

The Order of the Set of \mathcal{L} -Class for Semigroup of Order-Preserving and Order-Decreasing Partial Transformation

Mbah, Moses Anayo¹, Ugbene, Ifeanyichukwu Jeff², Bakare, Getta Naimat³

¹Department of Mathematics, Federal University Lafia, P.M.B 146 Lafia, Nasarawa State, Nigeria.

²Department of Mathematics, Federal University of Petroleum Resources, Effurun, Delta State

³Department of Mathematics, University of Ilorin, Ilorin, Kwara State

Abstract: The major problems usually encountered by transformation semigroup is that of enumeration Umar, 2010. Here we studied the elements of partial transformation and extracted those that are order-preserving and order-decreasing which is a subsemigroup of partial transformation. Thus we listed the elements that satisfy the \mathcal{L} -class for PC_n . A general order of obtaining the set of \mathcal{L} -class for PC_n was obtained.

Keyword: \mathcal{L} -class, Green's relation, order-preserving and order-decreasing partial transformation semigroup, partial transformation semigroup.

1. Introduction

Let $[n] = \{1, 2, 3, \dots, n\}$, then a (partial) transformation $\alpha: \text{Dom } \alpha \subseteq [n] \rightarrow \text{Im } \alpha$ is said to be full or total if $\text{Dom } \alpha = [n]$ otherwise it is called strictly partial. The set of all partial transformation on n -object forms a semigroup under the usual composition of transformations. Here we shall call a partial transformation $\alpha: [n] \rightarrow [n]$ order-preserving if $(\forall x, y \in \text{Dom } \alpha) x \leq y \Rightarrow x\alpha \leq y\alpha$ and order-decreasing if $(\forall x \in \text{Dom } \alpha) x\alpha \leq x$. Thus we denote the order for order-preserving and order-decreasing partial transformation as PC_n . Obviously P_n has order $(n+1)^n$.

Green's relations are fundamental tools in the study of the structure of semigroups especially regular semigroups. In Mathematics, Green relations are five equivalence relations that characterize the element of a subgroup in terms of the principal Ideals they generate. The relations were named after James Alexander Green, who introduced them in a paper in 1951. Howie (2002) describe the work as "So all pervading" that on encountering a new subgroup; almost the first question we ask is what are the Green's relation like on this semigroup. So we decided to employ one of the five equivalence relation that is the \mathcal{L} -class in other to determine its behaviour in the subsemigroup of order-preserving and order-decreasing partial transformation semigroup. Two transformation γ and λ are said to be in \mathcal{L} -class if and only if they have the same set of image(s) that is $(\gamma, \lambda) \in \mathcal{L}$ -class iff $\text{rank}(\gamma) = \text{rank}(\lambda)$

2. Partition Structure for \mathcal{L} -Class

A partition of a set X is a collection P of pairwise disjoint, non-empty subsets of X whose union is X . The elements of P are called the blocks of the partition.

In the methodology we studied the semigroup of partial transformation and extracted the elements of the subsemigroup of order-preserving and order-decreasing. Thus we partitioned the set of elements that satisfies the \mathcal{L} -class from $n = 1, 2, 3$ where L is the set of elements for the \mathcal{L} -class and we put the result in tabular format for $n = 1 - 10$ as fo

Partition of \mathcal{L} -class for PC_1

L_i	Structure	$ L_i $
L_1	$\begin{pmatrix} 1 \\ - \end{pmatrix}$	1
L_2	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	1

Total number of \mathcal{L} -class in $PC_1 = 2$,
 $|PC_1| = \sum |L_i| = 2$

Partition of L -class for CP_2

L_i	Partition Structure	$ L_i $
L_1	$\begin{pmatrix} 12 \\ - - \end{pmatrix}$	1
L_2	$\begin{pmatrix} 12 & 12 & 12 \\ 1- & -1 & 11 \end{pmatrix}$	3
L_3	$\begin{pmatrix} 12 \\ - 2 \end{pmatrix}$	1
L_4	$\begin{pmatrix} 12 \\ 12 \end{pmatrix}$	1

Total number of L -class in $PC_2 = 4$,
 $|PC_2| = \sum |L_i| = 6$

Partition of L -class for PC_3

L_i	Partition class	$ L_i $
L_1	$\begin{pmatrix} 123 \\ - - - \end{pmatrix}$	1
L_2	$\begin{pmatrix} 123 & 123 & 123 & 123 & 123 \\ 111 & 1- - & -1- & - - 1 & 11- \end{pmatrix}$ $\begin{pmatrix} 123 & 123 \\ - 11 & 1- 1 \end{pmatrix}$	7
L_3	$\begin{pmatrix} 123 & 123 & 123 \\ - 2- & - - 2 & - 2 2 \end{pmatrix}$	3
L_4	$\begin{pmatrix} 123 \\ - - 3 \end{pmatrix}$	1
L_5	$\begin{pmatrix} 123 & 123 & 123 & 123 \\ 12- & 1- 2 & - 12 & 112 \end{pmatrix}$ $\begin{pmatrix} 123 \\ 12 2 \end{pmatrix}$	5
L_6	$\begin{pmatrix} 123 & 123 & 123 \\ 1- 3 & - 13 & 113 \end{pmatrix}$	3
L_7	$\begin{pmatrix} 123 \\ - 2 3 \end{pmatrix}$	1
L_8	$\begin{pmatrix} 123 \\ 12 3 \end{pmatrix}$	1

Total number of L -class in $PC_3 = 8$ $\sum |L_i| = 22$

Table Order of set of L -class

n	L
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

Theorem 1 Let PC_n be the subsemigroup of P_n and $L(PC_n)$, denote the L -class of PC_n then $L(PC_n) = 2^n$

proof:

Let $L(PC_n)$ be a subsemigroup of P_n . Thus $L(PC_n) \subseteq PC_n$. Also for element γ and λ of S the Green's relation L is defined by $\gamma L \lambda$ iff $S\gamma' = S\lambda'$ that is γ and λ are L -related if they generate the same left ideal, in other words $(\gamma, \lambda \in L)$ -class iff $rank(\gamma) = rank(\lambda)$.

Hence from the sequence 2,4,8,16,32... generated for the set of L -class of PC_n which we can likened to the power set which are subset of a set, we could also see that this situation is same as most idempotent elements of some subsemigroup. Thus we have

$$|L(PC_n)| = 2^n$$

3. Conclusion

It has been shown that the number for the set of L -class for semigroup of order-preserving and order-decreasing partial transformation can be calculated using the formula:

$$|L(PC_n)| = 2^n \tag{1}$$

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