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Approximation in Linear Stochastic Programming Using L-Shaped Method

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Abstract: ***Approximation algorithm is inveterate solution method in stochastic programming because the problem of stochastic programming is very difficult solved. Therefore, most research in this case is focused for designing the solution method which approximate optimal solution. This research tells a strategy to finish the problem of stochastic linear programming by approximation with L-Shaped method. This thesis gets result of approximation for stochastic model by fulfilling all linear constraint where optimal solution reached depend on selected partition.***

Keywords: Stochastic linear program, Convex approximation, L Shaped Method.

1. Introduction

Stochastic programming models arise as reformulations or extensions of optimization problems with random parameters. To set the stage for this review of approximation in stochastic programming. First introduce the models and give an overview of relevant mathematical properties. Consider the optimization problem.

$$\min_x \{cx\}$$
$$s.t$$
$$Ax = b$$
$$Tx = h$$
$$x \in X$$

Where $X \subseteq R^n$ specifies non negativity of and possibly integrality constraints on the decision variables x. In addition to the m_1 deterministic constraints $Ax = b$, there is a set of m constraints $Tx = h$, whose parameters T and h depend on information which becomes available only after a decision x is made. The stochastic programming approach to such problems is to assume that this uncertainty can be modeled by random variables with known probability distribution, and then to reformulate the model to obtain a meaningful and well-defined optimization problem. In this chapter we will use bold face characters for random variables, and plain face to indicate their realizations.

Birge and Tang [2] had modeled convex stochastic program with recourse. The algorithm had simple implementation that had certain converge. However some application was only effective for convex quadratic programming, not for linear stochastic program.

Shiina [6] had modelled a concentrator location problem in which traffic demand at each terminal location was uncertain. The problem was formulated as stochastic multi-stage integer linear program, with first stage binary variables concerning network design and continuous recourse variables concerning expansion of capacity. The new algorithm was gotten which combined an L-Shaped method and branch-and-bound method and was demonstrated by computational efficiency of multi-stage model.

Liu et.al [5] had modelled transportation network protection by using two-stage programming and approximation with L-Shaped method and benders decomposition. They focused resource protection allocation for improving resilience and dynamic transportation system.

Alvarado and Ntaimo [1] had modelled a stochastic integer programming extended attack response model for large-scale wildfires. This research determined the location and optimal time the source of fire in uncertainty weather by using lower and upper bound methods.

Approximation in stochastic programming of this research assumes that the uncertainty can be modelled by random variable with probability distribution known. This research uses lower-upper bound and L-Shaped method.

2. Bound

Kall and Wallace [4] explained that bounding technique can identify the value interval. The bounding technique is the part of approximation advance using stochastic programming.

2.1 Jensen Lower Bound

To get lower bound, use $\varphi(\boldsymbol{\xi}) = Q(\hat{x}, \boldsymbol{\xi})$ for fixed $\hat{x}$. Figure 1 illustrated that there are two lower bound function which can made from constraint function.

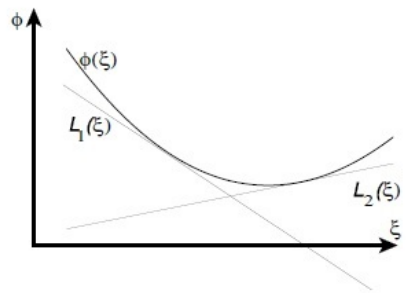


Figure 1: Lower Bound Illustration

Because $\varphi(\boldsymbol{\xi})$ is convex, so lower bound can be given as linear function $L(\boldsymbol{\xi}) = c\boldsymbol{\xi} + d$. Because of unlimited lower

bound candidate, so it must be given condition that the linear lower bound is the bound that has tangensial value for $\varphi(\xi)$ of some points $\hat{\xi}$. Figure 1 shows example of lower bound function. The question is which better one, $L_1(\xi)$ or $L_2(\xi)$? If lower bound function $L(\xi)$ has tangensial for $\varphi(\xi)$ of $\hat{\xi}$, the gradient is $\varphi'(\xi)$, so

$$\varphi(\hat{\xi}) = \varphi'(\hat{\xi})\hat{\xi} + b,$$

Because $\varphi(\hat{\xi}) = L(\hat{\xi})$, so lower bound function becomes:

$$L(\xi) = \varphi(\hat{\xi}) + \varphi'(\hat{\xi})(\xi - \hat{\xi}).$$

Because this is the linear function, so expected value can be calculated:

$$EL(\hat{\xi}) = \varphi(\hat{\xi}) + \varphi'(\hat{\xi})(E\xi - \hat{\xi}) = L(E\xi)$$

By changing $\hat{\xi} = E\xi$, so the best lower bound is gotten.

2.2 Edmundson-Madansky Upper Bound

If $\xi$is random variable, $\Xi = [a, b]$, and define $\Phi(\xi) = Q(\hat{x}, \xi)$ so the linear function can be drawed $U(\xi)$ between two points $(a, \Phi(a))$ and $(b, \Phi(b))$. Clearly the line is the over $\Phi(\xi)$ for all $\xi \epsilon \Xi$.

If the straight line of upper bound $c\xi + d$ there are two points, so:

$$c = \frac{\Phi(b) - \Phi(a)}{b - a},$$

$$d = \frac{b}{b-a}\Phi(a) - \frac{a}{b-a}\Phi(b),$$

$$EU(\hat{\xi}) = \frac{\Phi(b) - \Phi(a)}{b-a}E\xi + \frac{b}{b-a}\Phi(a) - \frac{a}{b-a}\Phi(b),$$

$$= \Phi(a)\frac{b - E\xi}{b-a} + \Phi(b)\frac{E\xi - a}{b-a}$$

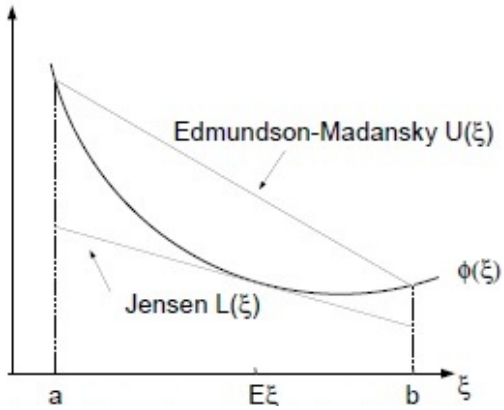


Figure 2: Illustration of bound partition influence

And the other hand, if the convex function of ξ is in the interval $\Xi = [a, b]$, so its distribution can be determined as two points flanked distribution so the upper bound is gotten by important parameter:

$$p = \frac{E\xi - a}{b - a}, so$$

$$P(\xi = a) = 1 - p, and\ P(\xi = b) = p$$

2.3 Kombinasi

Though the distribution of Edmundson –Madansky is useful, but $Q(x, \xi)$ is evaluated for exponential points. That means if there is k random variables, it must check as many as 2^k points that means if there are ten rendom variables, so it is out of the topic. Therefore the sum of random variables must be designed to get the precise upper bound. As example there is recourse function $Q(\hat{x}, \xi)$ convex for ξ that described as a opened bowl so the lower bound is y_p. if $\Phi(\xi)$ is defined by,

$$\Phi(\xi) = min_y\{q^T y | Wy = b + \xi, 0 \le y \le c\}$$

If given $\Xi(\xi) = [A, B]$ and assume $\Xi(\xi) = 0$ so convex function for ξ is:

$$U(\xi) = \varphi(0) + E_i \begin{Bmatrix} d_i^+ & if\ \xi_i \ge E\xi_i - 0 \\ d_i^- & if\ \xi_i < E\xi_i = 0 \end{Bmatrix}$$

To get the best bound, it needs calculate some sets of points and choose the best. First , calculate $\Phi(E\xi) = \Phi(0)$ so:

$$\Phi(0) = min_y\{q^T y | Wy = b, 0 \le y \le c\} = q^T y^0.$$

Then define r:=1 and $\xi_r > 0$ so:

$$min_y\{q^T y | Wy = \xi_r B_r, \alpha^r \le y \le \beta^r\} = q^T y^{r+} = d_r^+ B_r$$

It reminds that d_r^+ represent the cost per unit by increasing right hand side from 0 until $\xi_r B_r$. And the contrary for $\xi_r > 0$:

$$min_y\{q^T y | Wy = \xi_r A_r, \alpha^r \le y \le \beta^r\} = q^T y^{r+} = d_r^- A_r$$

Then the next random variable can be gotten to get the optimal bound:

$$\alpha_i^{r+1} = \alpha_i^r - \min\{y_i^{r+}, y_i^{r-}, 0\}$$

$$\beta_i^{r+1} = \beta_i^r - \max\{y_i^{r+}, y_i^{r-}, 0\}$$

Where β_i^{r-1} shows how many bound I available to positive direction. After advance linear upper bound for constraint, then add slack variable explicitly. Then do,

$$U(\xi_1, \xi_2) = \varphi(0,0) + \begin{Bmatrix} d_1^+ \xi_1 if\ \xi_1 \ge 0 \\ d_1^- \xi_1 if\ \xi_1 < 0 \end{Bmatrix} + \begin{Bmatrix} d_2^+ \xi_2 if\ \xi_2 \ge 0 \\ d_2^- \xi_2 if\ \xi_2 < 0 \end{Bmatrix}$$

3. L-Shaped Method

The L-shaped method is a decomposition method that is useful for solving problems that have the form of a master problem and several subproblems represented by the side model. The complete problem with both the master and subproblems may be very large and beyond the capabilities of the available Excel solver. The L-shaped method is a process that solves a sequence of much smaller problems. The combined solution converges in a finite number of iterations to the optimum. Because the number of required iterations may be large, we can terminate the process when the lower and upper bound values differ by a specified minimum tolerance. We have implemented the L-shaped method in the Jensen LP/IP Solver.

Cerisola et.al[3] explained that Benders or L-Shaped decomposition considered two-stage optimization can be formulated in the following form:

$$min\{cx + qy\}$$

$$Tx + Wy = h$$

$$x \in X, y \epsilon Y$$

Where x represents first-stage decisions and y comprises second-stage variables. And the decomposition algorithm solves in each iteration a relaxed master problem given by:

$$\min(cx + \theta)$$

$$0 \geq \theta^j + \pi^j T(x_0^j - x), j \in J,$$
$$0 \geq \theta^i + \pi^i T(x_0^i - x), i \in I,$$
$$x \in X.$$

Each iteration of the method starts with the solution of master problem and the proposal of a first stage solution, x_0. This first stage solution is then used to evaluate the recourse function by solving the corresponding subprolem. The description of the recourse function in master problem is enhanced with an optimality cut i in case of subproblem feasibility. In the other case, the feasibility region of master problem is constrained with a feasibility cut j. the algorithm will stop when the relative difference is less than an appropriate tolerance.

1. Solve the master problem with π not included. The solution provides the initial upper bound.
2. With the solution z from the master problem, solve the subproblem. If the solution to the master and the optimum solution to the subproblem, provides a better upper bound, replace the bound and the incumbent solution. The solution to the subproblem provides a new optimality cut.
3. Add the optimality cut to the master problem and solve it. The value of the objective provides a better lower bound.
4. If the difference between upper and lower bounds is less than some tolerance the stop. If the difference is zero. the optimum has been found. If it is not zero, then report the incumbent solution. If the bounds are not within the specified tolerance return to step 1 for another iteration.

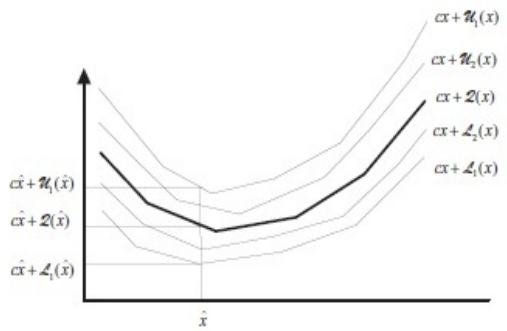


Figure 3: Illustration of using bound for L-Shaped method

4. Approximation Scheme with L-Shaped Method

Assume that $x = \hat{x}$ and before define $\varphi(\xi) = Q(\hat{x}, \xi)$ by using bound method, so there are the lower and upper bound function as recursive function,

$$L \leq E\varphi(\bar{\xi}) \leq U$$

After getting lower and upper bound function Q(x) by dividing $\bar{\xi}$, then search expected value for each partition. Give expected values equal to opportunity value for each partition. Then look at the distribution. After that apply L-Shaped method.

If ξ^i is conditional expectation $\bar{\xi}$ and every partition is consists of $\bar{\xi}$, so for $\xi^1, \ldots \xi^L$

$$\min\{c^T x + L(x)\}, s.t\ Ax = b, x \leq 0$$

where,

$$L(x) = \sum_{j=1}^{L} p^j Q(x, \xi^j)$$

Kall and Wallace [4] illustrate a partition correspondent to lower bound function $L_1(x)$ and upper bound function $U_1(x)$ so for each x effect $L_1(x) \leq Q(x) \leq U_1(x)$ (Figure 4). Minimize $cx + L_1(x)$ to get $\hat{x}$. Because of getting error for $U_1(\hat{x}) - L_1(\hat{x})$, so it must be made the new partition so that L_1 becomes L_2 and U_1 becomes U_2. If $\bar{x}$ is the optimal solution, so

$$c^T\hat{x} + L(\hat{x}) \leq c^T\bar{x} + L(\bar{x}) \leq c^T\bar{x} + Q(\bar{x})$$
$$c^T\bar{x} + L(\bar{x}) \leq c^T\hat{x} + L(\hat{x}) \leq c^T\hat{x} + Q(\hat{x})$$

Finally, $\hat{x}$ is an optimal solution and error as much as $U(\hat{x}) - L(\hat{x})$.

Consider a product mix problem. A furniture maker makes four products: P1, P2, P3 and P4. Two manufacturing resources are required: carpentry and finishing. The requirements measured in hours per unit are known and shown in the table below along with the profit per unit of product.

Table 1: Capacity problem

Product Parameters	P1	P2	P3	P4
Carpentary Hours per Unit	4	9	7	10
Finishing Hours per Unit	3	1	3	4
Profit per Unit	15	25	21	31

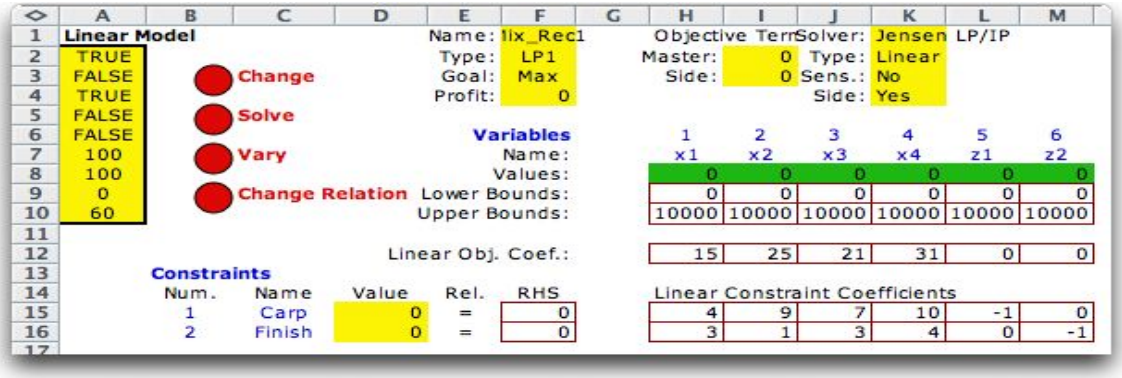


Figure 4: Mix Master

The problem is to select the product mix to maximize total profit, but the availability of the resources are not known. Rather we have four equally likely estimates of the hours available for each resource.

Table 2: Furniture Distribution

Resource Distribution	1	2	3	4
Carpentry Hours Available	4800	5500	6050	6150
Finishing Hours Available	3936	3984	4016	4064
Probability	0.25	0.25	0.25	0.25

The mix chosen will require some number of hours for the resources. Depending on the amount that is available we will pay for extra hours to obtain the necessary number of hours.

There are 16 scenarios for the stochastic programming model represented on the side model. An important restriction to this method is that all subproblems be linear programs. This makes nonlinear or integer forms inapplicable. Also the subproblems must have feasible solutions for every feasible solution for the linking variables. It is always possible to create models for which this is true, but the method will terminate if a subproblem happens not to have a feasible solution for some solution for the linking variables encountered during the solution process.

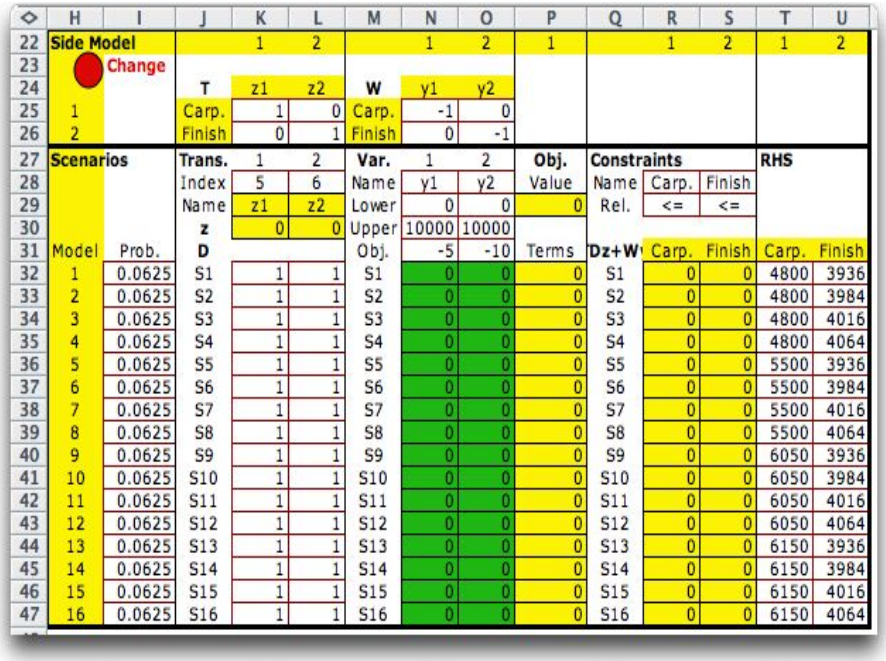


Figure 5: Mix_side

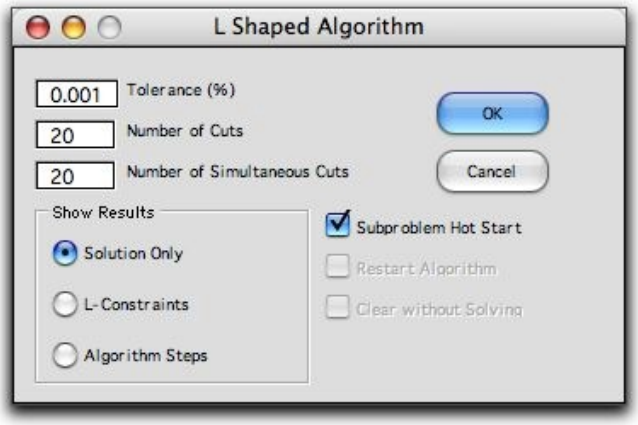
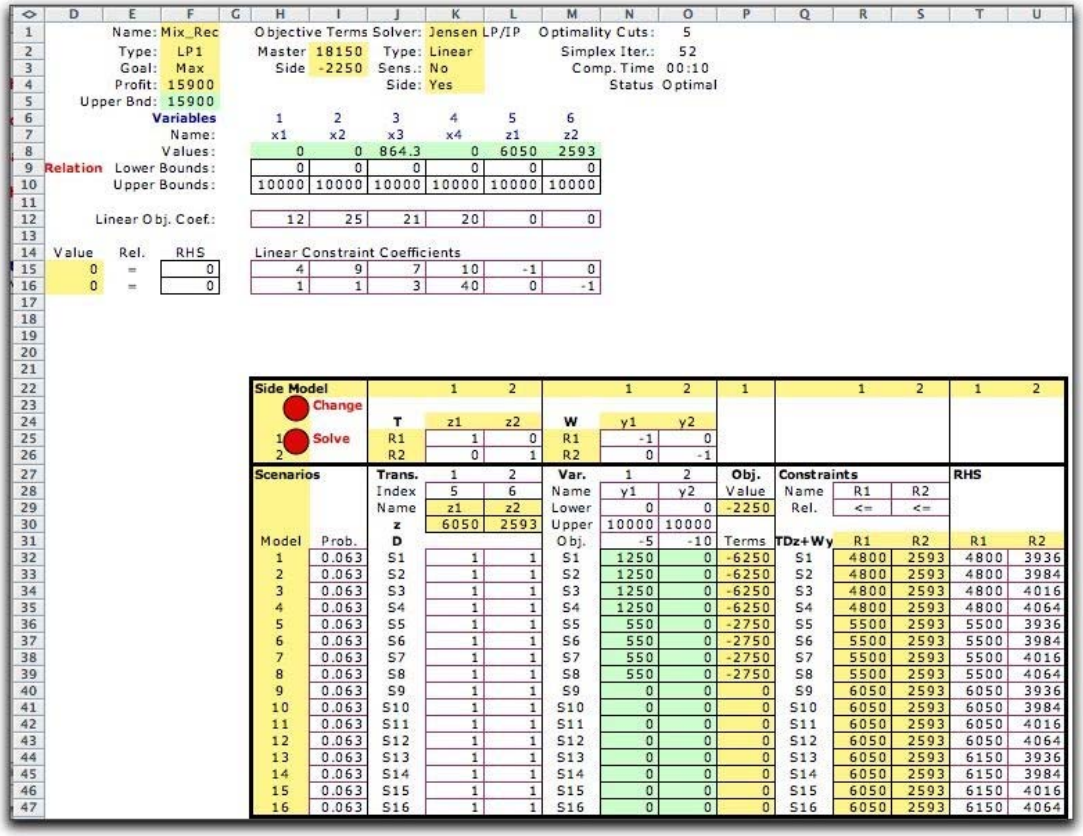


Figure 6: Caption for l_method dial

The fields at the top of the form are parameters for the method that will become meaningful as the example progresses. The buttons at the bottom indicate how much of the solution process is displayed. The Solution Only button solves the problem and then eliminates all intermediate information from the worksheet except a Lower Bound (or Upper Bound) cell just below the objective value. The L-Constraints button shows the extra constraints added to the master problem by the method. The Algorithm Steps button stops the algorithm at each step of the process so the student can review the current status and perhaps terminate the algorithm. Solving the problem with the options above results in the solution below.



1			Name:	Mix_Rec		Objective Terms		Solver:	Jensen LP/IP		Optimality Cuts:		5										
2			Type:	LP1		Master	18150	Type:	Linear		Simplex Iter.:		52										
3			Goal:	Max		Side	-2250	Sens.:	No		Comp. Time		00:10										
4			Profit:	15900				Side:	Yes		Status		Optimal										
5			Upper Bnd:	15900																			
6				Variables		1	2	3	4	5	6												
7				Name:		x1	x2	x3	x4	z1	z2												
8				Values:		0	0	864.3	0	6050	2593												
9	Relation			Lower Bounds:		0	0	0	0	0	0												
10				Upper Bounds:		10000	10000	10000	10000	10000	10000												
11																							
12				Linear Obj. Coef.:		12	25	21	20	0	0												
13																							
14	Value		Rel.	RHS		Linear Constraint Coefficients																	
15	0		=	0		4	9	7	10	-1	0												
16	0		=	0		1	1	3	40	0	-1												
17																							
18																							
19																							
20																							
21																							
22					Side Model		1	2		1	2	1		1	2	1	2						
23					Change																		
24						T	z1	z2	W	y1	y2												
25					1 Solve	R1	1	0	R1	-1	0												
26					2	R2	0	1	R2	0	-1												
27					Scenarios	Trans.	1	2	Var.	1	2	Obj.	Constraints			RHS							
28						Index	5	6	Name	y1	y2	Value	Name	R1	R2								
29						Name	z1	z2	Lower	0	0	-2250	Rel.	<=	<=								
30						z	6050	2593	Upper	10000	10000												
31					Model	Prob.	D		Obj.	-5	-10	Terms	TDz+Wy	R1	R2	R1	R2						
32					1	0.063	S1	1	1	S1	1250	0	-6250	S1	4800	2593	4800	3936					
33					2	0.063	S2	1	1	S2	1250	0	-6250	S2	4800	2593	4800	3984					
34					3	0.063	S3	1	1	S3	1250	0	-6250	S3	4800	2593	4800	4016					
35					4	0.063	S4	1	1	S4	1250	0	-6250	S4	4800	2593	4800	4064					
36					5	0.063	S5	1	1	S5	550	0	-2750	S5	5500	2593	5500	3936					
37					6	0.063	S6	1	1	S6	550	0	-2750	S6	5500	2593	5500	3984					
38					7	0.063	S7	1	1	S7	550	0	-2750	S7	5500	2593	5500	4016					
39					8	0.063	S8	1	1	S8	550	0	-2750	S8	5500	2593	5500	4064					
40					9	0.063	S9	1	1	S9	0	0	0	S9	6050	2593	6050	3936					
41					10	0.063	S10	1	1	S10	0	0	0	S10	6050	2593	6050	3984					
42					11	0.063	S11	1	1	S11	0	0	0	S11	6050	2593	6050	4016					
43					12	0.063	S12	1	1	S12	0	0	0	S12	6050	2593	6050	4064					
44					13	0.063	S13	1	1	S13	0	0	0	S13	6050	2593	6150	3936					
45					14	0.063	S14	1	1	S14	0	0	0	S14	6050	2593	6150	3984					
46					15	0.063	S15	1	1	S15	0	0	0	S15	6050	2593	6150	4016					
47					16	0.063	S16	1	1	S16	0	0	0	S16	6050	2593	6150	4064					

Figure 7: Caption for mix_sol

The master problem solution variables are in row 8 and the subproblem solution variables are in columns N and O. The L-shaped method does not guarantee an optimum solution. In the original dialog there is a tolerance percentage (in this case (0.001 percent). The program computes lower and upper bounds to the optimum solution and stops when difference between these bounds is less than this tolerance (relative to the largest absolute value of the lower or upper bounds). The solution presented is the best solution obtained during the process and its objective value is a lower bound to the optimum. The upper bound provides a measure of quality for the solution presented. In this case the upper bound and lower bounds are equal, so the solution is optimum. For a minimization problem the lower and upper bounds have reversed purposes.

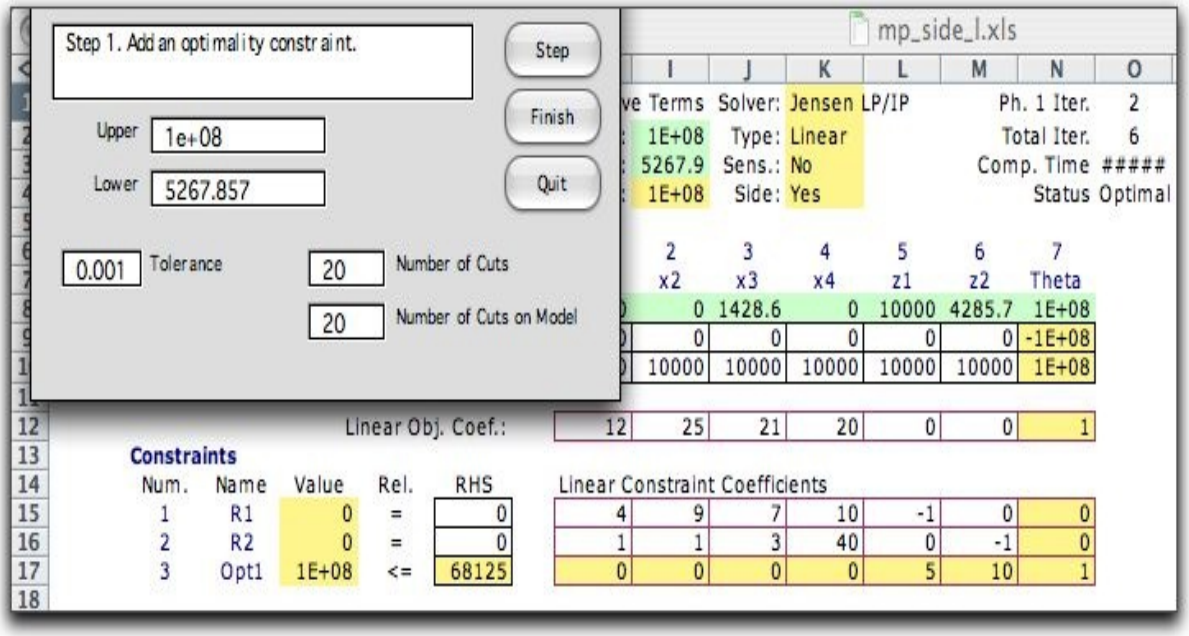


Figure 8: Step 1

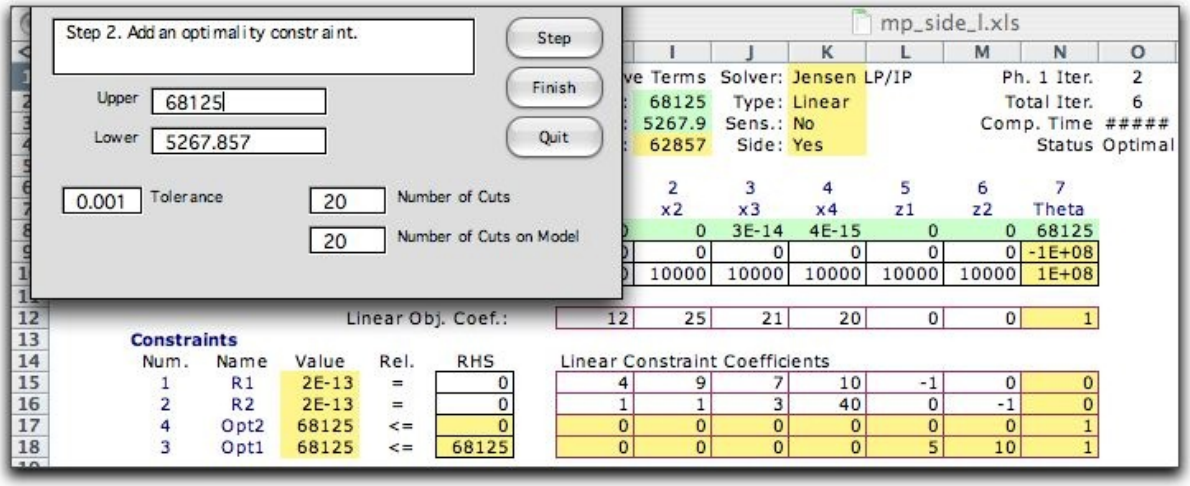


Figure 9: Step 2

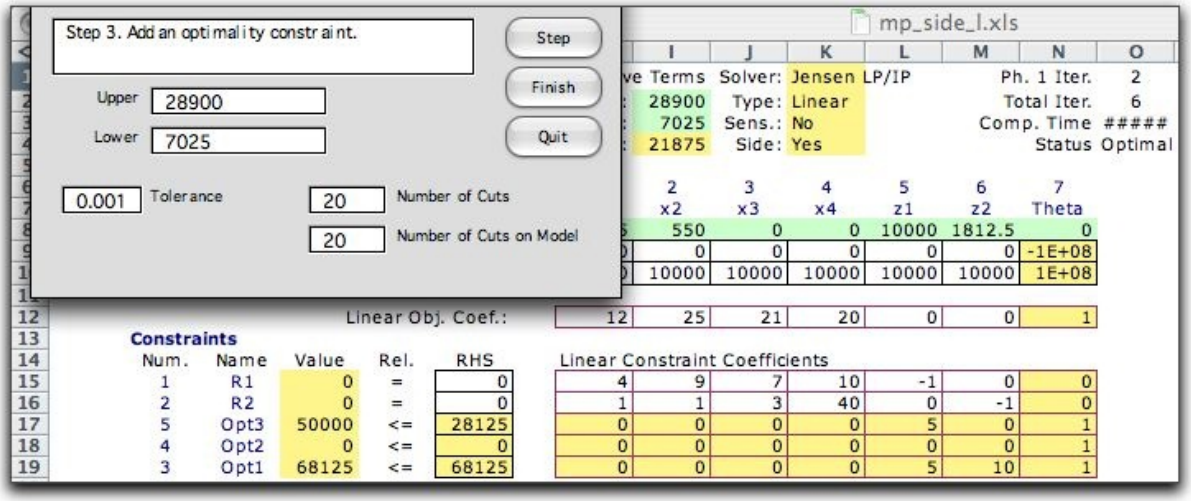


Figure 9: Step 3

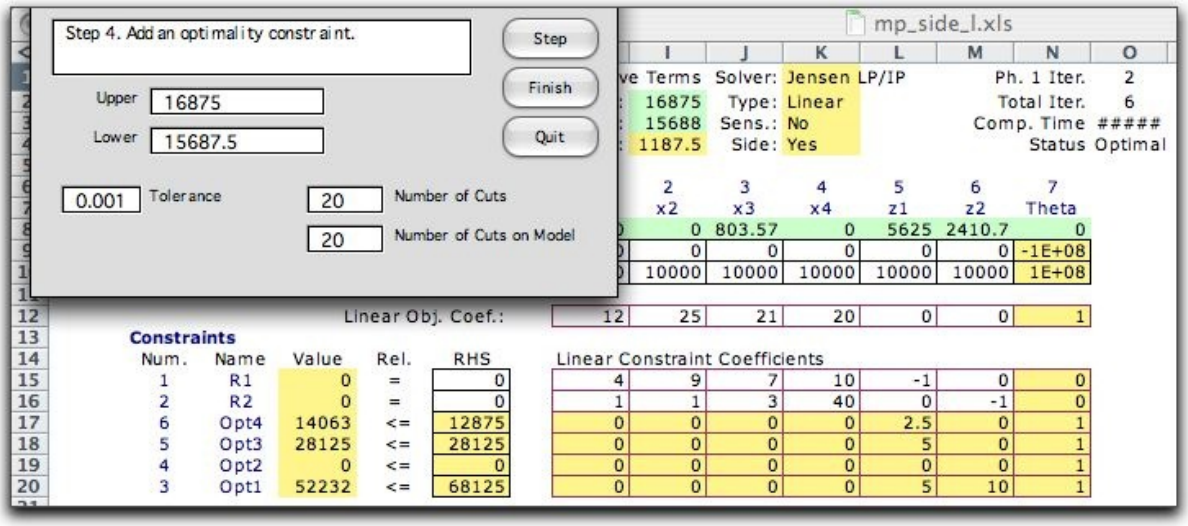


Figure 10: Step 4

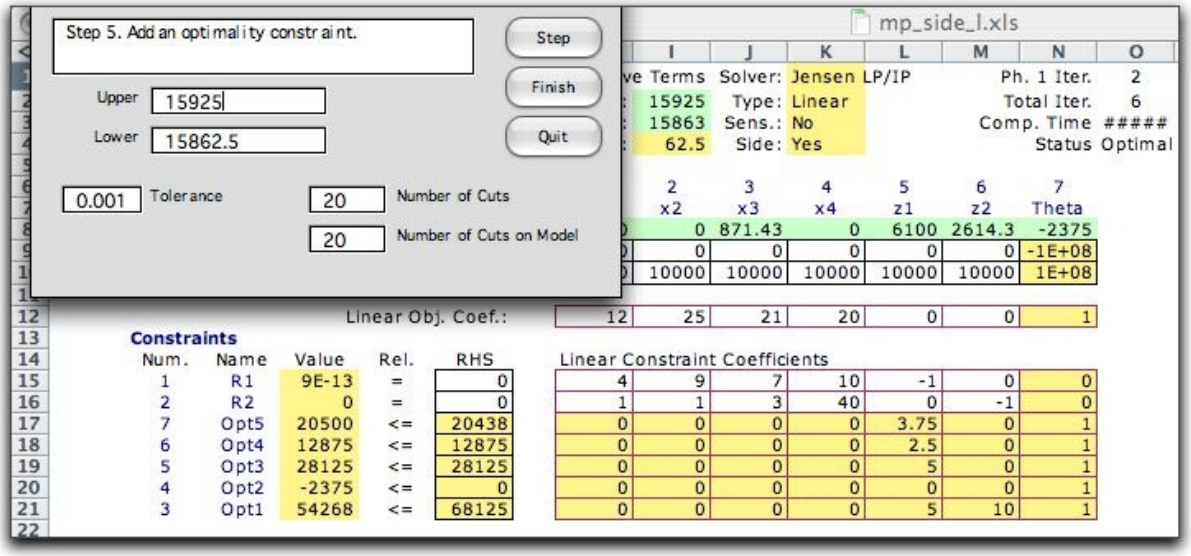


Figure 11: Step 5

4. Concluding Remarks

This paper uses two different partition methods. The first stage, partition is done by getting lower and upper bound, and the second stage use L-Shaped method. But L-Shaped method can get the optimal solution if the first partition is selected well. It means it will be optimal according to determining interval for random parameter distribution so error will decrease. The bigger we use first partition, the more next partition can be made so the final iteration comes near function to get the optimal solution.

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