is described below:

Let X be a r.v. having p.d.f. f(x,θ) with x $\in \mathfrak{X}$ and $\theta \in \mathbf{\Theta}$. Let ($\mathfrak{X}$, $\mathbb{F}$, μ) be the measure space with a σ -finite measure μ. An estimator T of θ or its function m(θ) is a measurable function T: $\mathfrak{X} \longrightarrow \mathbb{R}$: the real line. Let $\mathbf{\Theta}$ be an interval of $\mathbb{R}$. For each $\theta \in \mathbf{\Theta}$ let $_{\theta} = \{ h; (\theta + h) \in \mathbf{\Theta} \}$. For a fixed θ, let G_1 and G_2 be any two probability measures ∊ G defined on $\mathbf{\Theta}_0$ such that $E_i(h)$ = expectation of h w.r.t. G_i, i= 1,2. Then Kiefer, J.(1952) has proved , for the variance of estimator T of θ, that

$$Var(T(X)) \geq \sup_{G_1, G_2 \in G} K(G_1(h), G_2(h), \theta)$$

$$= K(\theta) \quad (2.2)$$

Where,

$$K(G_1, G_2, \theta) = \frac{\{E_1(h) - E_2(h)\}^2}{\int_{\mathfrak{X}} \left[\frac{\left\{ \int_{\Theta_\theta} f(x;\theta+h) d[G_1(h) - G_2(h)] \right\}}{f(x;\theta)} \right]^2 d\mu(x)}$$

Inequality (2.2) is called Kiefer inequality and its R.H.S. K(θ) is called Kiefer bound.

Minimum variance estimation in non-regular situations is previously treated in a systematic manner by Blischke et al. in a (1969) paper and in their reports of (1965), (1966) and (1968). Polfeldt (1970) considered attainment of Kiefer bound only in the asymptotic situation. Bartlett (1982) investigates attainment of Kiefer bound through ideal estimation equation involving generalized difference for certain probability distributions. Jadhav and Prasad (1986-87) provided a necessary and sufficient condition for the attainment of Kiefer bound for finite sample and extended Kiefer's results for a class of non-regular family of distributions to estimate the parameter and its r^{th} power.

The lower bounds on the variance of an estimator provided by different inequalities may not be the same. This raises a problem of comparison of the various bounds. Though various authors like Bhattacharyya, Chapman and Robbins compared the bounds obtained by them with Cram$\acute{e}$r-Rao lower bound, up to some extent detailed study of relative magnitudes of bounds is done for the first time by Sen and Ghosh(1976). They compared bounds due to Fr$\acute{e}$chet-Cram$\acute{e}$r-Rao, Bhattacharyya, and Chapman and Robbins. Jadhav and Prasad (1986-87) compared Kiefer bound with the bounds due to Fr$\acute{e}$chet-Cram$\acute{e}$r-Rao, Bhattacharyya, and Chapman and Robbins.

In this discussion we restricted ourselves only to one parameter. For the cases of vector parameters, sequential procedures and other related problems one may refer to the extensive (but not exhaustive) list of references given at the end. To describe the present and future of lower bounds on variance, we restrict ourselves to our work on Kiefer bound only.

3. The Present

Jadhav and Shanubhogue(2014) deal with the attainment and comparison of Kiefer bound in truncated families of distributions. Identifying the prior distributions, it is proved that the variances of unbiased estimators based upon the single sufficient statistics attain Kiefer bound in left as well as right truncated families to estimate the parametric functions involved in the densities, their linear functions and the r^{th} powers. They considered these functions of the parameter which is a step ahead of the other research workers. Treating Kiefer bound itself as a parametric function to be estimated, the attainment is investigated. The results for complete, Type 2 and doubly censored samples are also established. Thus, given the truncated distribution admitting single sufficient statistic, we have the expressions for estimable parametric functions, their UMVUE s and their variances attaining Kiefer bound.

4. The Future

The results may be extended to some more parametric functions, their estimators and some more probability distributions for complete as well as censored samples. The values of various bounds would be compared. Some more prior distributions would be identified enhancing the applicability of Kiefer bound. The situations involving two or more parameters might be dealt. Kiefer inequality may be generalized to deal with any estimable function, say, m(θ). One may try similar things for vector parameters, remaining bounds and bounds in sequential and other related situations. The added results might be used to get confidence intervals etc.

References

[1] Aitken, A.C., Silverstone, H.(1942). On the estimation of statistical parameters. Proc.Roy. Soc.Edinburg.Sect.A.61, 186-194.

[2] Barankin, E. W. (1949).Locally best unbiased estimates. Ann. Math. Statist. 20, 477-501.

[3] Barankin, E. W. (1951). Concerning some inequalities in the theory of statistical estimation. Skand. Aktuar, 34,35-40.

[4] Bartlett, M. S.(1982). The ideal estimation equation. Essays In Statistical Science. Papers in honor of P.A.P. Moran. Edited by J. Gani and E. J. Hannan. Applied Probability Trust. Journal of Applied Probability Vol.19A-1982, 187-200.

[5] Bhattachryya, A.(1946-48). On some analogues of the amount of information and their use in statistical estimation. Sankhya.8,1-14,201-218,315-328.

[6] Bhattachryya, A.(1950).Unbiased statistics with minimum variance. Proc. Royal Soc. Edinburgh.63-69.

[7] Blackwell, D., Girshick, M. A.(1947).A lower bound for the variance of some unbiased sequential estimates. Ann. Math. Statist. 18,277-280.

[8] Blight, B.J.N. and Rao,P.V.(1974). The convergence of Bhattacharyya bounds. Biometrika.61,137-142.

[9] Blischke, W.R., Truelove, A.J., Mundle, P.B.(1969). On non-regular estimation I. variance bounds for estimators of location parameters. J. Amer. Statist. Assoc. 64,1056-1072.

[10] Blischke, W.R. (1974). On non-regular estimation II.Estimation of the location parameter of the gamma and Weibull distributions. Comm. Statist.3,1109-1129.

[11] Blyth, C.R. and Roberts, D.M.(1972). On inequalities of Cramér-Rao type and admissibility proofs. Proc. Sixth Berkeley Symp. Math. Statist. Prob.1, 17-30. Univ. of California Press.

[12] Blyth, C.R.(1974). Necessary and sufficient condition for inequalities of Cramér-RaoType. Ann. Statist.2, 464-473.

[13] Compos, A.D.(1979). Extension of Cramér-Rao inequality to sequential case. Trab. Estadist. 30,65-70.

[14] Chapman, D.G. and Robbins,H.(1951). Minimum variance estimation without regularity assumptions. Ann.Math. Statist. 22,581-586.

[15] Darmois, G.(1945). Sur les limites de la dispersion de certaines estimations. Rev. Inst. Statist. 13,9-15.

[16] Fabian,B. and Hanan, J. (1977). On Cramér-Rao inequality. Anls Statist.5,197-205.

[17] Fend, A.V.(1959). On the attainment of Cramér-Rao and Bhattacharyya bounds for the variance of an estimate. Ann. Math. Statist. 30,381-388.

[18] Fraser, D.A.S. and Guttman, I (1952). Bhattacharyya bounds without regularity assumptions. Ann. Math. Statist.23,629-632.

[19] Hammersley, J.M.(1950). On estimating restricted parameters. J. Roy. Statist. Soc. Ser.B.12, 192-240.

[20] Jadhav, D. B. and Prasad, M.S.(1986-87).Some Results On Kiefer Bound. *Journal of Shivaji University*(Science),23,209-214.

[21] Jadhav,D.B. and Shanubhogue,A. (2014). Kiefer Bound in Truncated Distributions. Interantional Journal of Mathematics and Computer Research,2(6), 469-483.

[22] Joshi, V.M.(1976).On the attainment of Cramér-Rao lower bound. Annals. Statist.4,998-1002.

[23] Kiefer,J.(1952). On minimum variance estimators. Ann. Math. Statist.23,627-628.

[24] Pitman,E.J.G.(1978).The Cramér-Rao inequality. Australian J. Statist. 20,60-74.

[25] Polfeldt,T.(1970).Asymptotic results in Non-Regular Estimation.Skand.Akt.1-2 supplement,1-78.

[26] Roy, J. and Mitra, S.K. (1957). Unbiased minimum variance estimation in a class of discrete distributions. Sankhya, 18, 371-378.

[27] Schutzenberger, M.P.(1958). Concerning the Fréchet-Cramér inequality to the case of Bayes' estimation. Bull. Amer. Math. Soc., 63, 142.

[28] Sen, P.K. and Ghosh, B.K.(1976). Comparison of some bounds in estimation theory. Anls. Statist. 4, 755-765.

[29] Tate, R.F. (1959). Unbiased estimation : functions of location and scale parameters. Ann. Math. Statist. 30, 341-366.

[30] Vincze, I. (1979). On Cramér- Fréchet -Rao inequality in the non-regular case. Contributions to statistics. Jaroslav Hajek Memorial Volume edited by Jana Jureckova. Reidel Pub.Co. Dordrecht.

[31] Wijsman,R.A. (1973). On the attainment of Cramér-Rao lower bound. Anls. Statist.,1,538-542.

[32] Zacks, S.(1971). The Theory of Statistical Inference. J. Wiley and sons, Inc.