Coincidence and Common Fixed Point Theorems for Nonlinear Contractive in Intuitionistic Fuzzy Metric Space

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Abstract: The purpose of this paper is to obtain a new common fixed point theorem by using a new contractive condition and properties in Intuitionistic fuzzy metric spaces.

Keyword: Triangular norm, triangular co-norm, intuitionistic fuzzy metric space, fuzzy metric space, fixed point.

1. Introduction


2. Preliminaries

Definition 2.1 A binary operation * : [0,1] × [0,1] → [0,1] is a continuous t-norm if it satisfies the following conditions:
(a) * is commutative and associative;
(b) * is continuous;
(c) a * 1 = a for all a ∈ [0,1];
(d) a * (b * c) = (a * b) * c whenever a ≤ c and b ≤ d for each a, b, c, d ∈ [0,1].

Definition 2.2. A binary operation ± : [0,1] × [0,1] → [0,1] is a continuous t-conorm if it satisfies the following conditions:
(a) ± is commutative and associative;
(b) ± is continuous;
(c) a ± 0 = a for all a ∈ [0,1];
(d) a ± (b ± c) = (a ± b) ± c whenever a ≤ c and b ≤ d for each a, b, c, d ∈ [0,1].

Definition 2.3. A three tuple (X, M, N, ≤, ±) is said to be a fuzzy metric space if X is an arbitrary set, a continuous t-norm and M a fuzzy set on [x, y, z] satisfying the following condition, for all x, y, z ∈ X and t, s > 0:
(a) M(x, y, z) = 0 #
(b) M(x, y, z) = 1 for all x, y, z ∈ X and t, s > 0 if and only if x = y, #
(c) M(x, y, z) = M(y, x, z), #
(d) M(x, y, z) ≤ M(x, y, z + t) #
(e) M(x, y, z + t) = M(x, y, z) ± M(x, y, z + t) #
(f) M(x, y, z) is left continuous,
(g) f(x, y, z) : M(x, y, z) → [0,1] is left continuous.

Definition 2.4. A 5-tuple (X, M, N, ≤, ±, ≤, ±) is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if X is an arbitrary set, a is a continuous t-norm, ± is a continuous t-conorm and M, N are fuzzy sets on X² × [0,∞) satisfying the following conditions:
(a) M(x, y, t) + M(x, y, t) ≤ 1 for all x, y, t ∈ X and t > 0;
(b) M(x, y, t) = 0 for all x, y, t ∈ X and t ≤ 0;
(c) M(x, y, t) = 1 for all x, y, t ∈ X and t > 0 if and only if x = y,
(d) M(x, y, t) = M(y, x, t) for all x, y ∈ X and t > 0;
(e) M(x, y, t) ≤ M(x, y, t + s) for all x, y, t, s ∈ X and t, s > 0.
\[ f(x, y, \cdot) : [0, \infty) \rightarrow [0, 1] \] is left continuous for all \( x, y \in X \)

\[ M(x, y, \cdot) = 1 \]

\[ M(x, y, 0) = 1 \] for all \( x, y \in X \)

\[ M(x, y, t) = 0 \] for all \( x, y \in X \) and \( t > 0 \) if and only if \( x = y \).

\[ N(x, y, t) = N(y, x, t) \] for all \( x, y \in X \) and \( t > 0 \).

\[ M(x, y, t + s) \geq M(x, y, t) \] for all \( x, y, z \in X \) and \( s, t > 0 \).

\[ M(x, y, t) \rightarrow 0 \] as \( t \rightarrow 0^+ \) for all \( x, y \in X \).

Then \( M, N \) is called an intuitionistic fuzzy metric on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and degree of non nearness between \( x \) and \( y \) with respect to \( t \), respectively.

**Definition 2.5:** Let \((X, M, N, \Phi) \) be an intuitionistic fuzzy metric space. Then

(a) a sequence \( \{x_n\} \) in \( X \) is said to be Cauchy sequence if, for all \( \varepsilon > 0 \) and \( \delta > 0 \),

\[ \lim_{n,m \to \infty} M(x_n, x_m, \varepsilon) = 1, \lim_{n,m \to \infty} N(x_n, x_m, \delta) = 0 \]

(b) a sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if, for all \( \varepsilon > 0 \),

\[ \lim_{n \to \infty} M(x_n, x, \varepsilon) = 1, \lim_{n \to \infty} N(x_n, x, \delta) = 0 \]

**Definition 2.6:** An intuitionistic fuzzy metric space \((X, M, N, \Phi) \) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent.

**Definition 2.7:** An intuitionistic fuzzy metric space \((X, M, N, \Phi) \) is said to be compact if every sequence in \( X \) contains a convergent subsequence.

### 3. Main Results

**Theorem 2.1:** Let \((X, M, N, \Phi) \) be an intuitionistic fuzzy metric space. Let \( A, B, S \) and \( T \) be mappings from \( X \) into itself satisfying,

\[ \begin{align*}
(A(x) \in T(x)) & \text{ and } (B(x) \in S(x)) \text{ for all } x \in X \\
M(A(x), B(x), 0(t)) & \geq \min \{ M(S(x, A(x), t), M(T(x, B(x), t), M(T(x, A(x), t)) \} \ldots (2.1) \\
M(A(x), B(x), (2 - \beta)t) & \leq \max \{ N(S(x, A(x), t), M(T(x, B(x), t), N(T(x, A(x), t)) \} \ldots (2.2)
\end{align*} \]

for all \( x \in X \) and \( t \in (0, \infty) \). Then

\[ \begin{align*}
\text{if } & M((A(x) \in T(x)) \text{ and } (B(x) \in S(x)) \text{ for all } x \in X \\
\text{and } & M((A(x), B(x), (2 - \beta)t) \leq \max \{ N(S(x, A(x), t), M(T(x, B(x), t), N(T(x, A(x), t)) \} \ldots (2.2)
\end{align*} \]

for all \( x \in X \) and \( t \in (0, \infty) \). Also assume that there exist \( A(X) \neq \emptyset \) and \( B(X) \neq \emptyset \) and \( T(X) \neq \emptyset \) and \( S(X) \neq \emptyset \) and \( \beta \in (0, \infty) \).

**Proof.** Let \( t_0 \) be an arbitrary point in \( X \). Since \( A(t_0) \subseteq T(X) \) and \( B(t_0) \subseteq S(X) \), one can find a point \( t_1 \) in \( X \) with \( A(t_1) = T(t_1) = S(t_1) = B(t_1) \).

Inductively one can construct \( t_1 \) such that \( A(t_1) = T(t_1) = S(t_1) = B(t_1) \) and \( t_1 \rightarrow \infty \).

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let \( \max \left\{ \lambda(z_{2n-1}, z_{2n+1}), \lambda(z_{2n-1}, z_{2n+1}) \right\} \) reduce to.

\[ M \left( \alpha_n, \beta_n, \sigma(q) \right) \geq \min \left\{ M(z_{2n-1}, z_{2n+1}), M(z_{2n-1}, z_{2n+1}) \right\}, \]

and

\[ M \left( \alpha_n, \beta_n, \sigma(q) \right) \geq \max \left\{ M(z_{2n-1}, z_{2n+1}), M(z_{2n-1}, z_{2n+1}) \right\}. \]

Similarly, one can show that

\[ M(z_{2n+2}, z_{2n+2}) \geq \min \left\{ M(z_{2n+1}, z_{2n+1}), M(z_{2n+1}, z_{2n+1}) \right\}, \]

and

\[ M(z_{2n+2}, z_{2n+2}) \leq \max \left\{ M(z_{2n+1}, z_{2n+1}), M(z_{2n+1}, z_{2n+1}) \right\}. \]

therefore for all \( n \) (even and odd), we have,

\[ \lambda(z_{2n}, z_{2n+1}) \leq \max \left\{ \lambda(z_{2n-1}, z_{2n+1}), \lambda(z_{2n-1}, z_{2n+1}) \right\} \]

which is true yields,

\[ \lambda(z_{2n+1}, z_{2n+1}) \leq \min \left\{ \lambda(z_{2n}, z_{2n+2}), \lambda(z_{2n}, z_{2n+2}) \right\} \]

by repeated application of the above inequality (for \( m = 1, 2, 3, \ldots \), we get

\[ \lambda(z_{2n+2}, z_{2n+2}) \leq \min \left\{ \lambda(z_{2n+1}, z_{2n+2}), \lambda(z_{2n+1}, z_{2n+2}) \right\} \]

Thus for each \( \lambda \in (0,1) \), we have

\[ E_{n+1} = \inf \{ z > 0 : M(z_{2n+2}, z_{2n+2}) \geq 1 - \lambda \} \]

\[ \geq \inf \{ z > 0 : M(z_{2n+1}, z_{2n+2}) \geq 1 - \lambda \} \]

\[ \geq \inf \{ z > 0 : M(z_{2n}, z_{2n+1}) \geq 1 - \lambda \} \]

\[ \geq \max \{ M(z_{2n}, z_{2n+1}), M(z_{2n}, z_{2n+1}) \} \]

\[ \geq \max \{ M(z_{2n}, z_{2n+1}), M(z_{2n}, z_{2n+1}) \} \]

Now appearing to lemma 1.2. We conclude that \( \{z_n\} \) is a Cauchy sequence in \( X \).

Now suppose that \( S(X) \) is a complete subspace of \( X \), then by observing that the subsequence \( \{z_{2n+1}\} \) which is contained in \( S(X) \), must get a limit \( z \) in \( S(X) \). Let \( x \in S(X) \) then \( Ax = z \). As \( \{z_n\} \) is a Cauchy sequence containing a convergent subsequence \( \{z_{2n}\} \), therefore the sequence \( \{z_n\} \) also convergent implying thereby the convergence of \( \{z_n\} \) being a subsequence of the convergent subsequence \( \{z_{2n}\} \).

To prove \( Ax = z \), set \( y = u \) and \( q = e_{2n+2} \), with \( q = 1 \) in 2.1 and 2.2

\[ M(Au, e_{2n+1}, \sigma(q)) \leq \max \left\{ M(Te_{2n+1}, e), M(Su, e_{2n+1}, e) \right\} \]

\[ M(Au, e_{2n+1}, \sigma(q)) \leq \max \left\{ M(Te_{2n+1}, e), M(Su, e_{2n+1}, e) \right\}. \]
Which on letting $x \rightarrow \infty$ reduces to

\[ H(z, B_z, a(t)) \geq \min \{ M(z, z, t), M(z, B_z, t), M(z, z, t) \} \text{ and } \]

\[ H(z, B_z, a(t)) \leq \max \{ M(z, z, t), M(z, B_z, t), M(z, z, t) \} \]

implying thereby

\[ H(z, B_z, a(t)) \geq \min \{ H(z, B_z, t), H(z, B_z, t), H(z, z, t) \} \text{ and } \]

\[ H(z, B_z, a(t)) \leq \max \{ H(z, B_z, t), H(z, B_z, t), H(z, z, t) \} \text{ therefore } \]

\[ H(z, B_z, t) = C \text{ and } H(z, B_z, t) = D \cdot \]

Again in view of lemma 1.1, we have $H(z) = C$ and $H(z) = D$ for all $t > 0$ and hence $B_z = z$. Thus one gets $B_z = T_z = z$ which shows that the pair $(B, T)$ has a point of coincidence.

If $x$ assumes that $T(X)$ is a complete subspace of $X$, then analogous argument establish (1) and (2). The remaining cases pertain essentially to the previous cases. Indeed, if $B(X)$ is a complete subspace of $X$, then $z \in B(X) = S(X)$ and if $A(X)$ is complete then $z \in A(X) = T(X)$. Then (1) and (2) are completely established. Since the pair $(A, S)$ and $(B, T)$ are weakly compatible at $u$ and $v$ respectively, i.e. $z = A_1 = S_1 = B_1 = T_1$, therefore $A_z = A_{S_z} = S_{A_z} = S_z$ and $B_z = B_{T_z} = T_{B_z} = T_z$.

Which show that $z$ is a common coincidence point of both the pairs $(A, S)$ and $(B, T)$. Now it remains to show that $A_z = A_z = S_z = T_z = z$. To do this, we $p = t_{2z}, q = a$ with $\theta = 1$. In (2.1)

\[ M(z, B_z, a(t)) \geq \min \{ M(z, z, t), M(T_z, B_z, t), M(z, z, t) \} \text{ and } \]

\[ M(z, B_z, a(t)) \leq \max \{ M(z, z, t), M(T_z, B_z, t), M(z, z, t) \} \]

which on letting $n \rightarrow \infty$, reduces to

\[ M(z, B_z, a(t)) \geq \min \{ M(z, z, t), M(z, z, t) \} \text{ and } \]

\[ M(z, B_z, a(t)) \leq \max \{ M(z, z, t), M(z, z, t) \} \]

As $M(z, B_z, a(t)) \leq M(B_z, B_z, t)$ and $M(z, B_z, a(t)) \geq M(B_z, B_z, t)$, therefore $M(z, B_z, a(t)) = C$ and $N(z, B_z, a(t)) = D$, due to lemma 1.1, we get $H(z) = C$ and $H(z) = D$ for all $t > 0$ and $B_z = z$. Hence $B_z = T_z = z$. Thus $z$ is a common fixed point of $A, B, S$ and $T$.

References