procedure itself can be reduced if one of the polynomials is temporarily regarded as a monomial. This procedure is explained in the example below.

Example 11: In order to multiply the polynomials $x+y+z$ and $a+b$, each letter denotes a monomial, we will replace the polynomial $x+y+z$ with the monomial m. Then we get

$$m(a+b)=ma+mb=(x+y+z)a+(x+y+z)b$$
$$=xa+ya+za+xb+yb+zb.$$

Further, after we look at two or three more examples, we formulate the rule for polynomial multiplication.
Polynomial is multiplied by a polynomial when each term of the first polynomial is separately multiplied by each term of the second polynomial and the obtained products are added together.
Finally, as with multiplying a polynomial by a monomial, we assure the students in the correctness of the procedure by solving several examples where we calculate the numerical values of the factors and the obtained product.

Note. While learning polynomial multiplication, it is useful to point out to students the following: we first multiply the first term of the first polynomial by all the terms of the second polynomial, and then repeat the procedure with the second term of the first polynomial and so on. The last is very important because the students frequently perform the multiplication without any order thus resulting in multiplication errors.
The adoption of polynomial division must be preceded by first adopting division of a polynomial by a monomial and then moving on to the general case. With that, the initial considerations must include examples in which the result of the division is a polynomial i.e. the result is not a fractional rational expression. The following example provides the procedure for adopting division of a polynomial by a monomial.

Example 12: In the beginning, it is useful to remind the students of how we divide multi-digit number by a one-digit number, for example

$$375:5=(300+70+5):5=300:5+70:5+5:5.$$

Then similarly, we formulate the rule for division of a polynomial by a monomial:

Polynomial is divided by a monomial when each term of the polynomial is divided by the monomial and the obtained quotients are added together.

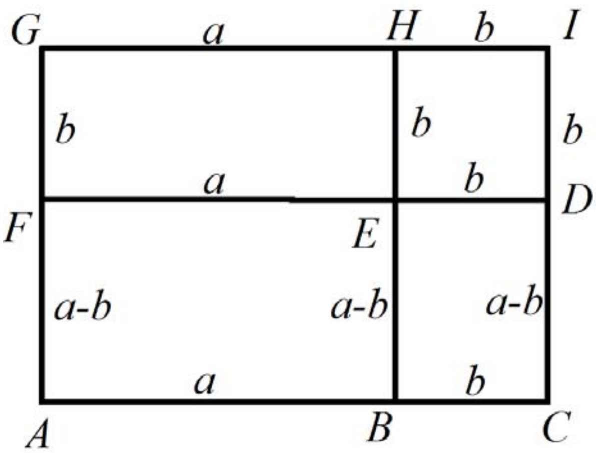
Further, we need to look at two or three examples where each term of the polynomial is divisible by the monomial and conclude that a given polynomial written in regular form is divisible by a given monomial if and only if each polynomial term is divisible by the monomial. In the end, we need to look at simple examples where the polynomial is not divisible by the monomial i.e. the result of the division is a fractional rational expression.

We can easily adopt polynomial division by using the similarity with division of natural numbers written down as polynomials. In brief, this is shown in the following example.

Example 13: First, we remind students how we divide two multi-digit numbers and we write the procedure in polynomial form, for example,

$$\begin{array}{l} 165:11=15 \\ \underline{-11} \\ \ \ 55 \\ \underline{\ -55} \\ \ \ \ 0 \end{array}$$

and



$$\begin{array}{l} (1\cdot 10^2+6\cdot 10+5):(1\cdot 10+1)=1\cdot 10+5 \\ \underline{\pm 1\cdot 10^2 \pm 1\cdot 10} \\ \ \ 5\cdot 10+5 \\ \ \ \underline{\pm 5\cdot 10+5} \\ \ \ \ \ \ \ \ \ 0 \end{array}$$

where for the polynomial entry we note that first

$$1\cdot 10^2:1\cdot 10=1\cdot 10,$$

and then $5\cdot 10:1\cdot 10=5$. At this point, we suggest to students, formally in the polynomial entry, to write down the power to the variable x instead to the number 10, and after they get

$$\begin{array}{l} (1\cdot x^2+6\cdot x+5):(1\cdot x+1)=1\cdot x+5 \\ \underline{\pm 1\cdot x^2 \pm 1\cdot x} \\ \ \ 5\cdot x+5 \\ \ \ \underline{\pm 5\cdot x+5} \\ \ \ \ \ \ \ 0 \end{array}$$

to multiply the polynomials $x+5$ and $x+1$. After we complete the multiplication of these two polynomials, we solve two or three more examples, and explain the operation for polynomial division. Finally, we have to look at simple examples where a polynomial cannot be divided by a polynomial and introduce division with a remainder. In this case, certainly, it is advisable to use the theorem for division with a remainder for the set of natural numbers. At that, we will come to the following conclusion.

Division with remainder of two polynomials is always possible, at that, dividing the polynomial $P(x)$ by the polynomial $Q(x)$ means that we have to find the polynomials $q(x)$ and $r(x)$, the polynomial exponent $r(x)$ is lower than the polynomial exponent $Q(x)$, where $P(x)=Q(x)q(x)+r(x)$.

2.2. Formulas of abridged multiplication

a) The study of formulas of abridged multiplication is an essential part of the content while studying algebraic rational expressions. In primary education, the most frequently adopted formulas include those for product of the sum and difference of two monomials, square of a binomial and factoring the sum and difference of cubes. The simplest way is to derive these formulas directly, which is performed in the following examples.

a1) We multiply the binomials $a-b$ and $a+b$ and we get:

$$(a-b)(a+b)=a^2+ab-ba-b^2=a^2-b^2.$$

Consequently, for the expression a^2-b^2 we introduce the concept *difference of squares* and we form the rule:

The product of the sum and difference of the two terms is equal to the difference of their squares.

we adopt the same by solving some examples. In addition, it is desirable to use the same rule for simpler numerical calculations, which requires looking into examples of the following kind:

$$187\cdot 213=(200-13)\cdot(200+13)=200^2-13^2$$
$$=40000-169=39831,$$

and illustrating the same, i.e. providing a geometric "proof", by which we point out the correlation between different content in mathematics curriculum. The last can be achieved by the drawing on the right, where

$$a^2-b^2=P_{ABHG}-P_{EDIH}=P_{ABEF}+P_{FEHG}-P_{EDIH}$$
$$=P_{ABEF}+P_{BCIH}-P_{EDIH}=P_{ABEF}+P_{BCDE}$$
$$=P_{ACDF}=(a-b)(a+b).$$

a2) When for the expression $(a+b)^2$, where a and b are two unlike terms, we introduce the concept *square of a binomial*, based on previous acquired knowledge we perform the following calculation

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2=a^2+2ab+b^2$$

Thus, we form the following rule:

The square of a binomial is the sum of the first term squared, two-times the product of the first and second terms and the second term squared.

Subsequently, we adopt the stated rule by solving several examples, bearing in mind that we have to solve numerical tasks of the following type as well:

$$213^2=(200+13)^2=200^2+2\cdot 200\cdot 13+13^2$$
$$=40000+5200+169=45369,$$

and illustrate the same, i.e. provide a geometric "proof", by which we point out the correlation between different content in mathematics curriculum. The last can be achieved by the drawing on the right, where:

$$(a+b)^2=P_{ABCD}=P_{AEFG}+P_{GFID}+P_{EBHF}+P_{FHCI}$$
$$=a^2+ab+ba+b^2=a^2+2ab+b^2.$$

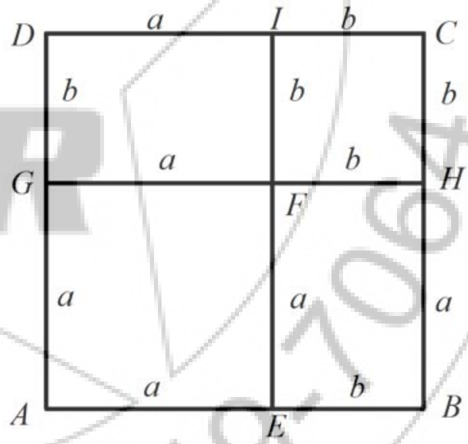
a3) Practice shows that, factoring the sum and difference of cubes of unlike terms a and b must be directly introduced. In addition, we have

$$(a-b)(a^2+ab+b^2)=a^3+a^2b+ab^2-ba^2-ab^2-b^3$$
$$=a^3-b^3,$$
$$(a+b)(a^2-ab+b^2)=a^3-a^2b+ab^2+ba^2-ab^2+b^3$$
$$=a^3+b^3.$$

and if we want the students to conscientiously adopt all the before mentioned formulas, we must prepare an appropriate system of tasks, which will facilitate adoption of formulas in the first place, and then enable students to use them for solving more complex tasks. Let us note that, as with the cases with the formulas in a1) and a2), we can present a geometrical illustration for the last two formulas as well. Thus, for example, in the case of the penultimate formula, the illustration can be made with the help of a cube with edges a and b, a cuboid with sides a,b and $a-b$ and a cuboid with sides a,a and $a-b$.

b) Factoring polynomials is an important part of the study of this subject matter. In addition, for the adoption of this material, we suggest three types of factoring:

It is advisable to use the analogy for factoring numbers while examining the question of factoring polynomials, thus emphasizing that one of the aims of factoring polynomials is reducing fractional-rational expressions. We need to mention here that, although students study this subject matter of reducing fractional - rational expressions later, it is appropriate to look at and explain several basic examples for this type of reducing, so that the students became aware of the need for factoring polynomials.



Students encounter numerous difficulties while factoring polynomials because while adopting this subject matter, unlike other algebraic subject matter, the successful adoption of the theory does not guarantee successful solving of tasks. These difficulties occur because successful factoring of polynomials requires not only knowledge of standard logarithms, but also most frequently, it is necessary to perform some prior transformations, which, in most cases, are not very simple. In order to overcome the mentioned difficulties, while studying polynomials, it is necessary to solve tasks of the following type.

Example 14: 1) Present the monomial $15x^3y^2$ as a product in three different ways, using $5x$ as one of the factors.

2) Calculate all the representations of $6x^2y^3 - 2x^3y^2$ as a product of a monomial and polynomial.

3) Write the polynomial equal to $-2xyz + \frac{x}{3}$, when in the given polynomial you replace the given terms with their opposite terms.

We will note here that, well-planned systems of tasks prepared by the teacher are a prerequisite for successful adoption of factoring polynomials. In this paper, we will not focus on the before-mentioned methods for factoring polynomials, nor on the systems of tasks, but we will point out to the possibility for correlation of the discussed subject matter with the number theory. For this purpose, we will use Sophie Germain's identity. Namely, by using formulas a1) and a2) we prove Sophie Germain's identity.

$$\begin{aligned} a^4 + 4b^4 &= (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2 \\ &= [(a^2)^2 + 2a^2 \cdot 2b^2 + (2b^2)^2] - (2ab)^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab), \end{aligned}$$

Then we move on to the following system of tasks:

Example 15: 1) Prove that for every natural number $n > 1$ the number $n^4 + 4$ is composite.

2) Prove that there are infinitely many natural numbers x for which $n \in \mathbf{N}$, the number $z = n^4 + x$ is composite.

3) Prove that the number $2^{10} + 5^{12}$ is composite.

4) Prove that the natural number $2^{2006} + 5^{2004}$ is composite.

5) Prove that the number $4n^4 + 1$, $n \in \mathbf{N}$ is a prime number only if $n = 1$.

6) Prove that for every $n > 1$ the natural number $n^4 + 4^n$ is composite.

7) Prove that the natural number $2005^4 + 4^{2005}$ is composite.

c) Identical transformations are an essential part of the subject of algebraic rational expressions. In most cases, the need for identical transformations is justified by the necessity for simpler presentation of complex expressions, which is motivated by the simplicity in calculating the numerical value of algebraic expressions. However, if students have adopted that the simplest type of a monomial or polynomial is its representation in regular form, and later notice that the numerical value of the expression $\frac{1}{2}ab \cdot 7ab^2$ where $a = \frac{1}{2}, b = 2$ can be directly calculated in a simpler way rather than reducing it to a regular form, then it is normal for them to doubt the appropriateness of the studied material.

The identical transformations can be associated with the representation of one and the same number in different forms, where the choice of form, in most cases, depends on the operations that need to be performed with that number. For example, if we consider the numerical expression $\frac{72 \cdot 25 \cdot 61}{9}$, then it is convenient to write the number 72 in the form $8 \cdot 9$, but if we have to multiply 72 by 13, then it is convenient to write the number 72 in the form $70 + 2$.

2.3. Fractional rational expressions

a) In primary education, the study of fractional rational expressions is reduced to adoption of operations involving the same. Before we start adopting the operations involving fractional rational expressions, we have to introduce the concept of *equivalent* fractional rational expressions (algebraic fractions) and prove the following theorem:

Let A, B, C and D be algebraic expression. Then $\frac{A}{B} = \frac{C}{D}$ if and only if $AD = BC$ and the fractional rational expressions $\frac{A}{B}$ and $\frac{C}{D}$ have same domains.

While studying this particular area, it is best to use the similarity with regular fractions. However, we have to explain to students that not all adopted concepts for regular fractions can be transferred to rational fractional expressions. This can be best illustrated by choosing well-suited examples.

Example 16. We assign the following algebraic fractions $\frac{3a+20}{2a+20}$ and $\frac{a+b}{2a+b}$ to the students, and we ask the following question: What can we say about these algebraic fractions? The most frequent answer is that the algebraic fraction $\frac{3a+20}{2a+20}$ is improper, and that the algebraic fraction $\frac{a+b}{2a+b}$ is proper.

The simplest way to eliminate this error is to put $a = -4$ in the first fraction, and then $a = -13$ in order to make the students see that we get the proper fraction $\frac{2}{3}$ in the first case, and the improper fraction $\frac{19}{6}$ in the second case. Further, for the $\frac{a+b}{2a+b}$ fraction it is sufficient to make algebraic rational expression calculation with $a = -3, b = 10$, and then with $a = -4, b = 2$ and to convince ourselves that we get the fractions $\frac{7}{4}$ and $\frac{1}{3}$.

Here we will mention the basic property of algebraic fractions which is derived from the equality $\frac{A}{B} = \frac{AP}{BP}$, where A, B and P are integral algebraic expressions, bearing in mind the domain. Further, writing down the last equality as $\frac{AP}{BP} = \frac{A}{B}$, students must adopt that in the case when the numerator and the denominator of the algebraic fraction have a common factor, then the fraction can be reduced and thus the value remains unchanged (once again bearing in mind the domain). At the end of this part, using the basic property of fractions, students must adopt the change of the sign in front of the fraction, i.e. they must adopt the equalities:

$$\frac{A}{B} = \frac{A(-1)}{B(-1)} = \frac{-A}{-B} \text{ and } \frac{-A}{B} = \frac{(-A)(-1)}{B(-1)} = \frac{A}{-B} = -\frac{A}{B}.$$

b) Addition and subtraction of fractions must be viewed as an identical transformation of a sum of fractions within one fraction. Before we move on to addition and subtraction of fractions, it is necessary to revise the rules for addition and subtraction of regular fractions. Then, by analogy, it is easier to introduce the rules of addition and subtraction of algebraic fractions. At that, looking at the sum and difference of two algebraic fractions with different denominators, the question arises to replace the same with fractions with equal denominators. At this point, we remind the students of the basic property of algebraic fractions and we replace the fractions $\frac{A}{B}$ and $\frac{C}{D}$ with the equal fractions $\frac{AD}{BD}$ and $\frac{BC}{BD}$, respectively, which have equal denominators and thus we get

$$\frac{A}{B} \pm \frac{C}{D} = \frac{AD}{BD} \pm \frac{BC}{BD} = \frac{AD \pm BC}{BD}.$$

We will mention here one more time, that by using the analogy with multiplication and division of regular fractions, in an identical manner, we can introduce the operations for multiplication and division of algebraic fractions. Therefore, we will not look into these operations in more detail. Regarding raising algebraic fractions to a power, with a natural number as an exponent, we will only mention that we need to analyze this as a partial case of multiplication of several equal algebraic fractions, which easily leads us to $(\frac{A}{B})^k = \frac{A^k}{B^k}$.

In the end of this section, we will note that, when adopting addition and subtraction of algebraic fractions, it is advisable to first adopt addition of algebraic fractions with monomials as denominators and abide by the following plan:

- First we look at simple tasks when the denominators do not have common factors, for example, $\frac{3a}{5b} + \frac{2a}{3c}$,
- Then we look at tasks where the denominator of one of the fractions is a multiple of the denominators of the other fractions, for example,
$$\frac{7a}{30b^3} + \frac{2a}{15b^2} + \frac{a}{10b}, \text{ and}$$
- Finally, we look at tasks when none of the denominators is a common multiple of the denominators, but some denominators have common multiples, for example,
$$\frac{2a}{15b^3c} + \frac{2a}{9b^2c^2} - \frac{4a}{12bc^3}.$$

It is advisable to abide by this suggested plan while adopting addition and subtraction of algebraic fractions with polynomials as denominators.

3. Conclusion

In the previous review we had addressed some issues related to the methodology of the study of rational algebraic expressions in primary education and presented some views on the same. Certain conclusions and methodological recommendations are the result of practical work with students of this age, with the described approach:

- Improve internal integration of rational algebraic expressions with: real numbers, number theory and geometry,
- Allow to increase the adoption of the subject with a greater understanding by the most of students, and
- Enable declarative knowledge that students gain easier, effectively to move into procedural knowledge, what actually is one of the goals of mathematics.

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