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Algebraic Rational Expressions in Mathematics

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Abstract: ***Algebraic rational expressions are a necessary component of the mathematics course in primary education. Hence, the need for appropriate methodical elaboration that enables enhanced acquisition of this abstract matter, which is the basis for improved adoption of numerous content areas in secondary education. This paper attempts to provide methodical guidelines for adoption of algebraic rational expressions with special attention to the possible intra-disciplinary integration with theory of numbers, geometry and writing numbers in expanded form.***

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1. Introduction

Study of algebraic rational expressions, as well as all other areas incorporated in the instruction of mathematics, has the purpose to enable accomplishment of educational, instructional and practical aims of mathematics teaching. The fundamental tasks while studying algebraic rational expressions are:

- adopting the concept of algebraic rational expression, as well as other concepts, propositions and algorithms directly related to the concept of algebraic rational expressions,
- developing logical thinking through conscientious adoption of calculation and identical transformations,
- developing functional thinking through calculating the values of algebraic rational expressions with different assigned values for the variable,
- adopting methods, approaches and operations which are necessary for the adoption of structural knowledge for algebraic rational expressions and concepts related to them,
- developing skills necessary for data processing,
- adopting the basic interpretations of the concept algebraic rational expression, the tasks which can be reduced to algebraic rational expressions, some of their fundamental applications in mathematics and other disciplines, and
- training students to apply acquired mathematical knowledge.

The following aims need to be achieved with students when studying algebraic rational expressions:
- adoption of letter symbols related to algebraic, arithmetic and geometric objects,
- understanding the concepts of algebraic rational expression and integral algebraic expression,
- adoption of operations concerning algebraic rational expressions and implementing the same with identical transformations of algebraic rational expressions, and
- adoption of the algorithm for calculating numerical values of algebraic rational expressions when the values for the variable are given.

The successful adoption of operational and structural knowledge for algebraic rational expressions is conditioned by the knowledge of:
- real number, constant, variable and allowed value of the variable,
- numerical expression and value of a numerical expression,
- operations concerning rational numbers and their properties,
- power and power operations,
- equations and solving the same,
- real function (linear function), and
- area of a square, rectangular and other plane figures.

By its structure, this subject matter is rather complex and covers adoption of several concepts and propositions concerning these concepts.

The following concepts have to be adopted when studying this subject matter: algebraic rational expression (integral and fractional), numerical value of an algebraic rational expression, monomial (regular form, monomial coefficient, monomial degree, like and opposite monomials), polynomial (regular form, polynomial degree and polynomial coefficient), factor and multiple of a polynomial, greatest common factor and least common multiple of two polynomials, domain of a fractional rational expression, equivalent algebraic rational expressions.

Apart from the adoption of the before mentioned concepts, the following should be adopted as well: operations concerning monomials and polynomials, formulas of abridged multiplication, factoring polynomials, calculating the greatest common factor and lowest common multiple of two polynomials, operations concerning fractional rational expressions.

2. Methodical notes on adoption of algebraic rational expressions

The introduction of the algebraic expression must be well motivated due to its abstractness. For that purpose we can use the already adopted formulas $i = \frac{Kp}{100}$ and $s = vt$, for percentage and calculating the distance depending on the speed and time at uniform motion, respectively. In these cases, we have to comment on the right sides of these formulas so that students recognize the common characteristics of $\frac{Kp}{100}$ and vt.

A starting point for the introduction of algebraic rational expressions are the concepts real numbers, variable, addition, subtraction, multiplication and division performed with numbers and variables. With that, the concept algebraic rational expression can be introduced in the following manner.

Every real number or variable is an algebraic rational expression, and if A and B are algebraic rational expressions then $A+B$, $A-B$, AB and $A:B$ are algebraic rational expressions as well.

Furthermore, after we list examples of algebraic rational expressions, depending whether the divisor contains a variable or not, the algebraic rational expressions can be divided into *integral* and *fractional* rational expressions, respectively. Practice shows that with the introduction of fractional rational expressions we also need to introduce the concept of *domain* of a fractional rational expression. It is best to introduce this concept by using the inductive method i.e. to look at examples such as the following.

Example 1. Let us look at the following fractional rational expression $\frac{2a}{a+2}$. For $a = 1$ the expression takes the value $\frac{2 \cdot 1}{1+2} = \frac{2}{3}$, for $a = 2$ value $\frac{2 \cdot 2}{2+2} = 1$, but for $a = -2$ the denominator $a + 2$ takes the value $-2 + 2 = 0$, which means that the expression $\frac{2a}{a+2}$ does not make any sense. Furthermore, if $a \neq -2$, then $a + 2 \neq 0$ which means for $a \neq -2$ the expression $\frac{2a}{a+2}$ makes sense and, in this case, we say that the domain of the expression $\frac{2a}{a+2}$ is a set of all real numbers other than -2. The last can be symbolically written as $D = \{a \mid a \neq -2\}$.

After we look at one or two more examples, we can give the following definition.

The domain of an algebraic rational expression with one variable is a set of values for the variable for which the expression makes sense.

Before we move further on, we need to introduce the concept of numerical value of an algebraic rational expression and using the same, provide the definition for equality of algebraic rational expressions. We can do this as shown in the following example.

Example 2. a) It is advisable to adopt the concept of numerical value of algebraic rational expression intuitively. For example, if in the expression $a^2 - 2ab$, we assign $a = 3$ and $b = 2$ for the variables, we obtain the numerical expression $3^2 - 2 \cdot 3 \cdot 2$ with value of -3. Then, we explain to students that the number -3 is called numerical value of the algebraic rational expression $a^2 - 2ab$ for the values of the variables $a = 3$ and $b = 2$. Hence, it is important for the students to adopt that the numerical value of the expression varies depending on the change of the variables, which is not the case with the numerical expressions.

b) Firstly, we look at the numerical values of algebraic expressions of type $5x + 3$ and $3x - 1$ with value of the variable $x = -2$ and we get that they are equal to -7, and then for the value of the variable, for example 0 , we get that the first has the numerical value 3 and the second -1.

In the consequent step, we calculate the numerical values of expressions of type $1 + \frac{1}{x^2+1}$ and $\frac{x^2+2}{x^2+1}$ for several values for the variable x, for example $-3, -2, \frac{1}{2}, \sqrt{2}, \sqrt{3}$ and 4, we conclude that for each value of the variable these algebraic rational expressions have equal numerical values. Further on, we inform the students that the expressions $1 + \frac{1}{x^2+1}$ and $\frac{x^2+2}{x^2+1}$ have equal numerical values for each value of the variable, which belongs to the domain, which is not the case with the expressions $5x + 3$ and $3x - 1$. Now we are prepared to give the following definition:

Two algebraic rational expression are equal if they have equal domains and if they accept same numerical values for every value of the variable that belongs to that domain.

The basic type of tasks, done in class with the aim to provide an answer whether students understand the definitions for concepts related to algebraic rational expression, integral or fractional, are the following:

i) Tasks where from a given set of mathematical objects students have to identify the algebraic expressions, and tasks where students independently write down algebraic expressions,
ii) Tasks where students find the domain and tasks where students compose expressions according to a given domain,
iii) Tasks where students calculate the numerical value of an expression and tasks where students compose an expression for which the numerical value has already been assigned,
iv) Tasks which bring students closer to understanding the concept of functional dependence, without mentioning the term itself but by pointing out the essence which stems from the actual task.

2.1 Monomials

The concepts *monomial* and *polynomial* are introduced as types of the concept integral algebraic expressions. At that, the separation of the two concepts depends on the involvement of the operations addition, subtraction and multiplication. Thus, for example, we have the following definition for a monomial.

A monomial is an expression that consists of constants (numbers), variables and the sign for multiplication.

After we introduce the monomial, students are also introduced to the following terms: *regular form of a monomial, monomial coefficient, main value of a monomial, degree of a monomial, like monomials and opposite monomials*, which must be illustrated with examples. We must choose these examples so that they can reveal the essence of these concepts thus facilitating students towards conscientious adoption of the mentioned concepts. Therefore, when introducing the concept of regular form of a monomial, it is best to use the similarity that exists with numbers. We can use this similarity in the following manner.

"We mention that we usually want to write the product $2 \cdot 5 \cdot 12 \cdot 25$ in a shorter form, therefore, we perform the calculation and write it down in the form

$$2 \cdot 5 \cdot 12 \cdot 25 = 2^3 \cdot 3 \cdot 5^3 = 3000 .$$

Then, we explain to students that by analogy to the previous entry, the monomial $2 \cdot 4 \cdot 6 \cdot a \cdot a \cdot a \cdot a \cdot a$ can be written in shorter form in the following manner:

$$2 \cdot 4 \cdot 6 \cdot a \cdot a \cdot a \cdot a \cdot a = 2^4 \cdot 3a^5 = 48a^5 ,$$

In this case, the given monomial is written in regular form. At that, it is desirable to pay attention that the regular form of the monomial is more appropriate if, for example, the given task is to calculate the numerical value of the monomial for $a = \frac{1}{2}$. However, we cannot insist on this too much because if, for example, we need to calculate the value of the monomial $24ababaab$ for $a = 4$ and $b = \frac{1}{2}$, then it is better to note that $ab = 2$ and continue the calculation."

Finding the monomial coefficient and main value is essential because the introduction and giving meaning to concepts related to the concept of monomial, coefficient and main value have vital meaning. We can successfully introduce these concepts using examples of the following type.

Example 3: a) Find the numerical factor and the product of the variables in the monomials

$$4ax,\ 4y, \tfrac{1}{3}x,\ -2x,\ x^2y,\ -x^2y,\ -0{,}532y^2zt .$$

Reduce the monomial to its regular form and find the numerical factor and the product of the variables:

$$3x^2 \cdot (-4)x\ ,\ 5abc \cdot 7ab\ ,\ \tfrac{4}{7}a^3bc \cdot 0{,}7ax\ .$$

After we look at the given example, we are prepared to give the following definition.

The numerical factor of a monomial in its regular form is called the monomial coefficient, and the product of its variables – main value of the monomial.

Before we move on to adoption of operations with monomials, we have to introduce the concepts of like and opposite monomials. For that purpose, we can look at the following example.

Example 4: a) Find the coefficients and the main values of the following monomials

$$4x^2,\ -3x^2,\ \tfrac{1}{3}x^2,\ -2{,}453x^2 .$$

b) Find the coefficients and the main values of the monomials $5xa^2$ and $-5xa^2$.

After we look at the given example, we can give the following definition.

Two monomials that have same main value are called like monomials. Two like monomials are opposite if their coefficients are opposite numbers.

Before we move further on, we need to introduce the concept for *degree of a monomial*. We will only mention here that students must become aware that for constants with the exception of zero, discussed as monomials, by agreement, we accept that they have zero degree, and the degree of other monomials is equal to the sum of all the exponents of the factors in the main value.

When introducing operations involving monomials, it is useful to use the similarity with operations involving numbers. We have to bear in mind that numbers are rational expressions as well. Monomial multiplication is defined by writing a monomial in its regular form and this fact is of great importance. Namely, multiplication cannot be defined as a sum of equal summands, but it is carried out by using algorithms for multiplication of numbers and the rules for power multiplication i.e. power properties. Therefore, based on the multiplication of numbers and the rules for power multiplication we have:

Product of two monomials is a monomial whose coefficient is equal to the coefficient product of the factors, and the main value is equal to the product of the main values of the factors written in regular form.

Our next step is to look at raising monomials to a power when the exponent is a natural number. The procedure that enables conscientious adoption of raising monomials to a power is presented in the following example.

Example 5: As with real numbers, the product

$$2x^2y^3 \cdot 2x^2y^3 \cdot 2x^2y^3$$

is written down as $(2x^2y^3)^3$. For this product we have

$$\begin{aligned}(2x^2y^3)^3 &= 2x^2y^3 \cdot 2x^2y^3 \cdot 2x^2y^3 \\ &= 2 \cdot 2 \cdot 2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot y^3 \cdot y^3 \cdot y^3 \\ &= 2^3(x^2)^3(y^3)^3 = 8x^6y^9.\end{aligned}$$

Then we look at one or two more examples, and we form the rule:

A monomial is raised to the power when each factor is raised to the same exponent and the obtained powers are multiplied.

When introducing monomial division, it is useful to use the similarity with power division. The adoption of monomial division is not accompanied by difficulties because the rule is simple: we calculate the quotient of the monomial coefficients and divide the exponents that have an equal base. Thus, we have to restrict ourselves to the case when the exponent of the powers with a same base in the dividend is higher than the exponent in the divisor because students have not adopted powers whose exponent is lower or equal to zero. Consequently, we should not discuss examples as $a^{3m} : a^{5m}$ or $x^{k-2} : x^{k+2}$.

Further on, by using suitable examples, we have to demonstrate to the students that division of monomials is reverse to the multiplication of monomials. This means that, side by side with solving examples where we calculate the quotient of two monomials we have to solve tasks as the ones provided in the following example.

Example 6: a) Check whether the division was correctly performed

1) $12a^2x^3y : 3ax = 4ax^2y$, 2) $12a^3x^5y^2 : 4ax = 3ax^3y$.

б) If $6,4 \cdot 3,12a^3b = 19,968a^3b$, without doing any calculation, find:

1) $19,968a^3b : 3,12a^3b$, 2) $19,968a^3b : 6,4a^3b$, and

3) $19,968a^3b : 6,4$.

We can draw analogous conclusions regarding addition and subtraction of monomials. Addition reduces to finding an expression identical to two or more monomials linked by the sign +, whereas, subtraction reduces to addition of monomials where the subtrahend is replaced by its opposite monomial. Addition of like monomials is performed separately. Namely, the sum of two like monomials is a monomial similar to the given ones, with a coefficient equal to the sum of the summands. Further on, while adopting addition and subtraction of monomials, it is useful to comment on how we can check the addition with the help of subtraction and vice versa. In addition, the link between addition and subtraction, as the one between multiplication and division, should be used for calculating the unknown parts (minuend, subtrahend, summand, factor, dividend and divisor).

2.1. Polynomials

A polynomial is introduced as an algebraic sum of monomials, with special emphasis on the concepts of binomial and trinomial. Further, students should adopt the reducing to like terms of a given polynomial through examples, and then we should introduce the concepts of regular term of a polynomial and polynomial degree. We can achieve all this in the manner shown in the following example.

Example 7. a) Let us look at the algebraic expressions

$a + 2bx + 3c + 5$, $x^2 + y$,

$4a^3 - 3a^4 + a^3 + \frac{1}{2}a^2 - 5a + a - a \cdot a + 5a^4$ and

$\frac{1}{2}x^2 - xy + 3y^2$.

As we can see, each of these expressions is an algebraic sum of several monomials. Thus, the expression $x^2 + y$ is an algebraic sum of two unlike monomials, and we call these expressions binomials, whereas the expression

$$\frac{1}{2}x^2 - xy + 3y^2$$

is an algebraic sum of three unlike monomials, and we call these expressions trinomials. Generally speaking, we have the following definition.

An algebraic expression is a polynomial if it is a monomial or an algebraic sum of two or more monomials that we call polynomial terms.

From the previously stated, we can say that binomials and trinomials are a special type of polynomials.

b) We can note that in the polynomial

$$4a^3 - 3a^4 + a^3 + \frac{1}{2}a^2 - 5a + a - a \cdot a + 5a^4 \quad (1)$$

all terms, except the term $a \cdot a$ are written down in regular form. We write $a \cdot a$ and get the polynomial:

$$4a^3 - 3a^4 + a^3 + \frac{1}{2}a^2 - 5a + a - a^2 + 5a^4 . \quad (2)$$

In the polynomial (2) we have like terms, so therefore, if we group them and then add them together, we get:

$$5a^3 + 2a^4 - \frac{1}{2}a^2 - 5a . \quad (3)$$

We call this transformation of the polynomial (2) *reducing to like terms of a polynomial*. We can note that in the polynomial (3), which is identical to polynomial (1), all its terms are in regular form and it does not contain like terms. Thus, we have the following definition:

A given polynomial is in its regular form if all its terms are in regular form and if the polynomial does not contain like terms.

c) As we can see, the highest degree in the polynomial (3) is 4. Similarly in the polynomial

$$7x^5 + 3x^3 - 2x^2 + 5x - 12 \quad (4)$$

the highest degree is 5, and in the polynomial

$$x^3a + 4x^2a^4 - 2x^2a + 5xa - 7 \quad (5)$$

the monomial $4x^2a^4$ has the highest degree equal to 6. Accordingly, we can determine the highest degree for every term in each polynomial, which we call *polynomial degree*. At that, polynomials with multiple variables are characterized by polynomial degree according to each variable. Therefore, for example, for the polynomial (5) the polynomial degree according to the variable x is equal to 3, and the polynomial degree according to the variable a is equal to 4.

Let us return to the polynomial (4). As we can see, its terms are arranged in descending order. Similarly, the polynomial (3) can be written as $2a^4 + 5a^3 - \frac{1}{2}a^2 - 5a$.

Further, the polynomial (5) can be written as

$$4x^2a^4 + x^3a - 2x^2a + 5xa - 7 ,$$

where the degrees of its terms are descending. According to this, we write down polynomials in a regular form with a descending order of the degree of the terms.

Before we move on to adoption of operations involving polynomials, we have to introduce the concept of opposite polynomial and rules for removing the parentheses. This can be done as in the example presented below.

Example 8. We look at the polynomial $P = 3x^2 - 2x + 4$, then we adjoin the polynomial $-P$, which we call the opposite to the polynomial P and whose terms are opposite to the terms of the polynomial P. Accordingly,

$$-P = -3x^2 + 2x - 4 \,. \quad (6)$$

Further, in (6) we replace the polynomial P and we get the following equality

$$-(3x^2 - 2x + 4) = -3x^2 + 2x - 4 \,. \quad (7)$$

We repeat the previous procedure with one or two more examples and then we look at equation (7). We explain to the students that by finding the opposite polynomial we have actually obtained the rule for removing parentheses, in the case when we have the sign "-" in front of the parentheses, and we formulate the following rule.

If we have a negative sign in front of the parentheses, then, we can omit the parentheses and the negative sign, and change the signs of the terms inside the parentheses into their opposites.

In the next step, we draw an analogy with real numbers, where we remind ourselves that for each $a > 0$ we have $+a = a$ and we explain that for each polynomial P we have $+P = P$. In the last equality, we replace the polynomial P and we get

$$+(3x^2 - 2x + 4) = 3x^2 - 2x + 4 \,. \quad (8)$$

We repeat the previous procedure with one or two more examples, and then we look at the equality (8). We explain to the students that in this case we have obtained the rule for removing parentheses when we have the sign "+" in front of the parentheses, and we formulate the following rule.

If there is no sign or there is a positive sign in front of the parentheses, then, we can omit the parentheses and the positive sign, and the terms inside the parentheses keep their existing signs.

In the next step, through several examples, the students should adopt the rule to take out the positive or negative sign in front of the parentheses.

Now we can introduce the operations for addition and subtraction of two polynomials by, firstly, using parentheses to write the needed sum in symbols, i.e. difference of the given polynomials, and then removing the parentheses and reducing the obtained polynomial to its regular form. At that, together with the students, we comment on the relations between the two operations, between the components of the different actions and try to find out when we can use them. Likewise, we have to explain the meaning of introducing the concept of opposite polynomial, which can be done as shown in the following example.

Example 9: As we have seen, the polynomial

$$-P = -3x^2 + 2x - 4$$

is opposite to the polynomial $P = 3x^2 - 2x + 4$. If we look for the polynomial opposite to the polynomial $-P$, we get the following polynomial

$$-(-P) = -(-3x^2 + 2x - 4) = 3x^2 - 2x + 4 = P \,.$$

After we look at two or three more examples, we can deduce that for each polynomial Q, the opposite polynomial $-Q$ is the actual polynomial Q. Then we calculate the sum

$$P + (-P) = 3x^2 - 2x + 4 + (-3x^2 + 2x - 4)$$
$$= 3x^2 - 2x + 4 - 3x^2 + 2x - 4 = 0.$$

After we look at two or three more examples, together with the students, we can make the following conclusion:

The sum of a polynomial and its opposite polynomial is equal to zero.

When learning about multiplication of polynomials, we first have to learn multiplying polynomial by a monomial. This can be done as in the example described.

Example 10: First, we revise that, when studying number sets, we have already adopted that, for example,

$$9 \cdot (40 + 7) = 9 \cdot 40 + 9 \cdot 7; \qquad (20 + 3) \cdot 6 = 20 \cdot 6 + 3 \cdot 6;$$
$$9 \cdot (40 - 7) = 9 \cdot 40 - 9 \cdot 7; \qquad (20 - 3) \cdot 6 = 20 \cdot 6 - 3 \cdot 6.$$

Further, we remind the students that if we write these equalities with the help of letters we get:

$$\begin{aligned} a(b + c) &= ab + ac; \qquad (a + b)c = ac + bc \\ a(b - c) &= ab - ac; \qquad (a - b)c = ac - bc. \end{aligned} \quad (9)$$

In the next step, we explain why we can replace a, b and c with monomials (see numerical value of a monomial). Now, we remind students that equalities (9) are correct even for unspecified finite number of summands in the parentheses and that again we can replace the letters with monomials. Thus, we formulate the rule for multiplying a monomial by a polynomial.

A polynomial is multiplied by a monomial when each term of the polynomial is multiplied by the given monomial, and the obtained products are added together.

Further, we look at examples like these:

$4x^2(3x + 5) = 4x^2 \cdot 3x + 4x^2 \cdot 5 = 12x^3 + 20x^2$ and

$$(2x^3 - 5x^2 + 3) \cdot (-3x^2) = 2x^3 \cdot (-3x^2) - 5x^2 \cdot (-3x^2) + 3 \cdot (-3x^2)$$
$$= -6x^5 + 15x^4 - 9x^2 \,.$$

We have to make sure the students are convinced that the multiplication algorithm provides a polynomial that is equal to the product of the monomial and polynomial. Therefore, it is advisable to calculate the numerical values of the factors and the numerical values of the obtained product for several values of the variable and state that they are equal for the appropriate values of the variable. Additionally, we can refer to the definition for equality of algebraic rational expressions in order to further convince the students in the correctness of the procedure. Thus, for the first polynomial where $x = -2, 0$ and 1 we have:

$4 \cdot (-2)^2 \cdot [3 \cdot (-2) + 5] = -16$ and

$12 \cdot (-2)^3 + 20 \cdot (-2)^2 = -16$;

$4 \cdot 0^2 \cdot [3 \cdot +5] = 0$ and $12 \cdot 0^3 + 20 \cdot 0^2 = 0$; and

$4 \cdot 1^2 \cdot [3 \cdot 1 + 5] = 32$ and $12 \cdot 1^3 + 20 \cdot 1^2 = 32$.

The product of two polynomials can be introduced similarly to the product of monomial and polynomial, but the procedure itself can be reduced if one of the polynomials is temporarily regarded as a monomial. This procedure is explained in the example below.

Example 11: In order to multiply the polynomials $x+y+z$ and $a+b$, each letter denotes a monomial, we will replace the polynomial $x+y+z$ with the monomial m. Then we get

$$m(a+b)=ma+mb=(x+y+z)a+(x+y+z)b$$
$$=xa+ya+za+xb+yb+zb.$$

Further, after we look at two or three more examples, we formulate the rule for polynomial multiplication.

Polynomial is multiplied by a polynomial when each term of the first polynomial is separately multiplied by each term of the second polynomial and the obtained products are added together.

Finally, as with multiplying a polynomial by a monomial, we assure the students in the correctness of the procedure by solving several examples where we calculate the numerical values of the factors and the obtained product.

Note. While learning polynomial multiplication, it is useful to point out to students the following: we first multiply the first term of the first polynomial by all the terms of the second polynomial, and then repeat the procedure with the second term of the first polynomial and so on. The last is very important because the students frequently perform the multiplication without any order thus resulting in multiplication errors.

The adoption of polynomial division must be preceded by first adopting division of a polynomial by a monomial and then moving on to the general case. With that, the initial considerations must include examples in which the result of the division is a polynomial i.e. the result is not a fractional rational expression. The following example provides the procedure for adopting division of a polynomial by a monomial.

Example 12: In the beginning, it is useful to remind the students of how we divide multi-digit number by a one-digit number, for example

$$375:5=(300+70+5):5=300:5+70:5+5:5.$$

Then similarly, we formulate the rule for division of a polynomial by a monomial:

Polynomial is divided by a monomial when each term of the polynomial is divided by the monomial and the obtained quotients are added together.

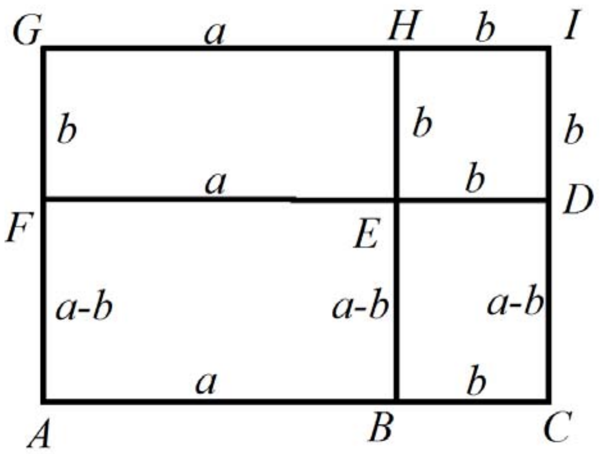
Further, we need to look at two or three examples where each term of the polynomial is divisible by the monomial and conclude that a given polynomial written in regular form is divisible by a given monomial if and only if each polynomial term is divisible by the monomial. In the end, we need to look at simple examples where the polynomial is not divisible by the monomial i.e. the result of the division is a fractional rational expression.

We can easily adopt polynomial division by using the similarity with division of natural numbers written down as polynomials. In brief, this is shown in the following example.

Example 13: First, we remind students how we divide two multi-digit numbers and we write the procedure in polynomial form, for example,

$$\begin{array}{l} 165:11=15 \\ \underline{-11} \\ \;55 \\ \underline{-55} \\ \;0 \end{array}$$

and



$$\begin{array}{l} (1\cdot 10^2+6\cdot 10+5):(1\cdot 10+1)=1\cdot 10+5 \\ \underline{\pm 1\cdot 10^2 \pm 1\cdot 10} \\ \;5\cdot 10+5 \\ \underline{\pm 5\cdot 10+5} \\ \quad 0 \end{array}$$

where for the polynomial entry we note that first

$$1\cdot 10^2:1\cdot 10=1\cdot 10,$$

and then $5\cdot 10:1\cdot 10=5$. At this point, we suggest to students, formally in the polynomial entry, to write down the power to the variable x instead to the number 10, and after they get

$$\begin{array}{l} (1\cdot x^2+6\cdot x+5):(1\cdot x+1)=1\cdot x+5 \\ \underline{\pm 1\cdot x^2 \pm 1\cdot x} \\ \;5\cdot x+5 \\ \underline{\pm 5\cdot x+5} \\ \quad 0 \end{array}$$

to multiply the polynomials $x+5$ and $x+1$. After we complete the multiplication of these two polynomials, we solve two or three more examples, and explain the operation for polynomial division. Finally, we have to look at simple examples where a polynomial cannot be divided by a polynomial and introduce division with a remainder. In this case, certainly, it is advisable to use the theorem for division with a remainder for the set of natural numbers. At that, we will come to the following conclusion.

Division with remainder of two polynomials is always possible, at that, dividing the polynomial $P(x)$ by the polynomial $Q(x)$ means that we have to find the polynomials $q(x)$ and $r(x)$, the polynomial exponent $r(x)$ is lower than the polynomial exponent $Q(x)$, where $P(x)=Q(x)q(x)+r(x)$.

2.2. Formulas of abridged multiplication

a) The study of formulas of abridged multiplication is an essential part of the content while studying algebraic rational expressions. In primary education, the most frequently adopted formulas include those for product of the sum and difference of two monomials, square of a binomial and factoring the sum and difference of cubes. The simplest way is to derive these formulas directly, which is performed in the following examples.

a1) We multiply the binomials $a-b$ and $a+b$ and we get:

$$(a-b)(a+b)=a^2+ab-ba-b^2=a^2-b^2.$$

Consequently, for the expression a^2-b^2 we introduce the concept *difference of squares* and we form the rule:

The product of the sum and difference of the two terms is equal to the difference of their squares.

we adopt the same by solving some examples. In addition, it is desirable to use the same rule for simpler numerical calculations, which requires looking into examples of the following kind:

$$187\cdot213=(200-13)\cdot(200+13)=200^2-13^2$$
$$=40000-169=39831,$$

and illustrating the same, i.e. providing a geometric "proof", by which we point out the correlation between different content in mathematics curriculum. The last can be achieved by the drawing on the right, where

$$a^2-b^2=P_{ABHG}-P_{EDIH}=P_{ABEF}+P_{FEHG}-P_{EDIH}$$
$$=P_{ABEF}+P_{BCIH}-P_{EDIH}=P_{ABEF}+P_{BCDE}$$
$$=P_{ACDF}=(a-b)(a+b).$$

a2) When for the expression $(a+b)^2$, where a and b are two unlike terms, we introduce the concept *square of a binomial*, based on previous acquired knowledge we perform the following calculation

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2=a^2+2ab+b^2$$

Thus, we form the following rule:

The square of a binomial is the sum of the first term squared, two-times the product of the first and second terms and the second term squared.

Subsequently, we adopt the stated rule by solving several examples, bearing in mind that we have to solve numerical tasks of the following type as well:

$$213^2=(200+13)^2=200^2+2\cdot200\cdot13+13^2$$
$$=40000+5200+169=45369,$$

and illustrate the same, i.e. provide a geometric "proof", by which we point out the correlation between different content in mathematics curriculum. The last can be achieved by the drawing on the right, where:

$$(a+b)^2=P_{ABCD}=P_{AEFG}+P_{GFID}+P_{EBHF}+P_{FHCI}$$
$$=a^2+ab+ba+b^2=a^2+2ab+b^2.$$

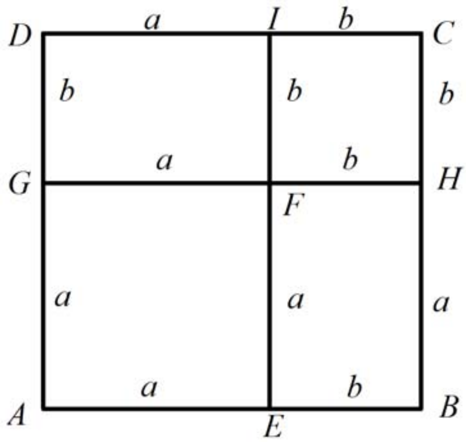
a3) Practice shows that, factoring the sum and difference of cubes of unlike terms a and b must be directly introduced. In addition, we have

$$(a-b)(a^2+ab+b^2)=a^3+a^2b+ab^2-ba^2-ab^2-b^3$$
$$=a^3-b^3,$$
$$(a+b)(a^2-ab+b^2)=a^3-a^2b+ab^2+ba^2-ab^2+b^3$$
$$=a^3+b^3.$$

and if we want the students to conscientiously adopt all the before mentioned formulas, we must prepare an appropriate system of tasks, which will facilitate adoption of formulas in the first place, and then enable students to use them for solving more complex tasks. Let us note that, as with the cases with the formulas in a1) and a2), we can present a geometrical illustration for the last two formulas as well. Thus, for example, in the case of the penultimate formula, the illustration can be made with the help of a cube with edges a and b, a cuboid with sides a,b and $a-b$ and a cuboid with sides a,a and $a-b$.

b) Factoring polynomials is an important part of the study of this subject matter. In addition, for the adoption of this material, we suggest three types of factoring:

It is advisable to use the analogy for factoring numbers while examining the question of factoring polynomials, thus emphasizing that one of the aims of factoring polynomials is reducing fractional-rational expressions. We need to mention here that, although students study this subject matter of reducing fractional - rational expressions later, it is appropriate to look at and explain several basic examples for this type of reducing, so that the students became aware of the need for factoring polynomials.



Students encounter numerous difficulties while factoring polynomials because while adopting this subject matter, unlike other algebraic subject matter, the successful adoption of the theory does not guarantee successful solving of tasks. These difficulties occur because successful factoring of polynomials requires not only knowledge of standard logarithms, but also most frequently, it is necessary to perform some prior transformations, which, in most cases, are not very simple. In order to overcome the mentioned difficulties, while studying polynomials, it is necessary to solve tasks of the following type.

Example 14: 1) Present the monomial $15x^3y^2$ as a product in three different ways, using $5x$ as one of the factors.

2) Calculate all the representations of $6x^2y^3 - 2x^3y^2$ as a product of a monomial and polynomial.

3) Write the polynomial equal to $-2xyz + \frac{x}{3}$, when in the given polynomial you replace the given terms with their opposite terms.

We will note here that, well-planned systems of tasks prepared by the teacher are a prerequisite for successful adoption of factoring polynomials. In this paper, we will not focus on the before-mentioned methods for factoring polynomials, nor on the systems of tasks, but we will point out to the possibility for correlation of the discussed subject matter with the number theory. For this purpose, we will use Sophie Germain's identity. Namely, by using formulas a1) and a2) we prove Sophie Germain's identity.

$$\begin{aligned} a^4 + 4b^4 &= (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2 \\ &= [(a^2)^2 + 2a^2 \cdot 2b^2 + (2b^2)^2] - (2ab)^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab), \end{aligned}$$

Then we move on to the following system of tasks:

Example 15: 1) Prove that for every natural number $n > 1$ the number $n^4 + 4$ is composite.

2) Prove that there are infinitely many natural numbers x for which $n \in \mathbf{N}$, the number $z = n^4 + x$ is composite.

3) Prove that the number $2^{10} + 5^{12}$ is composite.

4) Prove that the natural number $2^{2006} + 5^{2004}$ is composite.

5) Prove that the number $4n^4 + 1$, $n \in \mathbf{N}$ is a prime number only if $n = 1$.

6) Prove that for every $n > 1$ the natural number $n^4 + 4^n$ is composite.

7) Prove that the natural number $2005^4 + 4^{2005}$ is composite.

c) Identical transformations are an essential part of the subject of algebraic rational expressions. In most cases, the need for identical transformations is justified by the necessity for simpler presentation of complex expressions, which is motivated by the simplicity in calculating the numerical value of algebraic expressions. However, if students have adopted that the simplest type of a monomial or polynomial is its representation in regular form, and later notice that the numerical value of the expression $\frac{1}{2}ab \cdot 7ab^2$ where $a = \frac{1}{2}, b = 2$ can be directly calculated in a simpler way rather than reducing it to a regular form, then it is normal for them to doubt the appropriateness of the studied material.

The identical transformations can be associated with the representation of one and the same number in different forms, where the choice of form, in most cases, depends on the operations that need to be performed with that number. For example, if we consider the numerical expression $\frac{72 \cdot 25 \cdot 61}{9}$, then it is convenient to write the number 72 in the form $8 \cdot 9$, but if we have to multiply 72 by 13, then it is convenient to write the number 72 in the form $70 + 2$.

2.3. Fractional rational expressions

a) In primary education, the study of fractional rational expressions is reduced to adoption of operations involving the same. Before we start adopting the operations involving fractional rational expressions, we have to introduce the concept of *equivalent* fractional rational expressions (algebraic fractions) and prove the following theorem:

Let A, B, C *and* D *be algebraic expression. Then* $\frac{A}{B} = \frac{C}{D}$ *if and only if* $AD = BC$ *and the fractional rational expressions* $\frac{A}{B}$ *and* $\frac{C}{D}$ *have same domains.*

While studying this particular area, it is best to use the similarity with regular fractions. However, we have to explain to students that not all adopted concepts for regular fractions can be transferred to rational fractional expressions. This can be best illustrated by choosing well-suited examples.

Example 16. We assign the following algebraic fractions $\frac{3a+20}{2a+20}$ and $\frac{a+b}{2a+b}$ to the students, and we ask the following question: What can we say about these algebraic fractions? The most frequent answer is that the algebraic fraction $\frac{3a+20}{2a+20}$ is improper, and that the algebraic fraction $\frac{a+b}{2a+b}$ is proper.

The simplest way to eliminate this error is to put $a = -4$ in the first fraction, and then $a = -13$ in order to make the students see that we get the proper fraction $\frac{2}{3}$ in the first case, and the improper fraction $\frac{19}{6}$ in the second case. Further, for the $\frac{a+b}{2a+b}$ fraction it is sufficient to make algebraic rational expression calculation with $a = -3, b = 10$, and then with $a = -4, b = 2$ and to convince ourselves that we get the fractions $\frac{7}{4}$ and $\frac{1}{3}$.

Here we will mention the basic property of algebraic fractions which is derived from the equality $\frac{A}{B} = \frac{AP}{BP}$, where A, B and P are integral algebraic expressions, bearing in mind the domain. Further, writing down the last equality as $\frac{AP}{BP} = \frac{A}{B}$, students must adopt that in the case when the numerator and the denominator of the algebraic fraction have a common factor, then the fraction can be reduced and thus the value remains unchanged (once again bearing in mind the domain). At the end of this part, using the basic property of fractions, students must adopt the change of the sign in front of the fraction, i.e. they must adopt the equalities:

$$\frac{A}{B} = \frac{A(-1)}{B(-1)} = \frac{-A}{-B} \text{ and } \frac{-A}{B} = \frac{(-A)(-1)}{B(-1)} = \frac{A}{-B} = -\frac{A}{B}.$$

b) Addition and subtraction of fractions must be viewed as an identical transformation of a sum of fractions within one fraction. Before we move on to addition and subtraction of fractions, it is necessary to revise the rules for addition and subtraction of regular fractions. Then, by analogy, it is easier to introduce the rules of addition and subtraction of algebraic fractions. At that, looking at the sum and difference of two algebraic fractions with different denominators, the question arises to replace the same with fractions with equal denominators. At this point, we remind the students of the basic property of algebraic fractions and we replace the fractions $\frac{A}{B}$ and $\frac{C}{D}$ with the equal fractions $\frac{AD}{BD}$ and $\frac{BC}{BD}$, respectively, which have equal denominators and thus we get

$$\frac{A}{B} \pm \frac{C}{D} = \frac{AD}{BD} \pm \frac{BC}{BD} = \frac{AD \pm BC}{BD}.$$

We will mention here one more time, that by using the analogy with multiplication and division of regular fractions, in an identical manner, we can introduce the operations for multiplication and division of algebraic fractions. Therefore, we will not look into these operations in more detail. Regarding raising algebraic fractions to a power, with a natural number as an exponent, we will only mention that we need to analyze this as a partial case of multiplication of several equal algebraic fractions, which easily leads us to $(\frac{A}{B})^k = \frac{A^k}{B^k}$.

In the end of this section, we will note that, when adopting addition and subtraction of algebraic fractions, it is advisable to first adopt addition of algebraic fractions with monomials as denominators and abide by the following plan:

- First we look at simple tasks when the denominators do not have common factors, for example, $\frac{3a}{5b} + \frac{2a}{3c}$,
- Then we look at tasks where the denominator of one of the fractions is a multiple of the denominators of the other fractions, for example,
$$\frac{7a}{30b^3} + \frac{2a}{15b^2} + \frac{a}{10b}, \text{ and}$$
- Finally, we look at tasks when none of the denominators is a common multiple of the denominators, but some denominators have common multiples, for example,
$$\frac{2a}{15b^3c} + \frac{2a}{9b^2c^2} - \frac{4a}{12bc^3}.$$

It is advisable to abide by this suggested plan while adopting addition and subtraction of algebraic fractions with polynomials as denominators.

3. Conclusion

In the previous review we had addressed some issues related to the methodology of the study of rational algebraic expressions in primary education and presented some views on the same. Certain conclusions and methodological recommendations are the result of practical work with students of this age, with the described approach:

- Improve internal integration of rational algebraic expressions with: real numbers, number theory and geometry,
- Allow to increase the adoption of the subject with a greater understanding by the most of students, and
- Enable declarative knowledge that students gain easier, effectively to move into procedural knowledge, what actually is one of the goals of mathematics.

References

[1] J. R. Booth , B. Macwhinney, K. R. Thulborn, K. Sacco, J. Voyvodic, H. M. Feldman, “Functional organization of activation patterns in children: Whole brain fMRI imaging during three different cognitive tasks”, Elsevier, Volume 23, Issue 4, pp. 669–682, 1999

[2] S. Grozdev, For High Achievements in Mathematics. The Bulgarian Experience (Theory and Practice), Sofia, 2007.

[3] M.K. Stain, M.S. Smith, M.A. Henningsen, E.A. Silver, Implementing Standard – based on Mathematics Instructions, NCTM, USA, 2007.

[4] R. Malčeski, Methodics of teaching mathematics, FON University, Skopje, 2010.

[5] M. Henningsen, M. K. Stein, “Mathematical Tasks and Student Cognition: Classroom-Based Factors That Support and Inhibit High-Level Mathematical Thinking and Reasoning”, Journal for Research in Mathematics Education, Vol. 28, No. 5, pp. 524-549, 1997.

[6] V. Gogovska, “Promoting reflection-right key for getting structural knowledge”, In Proceedings of Anniversary International Conference on Synergetics and Reflection in Mathematics Education, Blagoevgrad, pp. 132-138, 2010.

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