Formula for Computation of Wigner Function for Two Mode Quantized Electromagnetic Field and its Application in Detection of Nonclassicality

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Abstract: **An expression for Wigner function is obtained for a density matrix which is expressible in photon number state representation. The result is then applied to detect the presence of nonclassicality. A conclusion is drawn that the magnitude of most negative value of Wigner function decreases with the decrement in degree of entanglement.**

Keywords: Wigner function, nonclassicality, entangled states, two mode electromagnetic field, negativity of states

1. Introduction

Nonclassical states in quantum mechanics have been a subject of intensive theoretical and experimental investigations due to their wide ranging applications particularly in quantum information processing. Various schemes have been proposed[1-4] towards this end. Quantum measure of nonclassical behaviour of electromagnetic field states has been shown to be based on negative regions of quasiprobability distributions[5]. To every density operator $\hat{\rho}$, there corresponds a quasi-probability $\phi(z,s)$. For normal ordering s=1, $\phi(z,1)$ which is Sudarshan's diagonal coherent state representation of $\hat{\rho}$ or Glauber's $P(\alpha)$ representation[6], is not easy because in many cases it is highly singular. In contrast to this situation, Wigner function $W(z) = \phi(z,0)$ is always well behaved corresponding to Weyl ordering.

The paper is divided into 2 sections. In section 2 we obtain an expression for Wigner function for two mode electromagnetic field. In section 3 we apply this formula to test nonclassicality.

2. Expression for Wigner function for two mode quantized electromagnetic field

The two mode density operator $\hat{\rho}$ in photon number state representation is given by,

$$\hat{\rho} = \sum_{n_1,n_2=0}^{M} \sum_{m_1,m_2=0}^{M} \rho_{n_1,n_2,m_1,m_2} |n_1 n_2\rangle\langle m_1 m_2|$$

with known $\rho_{n_1,n_2,m_1,m_2} = \langle n_1 n_2|\hat{\rho}|m_1 m_2\rangle$. The corresponding Wigner function $W(z_1,z_2)$ is[7]

$$W(z_1,z_2) = \frac{1}{\pi^4} \sum_{m_2 n_2} \sum_{m_1 n_1} \rho_{n_1,n_2,m_1,m_2} \int\int d^2\alpha_1 d^2\alpha_2 \langle m_1 m_2|\hat{D}(\alpha_1,\alpha_2)|n_1 n_2\rangle e^{-(\alpha_1 z_1^* - \alpha_1^* z_1)} e^{-(\alpha_2 z_2^* - \alpha_2^* z_2)}, \quad (2)$$

Using two mode displacement operator and references[8,9], we then get

$$W(z_1,z_2) = \frac{4}{\pi^2} e^{-2(|z_1|^2+|z_2|^2)} \Big[\sum_{n_1,n_2=0}^{M} (-1)^{n_1+n_2} L_{n_1}(4|z_1|^2) L_{n_2}(4|z_2|^2) \rho_{n_1,n_2,n_1,n_2} + \Big\{ \sum_{p_1,p_2=1}^{M} (2|z_1|)^{p_1} (2|z_2|)^{p_2} \times$$
$$e^{i(p_1\theta_1+p_2\theta_2)} \sum_{n_1,n_2=0}^{M} (-1)^{n_1+n_2} L_{n_1}^{p_1}(4|z_1|^2) L_{n_2}^{p_2}(4|z_2|^2) \rho_{n_1,n_2,n_1+p_1,n_2+p_2} \left(\frac{n_1!}{(n_1+p_1)!}\right)^{\frac{1}{2}} \left(\frac{n_2!}{(n_2+p_2)!}\right)^{\frac{1}{2}} + c.c. \Big\} +$$
$$\Big\{ \sum_{p_1,p_2=1}^{M} (2|z_1|)^{p_1} (2|z_1|)^{p_2} e^{i(p_1\theta_1-p_2\theta_2)} \sum_{n_1,n_2=0}^{M} (-1)^{n_1+n_2} L_{n_1}^{p_1}(4|z_1|^2) L_{n_2}^{p_2}(4|z_2|^2) \rho_{n_1,n_2+p_2,n_1+p_1,n_2} \times$$
$$\left(\frac{n_1!}{(n_1+p_1)!}\right)^{\frac{1}{2}} \left(\frac{n_2!}{(n_2+p_2)!}\right)^{\frac{1}{2}} + c.c. \Big\} + \Big\{ \sum_{p_1,p_2=1}^{M} (2|z_1|)^{p_1} (2|z_2|)^{p_2} e^{ip_1\theta_1} \sum_{n_1,n_2=0}^{M} (-1)^{n_1+n_2} L_{n_1}^{p_1}(4|z_1|^2) L_{n_2}(4|z_2|^2) \times$$
$$\rho_{n_1,n_2,n_1+p_1,n_2} \left(\frac{n_1!}{(n_1+p_1)!}\right)^{\frac{1}{2}} + c.c. \Big\} +$$
$$\Big\{ \sum_{p_1,p_2=1}^{M} (2|z_1|)^{p_1} (2|z_2|)^{p_2} e^{ip_2\theta_2} \sum_{n_1,n_2=0}^{M} (-1)^{n_1+n_2} L_{n_1}(4|z_1|^2) L_{n_2}^{p_2}(4|z_2|^2) \rho_{n_1,n_2,n_1,n_2+p_2} \left(\frac{n_2!}{(n_2+p_2)!}\right)^{\frac{1}{2}} +$$
$$c.c. \Big\} \Big] \quad (3)$$

where $L_n^p(x)$ is associated Laguerre function.

3. Detection of Nonclassicality via Wigner Function

Example 1. $|\psi\rangle = \frac{1}{\sqrt{\lambda^2+2}}(\lambda|00\rangle + |01\rangle + |10\rangle)$

and corresponding Wigner function is

$$W(z_1,z_2) = \frac{4}{\pi^2}e^{-2(x^2+y^2)}[\lambda^2 + 4x^2 + 4y^2 + 8xy\cos(\theta_1 - \theta_2) + 4$$

When $\theta_1 = 180^\circ, \theta_2 = 0^\circ \Rightarrow \cos(\theta_1 - \theta_2) = -1$

$$W(z_1,z_2) = \frac{4}{\pi^2}e^{-2(x^2+y^2)}[\lambda^2 + 4x^2 + 4y^2 - 8xy + 4y\lambda - 4x\lambda - 2$$

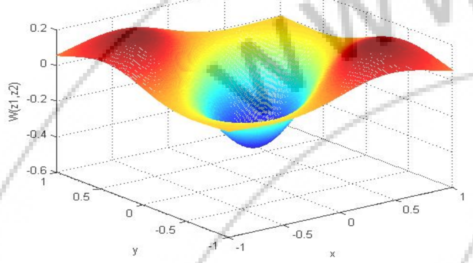

Figure 1: *Wigner function for (5) when λ=0*

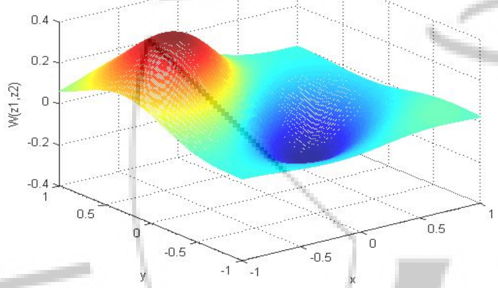

Figure 2: *Wigner function for (5) when λ =1*

Table 1: Negativity and most negative value of Wigner function with different values of λ for (4)

Value of λ	Value of negativity	Most negative value of Wigner function (5)
0.0	0.5	-0.4056
0.5	0.44445	-0.3457
1.0	0.33335	-0.2285

It is evident from Table 1 that the negativity increases as the value of λ decreases and from figure 1 and 2 the region of negativity increases as the value of λ decreases.

Example 2. Werner state:

$\hat{\rho} = p|\psi\rangle\langle\psi| + \frac{1-p}{4}\sum_{n_1,n_2}|n_1 n_2\rangle\langle n_1 n_2|$

With $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and corresponding Wigner function is

$$W(z_1,z_2) = \frac{2}{\pi^2}e^{-2(x^2+y^2)}p[4x^2 + 4y^2 - 8xy - 2] + \frac{1-p}{4\pi^2}$$

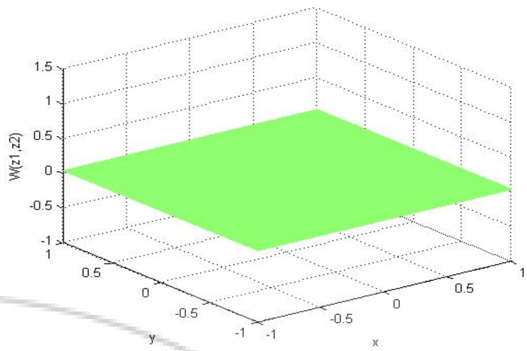

Figure 3: Wigner function for (6) when p=0

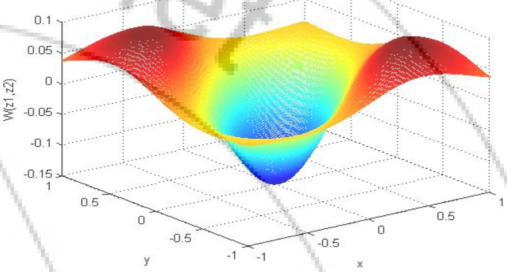

Figure 4: Wigner function for (6) when p=0.4

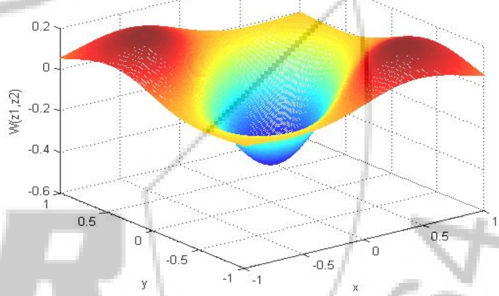

Figure 5: *Wigner function for (6) when p=1*

Table 2: The most negative value of Wigner function for different p.

Value of p	Most negative value of Wigner function (6)
0.0	--
0.2	-0.0608
0.4	-0.1470
0.6	-0.2332
0.8	-0.3194
1.0	-0.4056

From Table 2, it is observed that the range of negative values of Wigner function increases as p increases for Werner state.

4. Conclusion

Explicit formula for computing Wigner function has been obtained. It has been applied to different states to detect their nonclassicality. We can conclude from above data that the region of negativity as well as the magnitude of most negative value decreases as the degree of entanglement decreases.

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