International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Coincidence and Common Fixed Point Theorems for Nonlinear Contractive in Intuitionistic Fuzzy Metric Space

A.D. Singh[1], Ram Milan Singh[2], R. S. Chandel[3]

Abstract: ***The purpose of this paper is to obtain a new common fixed point theorem by using a new contractive condition and properties in Intuitionistic fuzzy metric spaces.***

Keyword: Triangular norm, triangular co-norm, intuitionistic fuzzy metric space, fuzzy metric space, fixed point.

1. Introduction

Since the introduction of the concept of fuzzy set by Zadeh [7] in 1965, many authors have introduced the concept of fuzzy metric in different ways. George and Veeramani [2] modified the concept of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space. Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets. There have been a much progress in the study of intuitionistic fuzzy sets by many authors. Park [5] using the idea of intuitionistic fuzzy sets, defined by the notation of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [2], O.Kramosil and J.Michalck [4] and S.Sharma and J.K.Tiwari [6] O.Kramosil and J.Michalck[3].

2. Preliminaries

Definition 2.1 *A binary operation* $*: [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

(a) $*$ *is commutative and associative;*

(b) $*$ *is continuous;*

(c) $a * 1 = a$ *for all* $a \in [0,1]$;

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

Definition 2.2. *A binary operation* $\blacklozenge : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-conorm if it satisfies the following conditions:*

(a) $\blacklozenge$ *is commutative and associative;*

(b) $\blacklozenge$ *is continuous;*

(c) $a \blacklozenge 0 = a$ *for all* $a \in [0,1]$;

(d) $a \blacklozenge b \leq c \blacklozenge d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

Definition 2.3. *A three tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *a continuous t-norm and M a fuzzy set on* $X^2 \times [0,\infty)$ *satisfying the following condition, for all* $x, y, z \in X$ *and* $t, s > 0$:

(a) $M(x,y,0) = 0$ #

(b) $M(x,y,t) = 1$ *for all* $t > 0$ *iff* $x = y$, #

(c) $M(x,y,t) = M(y,x,t)$, #

(d) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,

(e) $M(x,y,.) : [0,\infty) \longrightarrow [0,1]$ *is left continuous,*

(f) $\lim_{n\to\infty} M(x,y,t) = 1$.

Definition 2.4. *A 5-tuple* $(X, M, N, *, \blacklozenge)$ *is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if X is an arbitrary set,* $*$ *is a continuous t-norm,* $\blacklozenge$ *is a continuous t-conorm and M, N are fuzzy sets on* $X^2 \times [0,\infty)$ *satisfying the following conditions:*

(a) $M(x,y,t) + N(x,y,t) \leq 1$ *for all* $x, y \in X$ *and* $t > 0$;

(b) $M(x,y,0) = 0$ *for all* $x, y \in X$;

(c) $M(x,y,t) = 1$ *for all* $x, y \in X$ *and* $t > 0$ *if and only if* $x = y$,

(d) $M(x,y,t) = M(y,x,t)$ *for all* $x, y \in X$ *and* $t > 0$;

(e) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ *for all* $x, y, z \in X$ *and* $s, t > 0$,

(f) $M(x, y, .) : [0, \infty) \longrightarrow [0,1]$ *is left continuous for all* $x, y \in X$
(g) $\lim_{n\to\infty} M(x, y, t) = 1$,
(h) $N(x, y, 0) = 1$ *for all* $x, y \in X$
(i) $N(x, y, t) = 0$ *for all* $x, y \in X$ *and* $t > 0$ *if and only if* $x = y$,
(j) $N(x, y, t) = N(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$,
(k) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ *for all* $x, y, z \in X$ *and* $s, t > 0$,
(l) $M(x, y, .) : [0, \infty) \longrightarrow [0,1]$ *is right continuous for all* $x, y \in X$
(m) $\lim_{n\to\infty} N(x, y, t) = 0$ *for all* $x, y \in X$;

Then (M, N) *is called an intuitionistic fuzzy metric on X . The functions* $M(x, y, t)$ *and* $N(x, y, t)$ *denote the degree of nearness and degree of non nearness between x and y with respect to t, respectively.*

Definition 2. 5: *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. Then*
(a) *a sequence* $\{x_n\}$ *in X is said to be Cauchy sequence if, for all* $t > 0$ *and* $p > 0$,
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$
(b) *a sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$ *if, for all* $t > 0$,
$\lim_{n\to\infty} M(x_n, x_n, t) = 1$, $\lim_{n\to\infty} N(x_n, x_n, t) = 0$

Definition 2. 6: *An intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *is said to be complete if and only if every Cauchy sequence in X is convergent.*

Definition 2. 7: *An intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *is said to be compact if every sequence in X contains a convergent subsequence.*

3. Main Results

Theorem-2.1. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space. Let A, B, S and T be a mappings from X into itself satisfying,*

$$(a)\ A(X) \subset T(X) \text{ and } B(X) \subset S(X)\ (b) M(Au, Bv, \emptyset(t)) \geq \min\left\{\begin{matrix} M(Su, Av, t), M(Tv, Bv, t), M(Tv, Au, \beta t), \\ M(Su, Bv, (2-\beta)t), M(Su, Tv, t) \end{matrix}\right\} \dots (2.1)$$

$$\text{and } N(Au, Bv, \emptyset(t)) \leq \max\left\{\begin{matrix} N(Su, Av, t), N(Tv, Bv, t), N(Tv, Au, \beta t), \\ N(Su, Bv, (2-\beta)t), N(Su, Tv, t) \end{matrix}\right\} \dots (2.2)$$

for all $u, v \in X$ *and* $\beta \in (0,2)$ *and for all* $t > 0$ *where the function* $\emptyset : (0, \infty) \longrightarrow [0, \infty)$ *is onto strictly increasing and decreasing and satisfies condition* $(*)$ *Also assume that there exist* $t_0, t_1, t_2 \in X$ *with* $A_{t0} = T_{t1}, B_{t1} = S_{t2}$ *and*
$E_M(A_{t0}, B_{t1}) = \sup\{E_{Y,M}(A_{t0}, B_{t1}) : Y \in (0,1)\} < \infty$, *and*
$E_N(A_{t0}, B_{t1}) = \inf\{E_{Y,N}(A_{t0}, B_{t1}) : Y \in (0,1)\} > \infty$
(c) *One of A(X), B(X), S(X) or T(X) is a complete subspace of X. Then*
(1) the pair (A,S) has the coincidence point.
(2) the pair (B,T) has the coincidence point
(3) A, B, S and T have a unique common fixed point provided both the pair (A,S) and (B,T) are weakly compatible.

Proof- *Let* t_0 *be an arbitrary point in X. Since* $A(X) \subset T(X)$, *one can find a point* t_1 *in X with* $A_{t0} = T_{t1} = z_0$, *again* $B(X) \subset S(X)$, *one can also choose a point* t_2 *in X with* $B_{t1} = S_{t2} = z_1$ *and* $E_M(A_{t0}, B_{t1}) < \infty$, $E_N(A_{t0}, B_{t1}) > \infty$. *Inductively one can construct* z_n *such that*
$A_u = T_v = z_{2n}$ *and* $B_u = S_w$ *for* $n = 0,1,2,3, \dots\dots\dots$,
where $u = t_{2n}, v = t_{2n+1}, w = t_{2n+2}$. *First we show that the sequence* $\{z_n\}$ *described by* $\{A_{t0}, B_{t1}, A_{t2}, \dots\dots\dots, B_{t2n-1}, A_{t2n}, B_{t2n+1}, \dots\dots\dots\}$, *is a Cauchy sequence in X. To accomplish this, set* $p = t_{2n}$, $q = t_{2n+1}$, *for* $t > 0$ *and*

$$\beta = 1 - \lambda \text{ with } \lambda \in (0,1) \text{ so that, } M(A_u, B_v, \emptyset(t)) \geq \min\left\{\begin{matrix} M(S_u, A_v, t), M(T_v, B_v, t), M(T_v, A_u, \beta t), \\ M(S_u, B_v, (2-\beta)t), M(S_u, T_v, t) \end{matrix}\right\}$$

$$\geq \min\left\{\begin{matrix} M(z_{2n-1}, z_{2n}, t), M(z_{2n}, z_{2n+1}, t), M(z_{2n-1}, z_{2n+1}, (1+\lambda)t), \\ M(z_{2n-1}, z_{2n}, t) \end{matrix}\right\}$$

$$\geq \min\{M(z_{2n-1}, z_{2n}, t), M(z_{2n}, z_{2n+1}, t), M(z_{2n-1}, z_{2n+1}, (1+\lambda)t)\}$$

$$\geq \min\{M(z_{2n-1}, z_{2n}, t), M(z_{2n}, z_{2n+1}, t), M(z_{2n-1}, z_{2n+1}, (1+\lambda)t)\}$$

$$\geq \min\{M(z_{2n-1}, z_{2n}, t), M(z_{2n}, z_{2n+1}, t), M(z_{2n-1}, z_{2n+1}, (\lambda)t)\} \text{ ,and}$$

$$N(A_u, B_v, \emptyset(t)) \leq \max\left\{\begin{matrix} \{N(S_u, A_v, t), N(T_v, B_v, t), N(T_v, A_u, \beta t), \\ N(S_u, B_v, (2-\beta)t), N(S_u, T_v, t) \end{matrix}\right\}$$

$$\leq max\left\{\begin{matrix} N(z_{2n-1}, z_{2n}, t), N(z_{2n}, z_{2n+1}, t), N(z_{2n-1}, z_{2n+1}, (1+\lambda)t), \\ N(z_{2n-1}, z_{2n}, t) \end{matrix}\right\}$$

$$\leq max\{N(z_{2n-1}, z_{2n}, t), N(z_{2n}, z_{2n+1}, t), N(z_{2n-1}, z_{2n+1}, (1+\lambda)t)\}$$

$$\leq max\{N(z_{2n-1}, z_{2n}, t), N(z_{2n}, z_{2n+1}, t), N(z_{2n-1}, z_{2n+1}, (1+\lambda)t)\}$$

$$\leq max\{N(z_{2n-1}, z_{2n}, t), N(z_{2n}, z_{2n+1}, t), N(z_{2n-1}, z_{2n+1}, (\lambda)t)\}$$

which on letting $\lambda \rightarrow 1$, *reduce to.*

$M(A_w, B_v, \phi(t)) \geq min\{M(z_{2n-1}, z_{2n}, t), M(z_{2n}, z_{2n+1}, t)\}$, *and*

$N(A_w, B_v, \phi(t)) \leq max\{N(z_{2n-1}, z_{2n}, t), N(z_{2n}, z_{2n+1}, t)\}$

Similarly, one can show that

$M(z_{2n+1}, z_{2n+2}, t) \geq min\{M(z_{2n}, z_{2n+1}, t), M(z_{2n+1}, z_{2n+2}, t)\}$, *and*

$N(z_{2n+1}, z_{2n+2}, t) \leq max\{N(z_{2n}, z_{2n+1}, t), N(z_{2n+1}, z_{2n+2}, t)\}$

therefore for all n(even and odd), we have,

$M(z_n, z_{n+1}, \phi(t)) \geq min\{M(z_{n-1}, z_n, t), M(z_n, z_{n+1}, t)\}$, *and*

$N(z_n, z_{n+1}, \phi(t)) \leq max\{N(z_{n-1}, z_n, t), N(z_n, z_{n+1}, t)\}$

which is true yields,

$M(z_n, z_{n+1}, t) \geq min\{M(z_{n-1}, z_n, \phi^{-1}(t)), M(z_n, z_{n+1}, \phi^{-1}(t))\}$, *and*

$N(z_n, z_{n+1}, t) \leq max\{N(z_{n-1}, z_n, \phi^{-1}(t)), N(z_n, z_{n+1}, \phi^{-1}(t))\}$

by repeated application of the above inequality (for $m = 1, 2, 3, \ldots$*), we get*

$$M(z_n, z_{n+1}, t) \geq min\left\{\begin{matrix} M(z_{n-1}, z_n, \phi^{-1}(t)), M(z_{n-1}, z_n, \phi^{-2}(t)), \\ M(z_n, z_{n+1}, \phi^{-2}(t)) \end{matrix}\right\}$$

$$= min\{M(z_{n-1}, z_n, \phi^{-1}(t)), M(z_{n-1}, z_n, \phi^{-2}(t))\} \geq \ldots\ldots\ldots\ldots\ldots \geq$$

$min\{M(z_{n-1}, z_n, \phi^{-1}(t)), M(z_n, z_{n+1}, \phi^{-m}(t))\}$, *and*

$$N(z_n, z_{n+1}, t) \leq max\left\{\begin{matrix} N(z_{n-1}, z_n, \phi^{-1}(t)), N(z_{n-1}, z_n, \phi^{-2}(t)), \\ N(z_n, z_{n+1}, \phi^{-2}(t)) \end{matrix}\right\}$$

$$= max\{N(z_{n-1}, z_n, \phi^{-1}(t)), N(z_{n-1}, z_n, \phi^{-2}(t))\} \geq \ldots\ldots\ldots\ldots\ldots \geq$$

$max\{N(z_{n-1}, z_n, \phi^{-1}(t)), N(z_n, z_{n+1}, \phi^{-m}(t))\}$

thus for each $\lambda \in (0,1)$, *we have*

$E_{\gamma,M}(z_n, z_{n+1}) = inf\{t > 0 : M(z_n, z_{n+1}, t) \geq 1 - \gamma\}$

$$\geq inf\{t > 0 : min\{M(z_{n-1}, z_n, \phi^{-1}(t)), M(z_n, z_{n+1}, \phi^{-m}(t))\} \geq 1 - \gamma\} \geq$$

$$max\left\{\begin{matrix} inf\{t > 0 : M(z_{n-1}, z_n, \phi^{-1}(t)) \geq 1 - \gamma\}, \\ \{M(z_n, z_{n+1}, \phi^{-m}(t)) \geq 1 - \gamma\} \end{matrix}\right\} \geq max\{\phi(E_{\gamma,M}(z_{n-1}, z_n)), \phi^m(E_{\gamma,M}(z_n, z_{n+1}))\}$$

$\geq max\{\phi(E_{\gamma,M}(z_{n-1}, z_n)), \phi^m(E_M(z_n, z_{n-1}))\}$, *and*

$E_{\gamma,N}(z_n, z_{n+1}) = sup\{t > 0 : N(z_n, z_{n+1}, t) \leq 1 - \gamma\}$

$\leq sup\{t < 0 : min\{N(z_{n-1}, z_n, \phi^{-1}(t)), N(z_n, z_{n+1}, \phi^{-m}(t))\} \leq 1 - \gamma\}$

$$\leq min\left\{\begin{matrix} sup\{t < 0 : N(z_{n-1}, z_n, \phi^{-1}(t)) \leq 1 - \gamma\}, \\ \{N(z_n, z_{n+1}, \phi^{-m}(t)) \leq 1 - \gamma\} \end{matrix}\right\}$$

$\leq min\{\phi(E_{\gamma,N}(z_{n-1}, z_n)), \phi^m(E_{\gamma,N}(z_n, z_{n+1}))\}$

$\leq min\{\phi(E_{\gamma,N}(z_{n-1}, z_n)), \phi^m(E_N(z_n, z_{n+1}))\}$

which on making $m \rightarrow \infty$, *reduces to*

$E_{\gamma,M}(z_n, z_{n+1}) \leq \phi(E_{\gamma,M}(z_{n-1}, z_n)) \leq \phi^n(E_{\gamma,M}(z_0, z_1))$, *and*

$E_{\gamma,N}(z_n, z_{n+1}) \geq \phi(E_{\gamma,N}(z_{n-1}, z_n)) \geq \phi^n(E_{\gamma,N}(z_0, z_1))$

Now appearing to lemma 1.2.We conclude that $\{z_n\}$ *is a Cauchy sequence in X.*

Now suppose that S(X) is a complete subspace of X, then by observing that the subsequence $\{z_{2n+1}\}$ *which is contained in* $S(X)$ *must get a limit z in* $S(X)$. *Let* $u \in S^{-1}(z)$ *then* $Su = z$. *As* $\{z_n\}$ *is a Cauchy sequence containing a convergent subsequence* $\{z_{2n+1}\}$, *therefore the sequence* $\{z_n\}$ *also convergent implying thereby the convergence of* $\{z_n\}$, *being a subsequence of the convergent subsequence* $\{z_n\}$.

To prove $Au = z$, *set* $p = u$ *and* $q = t_{2n+1}$ *with* $\beta = 1$ *in 2.1and 2.2*

$$M(A_w, Bt_{2n-1}, \phi(t)) \geq min\left\{\begin{matrix} M(S_w, A_w, t), M(Tt_{2n-1}, Bt_{2n-1}, t), \\ M(Tt_{2n-1}, A_w, t), M(S_u, Bt_{2n-1}, t), M(S_w, Tt_{2n-1}, t) \end{matrix}\right\} \text{and}$$

$$N(A_w, Bt_{2n-1}, \phi(t)) \leq max\left\{\begin{matrix} N(S_w, A_w, t), N(Tt_{2n-1}, Bt_{2n-1}, t), \\ N(Tt_{2n-1}, A_w, t), N(S_w, Bt_{2n-1}, t), N(S_w, Tt_{2n-1}, t) \end{matrix}\right\}$$

Which on letting $n \to \infty$, *reduces to*

$M(z, Bv, \phi(t)) \geq min\{M(z,z,t), M(z, B_v, t), M(z,z,t), M(z, B_v, t), M(z,z,t)\}$, *and*

$N(z, Bv, \phi(t)) \leq max\{N(z,z,t), N(z, B_v, t), N(z,z,t), N(z, B_v, t), N(z,z,t)\}$

implying thereby

$M(z, B_v, \phi(t)) \geq M(z, B_v, t)$ *and* $N(z, B_v, \phi(t)) \leq N(z, B_v, t)$,*since*

$M(z, B_v, \phi(t)) \geq M(z, B_v, t)$ *and* $N(z, B_v, \phi(t)) \leq N(z, B_v, t)$, *therefore*

$M(z, B_v, t) = C$ *and* $N(z, B_v, t) = D$.

Again in view of lemma 1.1, we have $H(t) = C$ *and* $G(t) = D$ *for all* $t > 0$ *and hence* $B_v = z$. *Thus one gets* $B_v = T_v = z$ *which shows that the pair* (B, T) *has a point of coincidence.*

If one assumes that $T(X)$ *is a complete subspace of* X, *then analogous argument establish (1) and (2). The remaining cases pertain essentially to the previous cases. Indeed, if B(X) is a complete subspace of X, then* $z \in B(X) \subset S(X)$ *and if* $A(X)$ *is complete then* $z \in A(X) \subset T(X)$. *Then (1) and (2) are completely established .Since the pair* (A, S) *and* (B, T) *are weakly compatible at u and v respectively, i.e.* $z = A_u = S_u = B_v = T_v$,

therefore $Az = AS_u = SA_u = Sz$

and $Bz = BT_v = TB_v = Tz$,#

Which show that z is a common coincidence point of both the pairs (A, S) and (B, T). *Now it remains to show that* $Az = Bz = Sz = Tz = z$. *To do this, we* $p = t_{2n}$, $q = z$ *with* $\beta = 1$ *in* (2.1)

$M(At_{2n}, Bz, \phi(t)) \geq min\{ M(St_{2n}, At_{2n}, t), M(Tz, Bz, t),$

$M(Tz, At_{2n}, t), M(St_{2n}, Bz, t), M(St_{2n}, Tz, t)\}$,*and*

$N(At_{2n}, Bz, \phi(t)) \leq max\{ N(St_{2n}, At_{2n}, t), N(Tz, Bz, t),$

$N(Tz, At_{2n}, t), N(St_{2n}, Bz, t), N(St_{2n}, Tz, t)\}$

which on letting $n \to \infty$, *reduces to*

$M(z, Bz, \phi(t)) \geq min\{1, M(Bz, z, t)\} \geq M(Bz, z, t)$, *and* #

$N(z, Bz, \phi(t)) \leq max\{1, N(Bz, z, t)\} N(Bz, z, t)$

As $M(z, Bz, \phi(t)) \leq M(Bz, z, t)$ *and* $N(z, Bz, \phi(t)) \geq N(Bz, z, t)$, *therefore* $M(z, Bz, \phi(t)) = C$ *and* $N(z, Bz, \phi(t)) = D$. *Due to lemma 1.1, we get* $H(t) = C$ *and* $G(t) = D$ *for all* $t > 0$ *and* $Bz = z$. *Hence* $Bz = Tz = z$. *Thus z is a common fixed point of A, B, S and T.*

References

[1] K.Atanassov, intuititonistic fuzzy Sets and Systems **20** (1986).86, 96.
[2] A.Georgeand veeramani.On some results in fuzzy metric spaces,Fuzzy Sets and system **64**, (1994).
[3] O.Kramosil and J.Michalck, Fuzzy metric and statistical metric space, Kybernetika 11, (1975), 326-334.
[4] S.Kutukcu. D.Torkoglu and C. Yildiz, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, Commun. Korean Math. Soc. **21** (2006), no. 1.89-100.
[5] J.H. Park, Intuitionistic fuzzy metric spaces. Chhaos, Solitions and Fractals **22** (2004),1039- 1046.
[6] S.Sharma and J.K.Tiwari. Common fixed point in fuzzy metric spaces, J.Korean Soc. Math Educ: Ser. B: Pure Appl. Math. **12** (2005). no. 1, 17-31.
[7] L.A.Zadeh, Fuzzy sets Inform and Control **8** (1965), 338-353.