The Order of the Set of L-Class for Semigroup of Order-Preserving and Order-Decreasing Partial Transformation

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Abstract: ***The major problems usually encountered by transformation semigroup is that of enumeration Umar, 2010. Here we studied the elements of partial transformation and extracted those that are order-preserving and order-decreasing which is a subsemigroup of partial transformation. Thus we listed the elements that satisfy the*** L ***-class for*** PC_n ***. A general order of obtaining the set of*** L ***-class for*** PC_n ***was obtained.***

Keyword: L -class, Green's relation, order-preserving and order-decreasing partial transformation semigroup, partial transformation semigroup.

1. Introduction

Let $[n]=\{1,2,3,\ldots\ldots,n\}$, then a (partial) transformation $\alpha : Dom\,\alpha \subseteq [n] \to Im\,\alpha$ is said to be full or total if $Dom\,\alpha = [n]$ otherwise it is called strictly partial. The set of all partial transformation on n-object forms a semigroup under the usual composition of transformations. Here we shall call a partial transformation $\alpha : [n] \to [n]$ order-preserving if $(\forall x, y \in Dom\,\alpha) x \le y \Rightarrow x\alpha \le y\alpha$ and order-decreasing if $(\forall x \in Dom\,\alpha) x\alpha \le x$. Thus we denote the order for order-preserving and order-decreasing partial transformation as PC_n. Obviously P_n has order $(n+1)^n$.

Green's relations are fundamental tools in the study of the structure of semigroups especially regular semigroups. In Mathematics, Green relations are five equivalence relations that characterize the element of a subgroup in terms of the principal Ideals they generate. The relations were named after James Alexander Green, who introduced them in a paper in 1951. Howie (2002) describe the work as "So all pervading" that on encountering a new subgroup; almost the first question we ask is what are the Green's relation like on this semigroup.So we decided to employ one of the five equivalence relation that is the L -class in other to determine its behaviour in the subsemigroup of order-preserving and order-decreasing partial transformation semigroup. Two transformation γ and λ are said to be in L -class if and only if they have the same set of image(s) that is $(\gamma, \lambda) \in$ L -class iff $rank(\gamma) = rank(\lambda)$

2. Partition Structure for L -Class

A partition of a set X is a collection P of pairwise disjoint, non-empty subsets of X whose union is X. The elements of P are called the blocks of the partition.

In the methodology we studied the semigroup of partial transformation and extracted the elements of the subsemigroup of order-preserving and order-decreasing. Thus we partitioned the set of elements that satisfies the L -class from $n = 1,2,3$ where L is the set of elements for the L -class and we put the result in tabular format for $n = 1-10$ as fo

Partition of L -class for PC_1

L_i	Structure	$\lvert L_i \rvert$
L_1	$\begin{pmatrix} 1 \\ - \end{pmatrix}$	1
L_2	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	1

Total number of L -class in $PC_1 = 2$, $\lvert PC_1 \rvert = \sum \lvert L_i \rvert = 2$

Partition of L -class for CP_2

L_i	Partition Structure	$\lvert L_i \rvert$
L_1	$\begin{pmatrix}1&2\\-&-\end{pmatrix}$	1
L_2	$\begin{pmatrix}1&2\\1&-\end{pmatrix}\begin{pmatrix}1&2\\-&1\end{pmatrix}\begin{pmatrix}1&2\\1&1\end{pmatrix}$	3
L_3	$\begin{pmatrix}1&2\\-&2\end{pmatrix}$	1
L_4	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	1

Total number of L -class in $PC_2 = 4$, $\left|PC_2\right| = \sum \left|L_i\right| = 6$

Partition of L -class for PC_3

L_i	Partition class	$\lvert L_i \rvert$
L_1	$\begin{pmatrix}1&2&3\\-&-&-\end{pmatrix}$	1
L_2	$\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&-\end{pmatrix}$	7
	$\begin{pmatrix}1&2&3\\-&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&1\end{pmatrix}$	
L_3	$\begin{pmatrix}1&2&3\\-&2&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&2\end{pmatrix}\begin{pmatrix}1&2&3\\-&2&2\end{pmatrix}$	3
L_4	$\begin{pmatrix}1&2&3\\-&-&3\end{pmatrix}$	1
L_5	$\begin{pmatrix}1&2&3\\1&2&-\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&2\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}$ $\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}$	5
L_6	$\begin{pmatrix}1&2&3\\1&-&3\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}$	3
L_7	$\begin{pmatrix}1&2&3\\-&2&3\end{pmatrix}$	1
L_8	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	1

Total number of L -class in $PC_3 = 8$ $\sum \left|L_i\right| = 22$

Table Order of set of L -class

n	L
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

Theorem 1 *Let PC_n be the subsemigroup of P_n and $L(PC_n)$ denote the L -class of PC_n then $L(PC_n) = 2^n$*

proof:

Let $L(PC_n)$ be a subsemigroup of P_n . Thus $L(PC_n) \subseteq PC_n$. Also for element γ and λ of S the Green's relation L is defined by $\gamma L \lambda$ iff $S\gamma' = S\lambda'$ that is γ and λ are L -related if they generate the same left ideal, in other words $(\gamma, \lambda \in L)$ -class iff $rank(\gamma) = rank(\lambda)$.

Hence from the sequence $2,4,8,16,32...$ generated for the set of L -class of PC_n which we can likened to the power set which are subset of a set, we could also see that this situation is same as most idempotent elements of some subsemigroup. Thus we have

$$|L(PC_n)| = 2^n$$

3. Conclusion

It has been shown that the number for the set of L -class for semigroup of order-preserving and order-decreasing partial transformation can be calculated using the formula:

$$|L(PC_n)| = 2^n \qquad (1)$$

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