International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Independent Lict Subdivision Domination in Graphs

M. H. Muddebihal[1], Naila Anjum[2]

Department of Mathematics, Gulbarga University Gulbarga-585106, Karnataka, India

Abstract: *Let $S(G)$ be the subdivision graph of G. The lict graph $n[S(G)]$ of $S(G)$ is a graph whose vertex set is the union of the set of edges and set of cutvertices of $S(G)$ in which two vertices are adjacent if and only if the corresponding members are adjacent or incident. A dominating set D of the lict graph $n[S(G)]$ is an independent dominating set if $\langle D\rangle$ is independent in $n[S(G)]$. The minimum cardinality of the smallest independent dominating set of $n[S(G)]$ is called the independent lict subdivision dominating set of G and is denoted by $i_{ns}(G)$. In this paper many bounds on $i_{ns}(G)$ were obtained in terms of the vertices , edges and other different parameters of G and not in terms of the elements of $n[S(G)]$. Further , its relation with other different dominating parameters are also obtained.*

Subject Classification Number: AMS-05C69, 05C70

Keywords: Subdivision, Lict graph, domination number, independent domination number

1. Introduction

In this paper, all the graph considered here are simple, finite, nontrivial , undirected and connected . The vertex set and edge set of graph G are denoted by $V(G) = p$ and $E(G) = q$ respectively. Terms not defined here are used in the sense of Harary [1].

The degree, neighbourhood and closed neighbourhood of a vertex v in a graph G are denoted by $deg(v), N(v)$, and $N[v] = N(v) \cup \{v\}$ respectively. For a subset S of , the graph induced by $S \subseteq V$ is denoted by $\langle S\rangle$.

As usual, the maximum degree of a vertex (edge) in G is denoted by $\Delta(G)(\Delta'(G))$. For any real number x, $\lceil x\rceil$denotes the smallest integer not less than x and $\lfloor x\rfloor$denotes the greater integer not greater than x .

A vertex cover in a graph G is a set of vertices that covers all the edges of G.The vertex covering number $\alpha_o(G)$is the minimum cardinality of a vertex cover in G.A set of vertices/edges in a graph G is called independent set if no two vertices/edges in the set are adjacent. The vertex independence number $\beta_0(G)$ is the maximum cardinality of an independent set of vertices. The edge independence number $\beta_1(G)$of a graph G is the maximum cardinality of an independent set of edges.
A set D of graph $G = (V, E)$ is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in . The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating set of G.

A set F of edges in a graph G is called an edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number$\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of . Edge domination number was studied by S.L.Mitchell and Hedetniemi in [3].

A dominating set D of a graph G is a strong split dominating set if the induced subgraph$\langle V - D\rangle$ is totally disconnected with only two vertices. The Strong Split domination number $\gamma_{ss}(G)$ of a graph G is the minimum cardinality of a strong split dominating set of G. See [2].

A dominating set D of a graph $G = (V, E)$ is an independent dominating set if the induced subgraph $\langle D\rangle$ has no edges. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.

A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path (u, v, w). The graph obtained from a graph G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by$S(G)$.

Let $V' = V\big[n\big(S(G)\big)\big]$ .A set $D' \subseteq V'$is said to be dominating set of $n[S(G)]$, if every vertex in $V' - D'$ is adjacent to some vertex in D'. The domination number of $n\big(S(G)\big)$ is denoted by $\gamma_n\big(S(G)\big)$ and is the minimum cardinality of dominating set in $n\big(S(G)\big)$.

Analogously, we define Lict Independent Subdivision Domination Number as follows..
A dominating set D' of a Lict graph $n[S(G)]$ is an independent dominating set if the induced subgraph $\langle D'\rangle$ has no edges.
The independent lict subdivision domination number is the minimum cardinality of an independent dominating set of $n[S(G)]$ and is denoted by $i_{ns}(G)$.
In this paper , many bounds on $i_{ns}(G)$ were obtained and expressed in terms of the vertices , edges and other parameters of G but not in terms of members of $n[S(G)]$.
Also we establish independentlict subdivision domination number and express the results with other different domination parameters of G.
We need the following Theorems.
Theorem A [2]: For any connected graph$G$, with ≥ 4 , $G \neq K_p$,$\gamma_{ss}(G) = \alpha_0(G)$.
Theorem B [2]: For any connected graph G, $\left\lceil\frac{p}{1+\Delta}\right\rceil \leq \gamma(G)$.

2. Results

We list out the exact values of $i_{ns}(G)$ for some standard graphs .

Theorem 1:

a. For any path P_p with $p \leq 2$ vertices

$i_{ns}(P_p) = p - 1$.

b. For any star $K_{1,p}$,

$i_{ns}(K_{1,p}) = p - 1$.

c. For any wheel W_p ,

$i_{ns}(W_p) = p - 1$.

Theorem 2. Let G be a connected graph of order ≥ 2 , then $i_{ns}(G) \geq \gamma(G)$ .Equality holds if $G = K_2$.

Proof : Let $D = \{v_1, v_2, \dots \dots \dots \dots \dots \dots v_n\}$ be the minimal dominating set of G , such that $|D| = \gamma(G)$. Let $S = \{e_1, e_2 \dots \dots \dots \dots e_n\}$ be the minimal edge dominating set of $S(G)$ and $C = \{c_1, c_2 \dots \dots \dots \dots \dots \dots . c_n\} \subseteq V[S(G)]$ is the set of cutvertices in (G) . By the definition of $[S(G)]$,$S \cup C \subseteq V[n(S(G))]$. Now consider a minimal dominating set of vertices $D_1 \subseteq S \cup C$ in $n[S(G)]$ such that $N[D_1] = V[n(S(G))]$.Then $\langle D_1 \rangle$ is the minimal dominating set of $n[S(G)]$. Further if $E(D_1) = \emptyset$ in the subgraph $\langle D_1 \rangle$, then D_1 itself form an independent dominating set of $n[S(G)]$. Otherwise, let us take $I = D_2 \cup D_2'$, where $D_2 \subseteq D_1$ and $D_2' \subseteq V[n(S(G)) - D_1]$,such that no two vertices in $\langle D_2 \cup D_2' \rangle$are adjacent .Hence the subgraph$\langle D_2 \cup D_2' \rangle$ forms an independent dominating set of $n[S(G)]$. Obviously the number of edges of $S(G)$ are more than that of G ,and which gives $V(G) \subset V[n(S(G))]$.Hence clearly $|D_2 \cup D_2'| > |D|$ resulting in $i_{ns}(G) > \gamma(G)$.
For equality, let$G = K_2$, we have $\gamma(G) = 1$.Further $S(K_2) = P_3$ and $n[S(K_2)] = C_3$ and obviously $i_{ns}(K_2) = 1$. *Hence* $\gamma(G) = i_{ns}(K_2) = 1$.

The next Theorem gives a lower bound for $i_{ns}(G)$ in terms of the vertices and edges of G.

Theorem 3: For any connected graph G, $i_{ns}(G) \leq p + q - 3$.

Proof : Let G be a (p, q) graph , then $V[S(G)] = p + q$.Now let $F = \{v_1, v_2, \dots \dots \dots \dots \dots v_n\}$ be the set of vertices in $S(G)$ such that $|F| = \alpha_0(S(G))$and $E = \{e_1, e_2 \dots \dots \dots \dots \dots e_n\}$ be the set of edges in $S(G)$ such that $|E| = \beta_0(S(G))$, and by the definition of $n[S(G)]$, $V[n(S(G))] = E(S(G)) \cup C(S(G))$ where $C(S(G))$ is the set of cutvertices in $S(G)$. We consider a set$F' \subseteq E(S(G))$.If F' corresponds to such vertices $D = \{u_1, u_2, \dots \dots \dots \dots \dots u_i\}$ of $n[S(G)]$,which are independent and $N[D] = V[n(S(G))]$, then D forms the independent dominating set of $n[S(G)]$ with $|D| = i_{ns}(G)$ .If all $u_i \in D$ are such that $deg\ u_i \neq 0$ in $\langle D \rangle$, then we take another set $D_1 = D_2 \cup D_2'$ such that $D_2 \subseteq D$ and $D_2' \subseteq V[n(S(G))] - D$ .If all the vertices in $\langle D_2 \cup D_2' \rangle$ are such that $deg\ u_i = 0$ for each $u_i \in D_2 \cup D_2'$, then $\langle D_2 \cup D_2' \rangle$ forms an independent dominating set of $n[S(G)]$. Now we can easily verify that

$$\langle D_2 \cup D_2' \rangle \leq [\alpha_0(S(G)) - 2] + [\beta_0(S(G)) - 2] + 1$$
$$\leq \alpha_0(S(G)) + \beta_0(S(G)) - 3$$
$$= V(S(G)) - 3$$
$$= p + q - 3$$

Hence $i_{ns}(G) \leq p + q - 3$

Theorem 4: For any connected (p, q)graph G, $n[S(G)] \neq K_p$, $i_{ns}(G) + \gamma_{ssn}(S(G)) \leq 2q + C'$, where C' is the number of cutvertices in $S(G)$.

Proof: Suppose G has $p \leq 3$ then $\gamma_{ssn} -$ set does not exist. Now we consider any graph with$p \geq 4$, such that $n[S(G)] \neq K_p$.
Since $i_{ns}(G) \leq \beta_0[n(S(G))]$ and from Theorem B $\gamma_{ssn}(S(G)) = \alpha_0[n(S(G))]$.
Further $i_{ns}(G) + \gamma_{ssn}(S(G)) \leq \alpha_0[n(S(G))] + \beta_0[n(S(G))]$

$$= V[n(S(G))]$$
$$= E(S(G)) + C(S(G))$$
$$= 2q + C'$$

Hence $i_{ns}(G) + \gamma_{ssn}(S(G)) \leq 2q + C'$

Theorem 5:An independent lictsubdivision dominating set $D \subseteq V[n(S(G))]$ is minimal if and only if for each vertex $x \in D$, one of the following condition holds:

a. There exists a vertex $y \in V[n(S(G))] - D$ such that $N(y) \cap D = \{x\}$.

b. x is an isolated vertex in $\langle D \rangle$.

c. $\{V[n(S(G))] - D\} \cup \{x\}$is connected.

Proof: Suppose D is a minimal independent lict subdivision dominating set of G and there exists a vertex $x \in D$ such that x does not hold any of the above conditions. Then for some vertex v, the set $D_1 = D - \{v\}$ forms an independent dominating set of G, by the conditions (a) and (b). Also by (c), $\langle V[n(S(G))] - D \rangle$is disconnected This implies that D_1 is an independent lict subdivision dominating set of G , a contradiction.

Conversely, suppose for every vertex $x \in D$, one of the above statements hold. Further, if D is not minimal, then there exists a vertex $x \in D$ such that $D - \{x\}$ is an independent lict subdivision dominating set of G and there exists a vertex $y \in D - \{x\}$ such that y dominates x . That is $y \in N(x)$. Therefore, x does not satisfy (a) and (b), hence it must satisfy (c). Then there exists a vertex $y \in V[n(S(G))] - D$ such that $N(y) \cap D = \{x\}$. Since $D - \{x\}$ is an independent lict subdivision dominating set of G , then there exists another vertex $z \in D - \{x\}$ such that $z \in N(y)$. Therefore $w \in N(y) \cap D$, where $w \neq x$, a contradiction to the fact that $N(y) \cap D = \{x\}$. Clearly, D is a minimal independent lict subdivision dominating set of G .

Theorem 6: For any connected graph G, $\left\lceil \frac{p}{1+\Delta} \right\rceil \leq i_{ns}(G)$.

Proof : Since from Theorem B ,we have $\left\lceil \frac{p}{1+\Delta} \right\rceil \leq \gamma(G)$ and also from Theorem 2 , it follows $\left\lceil \frac{p}{1+\Delta} \right\rceil \leq i_{ns}(G)$.

Theorem 7: For any connected (p,q)graph G ,$i(L(G)) \leq i_{ns}(G)$.

Proof : Since $V[n(S(G))] \supseteq V[L(G)]$ and by the definition of lict graph the result follows.

Theorem 8: For any graph G , $i_{ns}(G) \leq p-1$.Equality holds if G is a tree but the converse may not be true.

Proof : We consider the following cases.

Case 1: Suppose G is a tree . Then clearly for any tree T ,$\beta_1(S(T)) = p-1$.

Any set of $(p-1)$ independent edges of $S(T)$is an independent dominating set of$n(S(T))$. Hence $i_{ns}(T) = p-1$.

Case 2: Suppose G is not a tree. Then we consider $V(G) = \{v_1, v_2, \dots \dots \dots \dots \dots \dots v_p\}$and $E(G) = \{e_1, e_2 \dots \dots \dots \dots \dots e_q\}$.Suppose $W = \{w_1, w_2, \dots \dots \dots \dots . w_q\}$ be the set of vertices subdividing the edges $e_1, e_2 \dots \dots \dots \dots \dots e_q$ respectively .Clearly $E[S(G)] = 2E(G)$. Now let $I = \{e_1, e_2 \dots \dots \dots \dots \dots e_n\} \subseteq E[S(G)]$ be the set of all end edges in $S(G)$and $I' = E[S(G)] - I$. Then there exist an independent set of edges $J = \{e_1, e_2 \dots \dots \dots \dots \dots e_k\} \subseteq I'$,such that J forms an edge dominating set of $S(G)$.Without loss of generality J forms a set $D_1 \subseteq V[n(S(G))]$ which is also a dominating set of $n[S(G)]$.If $\langle D_1 \rangle$ is such that $N[D_1] = V[n(S(G))]$ then D_1 is the independent dominating set of $n[S(G)]$. Otherwise ,we consider a set $D_2 = D_1' \cup D_2'$, where $D_1' \subseteq D_1$ and $D_2' \subseteq V[n(S(G))] - D_1$,with $deg(v_i) = 0$ for each $v_i \in D_1' \cup D_2'$,then $\langle D_1' \cup D_2' \rangle$ is an independent dominating set of $n[S(G)]$.It is known that $\langle D_1' \cup D_2' \rangle \subseteq V[n(S(G))]$ and $|V(G)| < |V[n(S(G))]|$ .Hence $|D_1' \cup D_2'| < |V(G)| - 1$,which gives $i_{ns}(G) \leq p-1$.

The following Theorem relates edge domination number of $G$and $i_{ns}(G)$.

Theorem 9: For any non-trivial connected graph ,$\gamma'(G) \leq i_{ns}(G)$.

Proof: Let D be the i_{ns} - set of G. Let $F = \{e_1, e_2 \dots \dots \dots \dots \dots e_q\} \subseteq E(G)$, such that for each $e_i \in F$, $i = 1,2, \dots \dots \dots \dots \dots \dots q$, $N(e_i) \cap F = \phi$.Then $|F| = \gamma'(G)$.Since $E(G) \subset E(S(G))$ and by the definition of lict graph ,$D \subseteq E(S(G)) \cup C(S(G))$,where $C(S(G))$ is the set of cutvertices in $S(G)$ .This implies that $|F| \leq |D|$ which gives $\gamma'(G) \leq i_{ns}(G)$.

Theorem 10: For any connected (p,q) graph G , $i_{ns}(G) \geq \left\lceil \frac{p}{2} \right\rceil$.

Proof: We consider the following cases

Case 1 : Suppose G is a tree. Then by the Theorem 8 , we have $i_{ns}(T) = p-1 > \left\lceil \frac{p}{2} \right\rceil$.Hence the result follows.

Case 2 : Suppose G is not a tree .Then there exists at least one edge joining two distinct vertices of a tree ,which forms a cycle.From the above case 1,$V\left(n(S(G))\right) \geq E(S(T)) \cup C(S(T)) + 1$ which gives $i_{ns}(G) \geq i_{ns}(T) + 1 = p - 1 + 1$ resulting in $i_{ns}(G) \geq p \geq \left\lceil \frac{p}{2} \right\rceil$.Hence the result.

Theorem 11: If G is a connected, non-trivial graph G , then $\left\lceil \frac{diam(G)+1}{2} \right\rceil \leq i_{ns}(G)$.

Proof: Suppose $E(S(G)) = \{e_1, e_2 \dots \dots \dots \dots \dots e_n\}$ and $C(S(G)) = \{c_1, c_2 \dots \dots \dots \dots \dots \dots . c_n\} \subseteq V[S(G)]$ be the edge set and cutvertex set of $S(G)$ respectively.Then $[n(S(G))] = E(S(G)) \cup C(S(G))$.

Let$S = \{e_1, e_2 \dots \dots \dots \dots \dots e_j\}$;$1 \leq j \leq n$ be the diametral path in G. Then $|S| = diam(G)$. Let D be the dominating set in $n[S(G)]$ .Ifeach$u_i \in D$ is such that $\deg u_i = 0$ then $|D| = i_{ns}(G)$. Otherwise , we take $D_1 = D_2 \cup D_2'$ where $D_2 \subseteq D$ and $D_2' \subseteq V[n(S(G))] - D$ such that for all $u_i \in \langle D_2 \cup D_2' \rangle$, $\deg u_i = 0$ .Hence $|D_2 \cup D_2'| = i_{ns}(G)$.Further since $S \subseteq E(S(G))$ which means $S \subseteq V[n(S(G))]$ and $D_2 \cup D_2'$ is an i_{ns} – set , the diameteral path includes atmost$i_{ns}(G) - 1$ vertices which belongs to the neighbourhood of $D_2 \cup D_2'$ in $n[S(G)]$. Hence$diam(G) \leq i_{ns}(G) + i_{ns}(G) - 1$, which gives $\left\lceil \frac{diam(G)+1}{2} \right\rceil \leq i_{ns}(G)$. One can easily verify the equality.

The next Theorem gives Nordan-Gaudass type of result

Theorem 12: Let G be a graph such that both G and $\bar{G}$ have no isolated edges then,

$$i_{ns}(G) + i_{ns}(\bar{G}) \leq 2\left\lceil \frac{p}{2} \right\rceil$$

$$i_{ns}(G) \, . \, i_{ns}(\bar{G}) \leq \left\lceil \frac{p}{2} \right\rceil^2$$

3. Future Scope

In this paper we surveyed selected results on independent dominating sets in lictgraphs.These results established key relationships between the independent lict subdivision domination number and other parameters including the domination number. Further , these results established optimal upper bounds on the independent domination number in terms of the order itself and the order and the maximum degree. The results established here are open for np problems , which if solved shed more light on the complexity of independent domination number.

References

[1] F.Harary, (1969), Graph Theory, Adison Wesley, Reading Mass (61-62).

[2] S.Mitchell and S.T.Hedetniemi, (1977), Edge Domination in Trees. In: Proc .8th S.E.Conf.OnCombinatorics,Graph Theory and Computing , utilitas Mathematica ,Winnipeg, 19 (489-509)

[3] V.R.Kulli,(2010),Theory Of Domination In Graphs. Vishwa International Publication, Gulbarga India.

Authors Profile

M. H. Muddebihal is Professor in Mathematics, Department of Mathematics, Gulbarga University, Gulbarga

Naila Anjum is Research Scholar, Department of Mathematics, Gulbarga University, Gulbarga .