A Space Having the Homotopy Type with Fuzzy Modules

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***Abstract:** Let Ω denotes the class of all homotopy type of fuzzy modules. In this paper first we construct a homotopy type invariant functor associated with fuzzy modules as well as we investigate function spaces F^M associated with this functor, where 'F' denotes the category of fuzzy left R-module and fuzzy R-map and 'M' denotes the category of R-modules and R-homomorphisms. Also we show that;*
i) if 'F' be the category of fuzzy left R-module and fuzzy R-map, then 'F' is a countable CW-Complex; and
ii) if 'M' denotes the category of left R-modules and R-homomorphisms, then 'M' is a compact metric space.

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1. Introduction

The concept of fuzzy modules was introduced by Nogoita and Ralescu[1] and the category of fuzzy sets was introduced by Goguen[2] in 1967. Also Category of fuzzy modules was introduced by P.K.Rana [9] in 2014. In this paper we study the properties of Ω, where Ω denote the class of all homotopy type with fuzzy modules. To do this we recall the following definitions and statements.

Definition1.1

Let R be a ring and M be left or right R-module. (M,λ) is called a **fuzzy left R-module** iff there is a map $\lambda : M \to [0,1]$ satisfying the following conditions:

i) $\lambda(a+b) \geq \min\{\lambda(a),\lambda(b)\},(\forall\ a,b\in M)$
ii) $\lambda(-a) = \lambda(a)\ ,\forall\ a\in M$
iii) $\lambda(0) = 1$
iv) $\lambda(ra) = \lambda(a)\ (\forall\ a\in M, r\in R)$

We write (M,λ) by λ_M

Definition1.2

Let λ_M and η_N be arbitrary fuzzy left R-modules. A **fuzzy R-map** $\widetilde{f} : \lambda_M \to \eta_N$ should satisfy the following conditions.

i) $f : M \to N$ is an R-map,
ii) $\eta(f(a)) \geq \lambda(a)\ ,\ \forall\ a\in M$

Definition 1.3

Let $f : M\to N$ and μ be a fuzzy subset of N . The fuzzy subset $f^{-1}(\mu)$ of M defined as follows; for all $x \in M$, $f^{-1}(\mu)(x) = \mu(f(x))$ is called fuzzy preimage of μ under f.

Definition 1.4

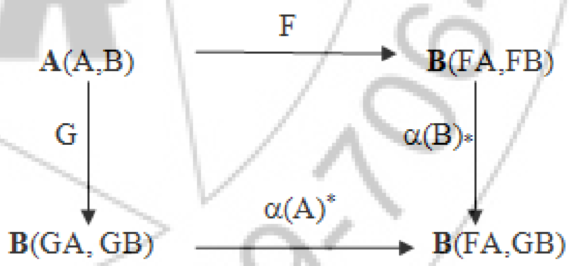
A **fuzzy submodules** of M is a fuzzy subset of M such that
i) $\mu(0) = 1$
ii) $\mu(rx)\geq\mu(x),\ \forall\ r\in R$ and $\forall x\in M$
iii) $\mu(x+y)\geq \min(\mu(x),\mu(y)),\ \forall x, y \in M$

Definition 1.5

Let **A** and **B** be **topologically enriched categories**. A functor $F : \mathbf{A}\to\mathbf{B}$ is called **continuous** if

$F : \mathbf{A}(A,B) \to \mathbf{B}(FA, FB)$ is continuous for all A and B in **A**.

A natural transformation $\alpha : F \to G$ of continuous functors $F, G : \mathbf{A}\to\mathbf{B}$ is called continuous if



is a commutative diagram of continuous maps for all A and B in **A**. A collection of morphisms $\{\beta(A): FA \to GA;\ A\in \text{ob}.\mathbf{A}\}$ is called a continuous natural transformation up to homotopy if the above diagram is homotopy commutative.

Definition 1.6

A pair (X, ε) consisting of a Hausdorff space X and a cell-decomposition ε of X is called a **CW-complex** if the following 3 axioms are satisfied:

Axiom 1: (**Characteristic Maps**) For each n-cell $e \in \varepsilon$ there is a map $\Phi e: D^n \to X$ restricting to a homeomorphism $\Phi e|\text{int}(D^n)$: $\text{int}(D_n) \to e$ and taking S^{n-1} into X^{n-1}.

Axiom 2: (**Closure Finiteness**) For any cell $e \in \varepsilon$ the closure $\overline{e}$ intersects only a finite number of other cells in ε.
Axiom 3: (**Weak Topology**) A subset $A \subseteq X$ is closed iff $A \cap \overline{e}$ is closed in X for each $e \in \varepsilon$.

Definition 1.7

Two spaces X and Y are said to have the same homotopy type if there exist maps $f:X\to Y$ and $g:Y\to X$ such that fg: $Y\to Y$ is homotopic to the identity, and gf: $X\to X$ is homotopic to the identity

Lemma1.8

Let $Hom_R(\lambda_M,\eta_N)$ denotes the set of all fuzzy R-maps from λ_M to η_N , then $Hom_R(\lambda_M,\eta_N)$ is an additive group. Moreover, if R is a commutative ring, then $Hom_R(\lambda_M,\eta_N)$ is a left R-modules.

Proof: Using [9], it follows.

Lemma1.9

For a commutative ring R, R-modules M and N, $Hom_R(M,N)$ is an R-module
Proof: Using [2], it follows.

Lemma1.10

Let $Hom_R(M,N)$ denotes the set of all fuzzy R-maps from R-modules M to R-modules N, then $Hom_R(M,N)$ is an R-module, if R is a commutative ring
Proof
Using Definition 1.1 and [6], it follows

Lemma 1.11
Given a fixed R-module M, the R-homomorphism $f:N\to P$ induces

a) an R-homomorphism $f_*:Hom_R(M,N)\to Hom_R(M,P)$ defined by
$f_*(\alpha) = f\circ\alpha$, $\forall\alpha\in Hom_R(M,N)$
b) an R-homomorphism f^* $:Hom_R(P,M) \to Hom_R(N,M)$ defined by
$f^*(\beta) = \beta\circ f, \forall\beta\in Hom_R(P,M)$

Proof:
Using [6,7], it follows

Lemma1.12
Let M,N,P be R-modules and $f:M\to N$ and $g:N\to P$ be R-homomorphisms

Then for any R- module A

i) $(g\circ f)_*$: $Hom_R(A,M) \to Hom_R(A,P)$ is an R-homomorphism such that
$(g\circ f)_* = g_*\circ f_*$;
ii) $(g\circ f)^*$: $Hom_R(P,A) \to Hom_R(M,A)$ is an R-homomorphism such that
$(g\circ f)^* = f^*\circ g^*$;

Proof:
Using [6,7], it follows

2. Functor with Fuzzy Modules

In this section we construct a homotopy type invariant functor with fuzzy modules.

In [10], Proposition 2.1 and 2.2, shown that '**M**' is a category of R-modules and R-homomrphisms and '**F**' is a category of fuzzy left R-module and fuzzy R-map**.**To find functor we recall the following Lemmas:

Lemma 2.1

Let R be a ring and M be a fixed R- module, then $Hom_R(M,N)$ is a fuzzy R-module, for any R-module N.
Proof
Using Definition 1.1 and [6], it follows.

Lemma 2.2
Let R be a ring and M be a fixed R- module, the R-homomorphism $f:N\to P$ induces

i) an fuzzy R-homomorphism f_* : $Hom_R(M,N) \to Hom_R(M,P)$ and
ii) an fuzzy R-homomorphism f^* : $Hom_R(P,M) \to Hom_R(N,M)$

Proof: Using Definition 1.1 and Lemma1.12, it follows.

Corollary 2.3

For any fixed R-module M, the fuzzy R module $Hom_R(M,N)$ and their fuzzy R-homomorphisms forms a category, for any R-module N; this category is denoted by 'F'

Proof

Using ref.[10] and Lemma1.12(a) , it follows

Proposition 2.4

'Hom_R' is a invariant functor in the sense that it is both a covariant and a contravariant functor
Hom_R: M→ F is an invariant functor, for any fixed R-module M.

Proof:

Define Hom_R : **M**→ **F** by
Hom_R(N) = $Hom_R(M,N)$, for any fixed R-module M, which is the object of **F**.
Let N,P are two R-modules in **M** and $f:N\to P$ be R-homomorphisms in **M**, then
Hom_R(f) = $f_*:Hom_R(M,N) \to Hom_R(M,P)$ in **F** and $f^*:Hom_R(P,M) \to Hom_R(N,M)$ are well defined mapping and so by **Definition 1.1, Lemma 1.12** and **Lemma 2.2**, the theorem follows.

Proposition 2.5

'Hom_R' is a Homotopy type functor in the sense that if f is a homotopy equivalence for any two R-modules M and N , then $Hom_R(f)$ is a isomorphism

Proof:
Since f is a homotopy equivalence for any two R-modules M and N, there exists f:M →N and g: N →M such that g.f $\simeq$ I_M and f.g $\simeq$ I_N, then Hom_R (f): $Hom_R(P,M)$ →$Hom_R(P,N)$ and Hom_R (f): $Hom_R(N,P)$ →$Hom_R(M,P)$ are fuzzy R- homomorphisms, then Hom_R satisfies the following conditions:

i) **f $\simeq$ g** $\Rightarrow Hom_R(f) = Hom_R(g)$

ii) g.f $\simeq I_M$

$\Rightarrow Hom_R(gf)= Hom_R(I_M) = Id.$

$\Rightarrow Hom_R(g).Hom_R(f) = Id.$

iii) f. g $\simeq I_N$

$\Rightarrow Hom_R(fg) = Hom_R(I_N) = Id.$

$\Rightarrow Hom_R(f).Hom_R(g) = Id.$

Thus $Hom_R(f)$ is isomorphic to $Hom_R(g)$

Corollary 2.6

Hom_R is also a Homotopy type invariant functor.

Proposition 2.7

All homotopy type invariant continuous functors form a function spaces from category M to the category F, it is denoted by F^M.

Proof:
Using the **Defination1.8**, **Proposition2.4**, **Proposition2.5** and **Corollary 2.6**, it follows.

In the next section we study generalize these results:

3. In this section we study the some properties of the class Ω , that can generalized the above result (Proposition2.7), where Ω denotes the class of all homotopy type with fuzzy modules.

Proposition3.1

Let F denotes the category of fuzzy left R-module and fuzzy R-map, then F is a countable CW-Complex.

Proof

Using the **Definition1.1**, **Definition1.2** and **Definition1.6**, the theorem follows.

Proposition3.2

Let 'M' denotes the category of left R-modules and R-homomorphisms, then 'M' is a compact metric space.

Proof

Using the Lemma1.11, Lemma1.12, ref.[10] and since **M** is a finite category, the theorem follows.

Proposition 3.3

The following statements are equivalent:

a) F belongs to the class Ω.
b) F is dominated by a countable CW-Complex
c) F has the homotopy type of a countable locally finite simplicial complex of fuzzy modules;
d) F has the homotopy type of an absolute neighbourhood retract

Proof:

Using the ref.[11], (a) ⇔(b) ⇔(c) it follows, but (c) ⇒(a) ⇒(b) are trivial.
In the sense of path connected (b) ⇒ (c). But if F is dominated by a countable CW-Complex, then each path component of F is an open set; and the collection of path components is countable.

Also (c)⇒(d)⇒(b) follows from O.Hanner [10]

Corollary3.4

Every separable manifold belongs to the class Ω.

Proof:

Using [11], it follows.

Proposition 3.5

If 'F' belongs to Ω and M is compact metric, then the function spaces F^M belongs to Ω.

Proof:

Using **Proposition 3.3,** we may assume that **F** is absolute neighborhoods retract and by **Kuratowski lemma** this implies that $\mathbf{F^M}$ is also a absolute neighborhood retract, this completes the proof.

From these results we have

Corollary 3.6

All Homotopy type invariant functors of fuzzy modules form **Function Spaces.**

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Prof. Pravanjan Kumar Rana obtained his MSc in Pure Mathematics and his PhD in Algebraic Topology. He has published, since 2005, more than 20 papers in peer-reviewed journals. He was the first Head of Mathematics Department in Berhampore Girls' College, Berhampore, Murshidabad and performs his research at Algebraic Topology and Category Theory.