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# A Space Having the Homotopy Type with Fuzzy Modules

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Abstract: Let  $\Omega$  denotes the class of all homotopy type of fuzzy modules. In this paper first we construct a homotopy type invariant functor associated with fuzzy modules as well as we investigate function spaces  $F^M$  associated with this functor, where 'F' denotes the category of fuzzy left R-module and fuzzy R-map and 'M' denotes the category of R-modules and R-homomorphisms. Also we show that; i) if 'F' be the category of fuzzy left R-module and fuzzy R-map, then 'F' is a countable CW-Complex; and ii) if 'M' denotes the category of left R-modules and R-homomorphisms, then 'M' is a compact metric space.

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#### 1. Introduction

The concept of fuzzy modules was introduced by Nogoita and Ralescu[1] and the category of fuzzy sets was introduced by Goguen[2] in 1967. Also Category of fuzzy modules was introduced by P.K.Rana [9] in 2014. In this paper we study the properties of  $\Omega$ , where  $\Omega$  denote the class of all homotopy type with fuzzy modules. To do this we recall the following definitions and statements.

#### Definition1.1

Let R be a ring and M be left or right R-module.  $(M,\lambda)$  is called a **fuzzy left R-module** iff there is a map  $\lambda : M \rightarrow [0,1]$  satisfying the following conditions:

- i)  $\lambda(a+b) \ge \min{\{\lambda(a),\lambda(b)\}, (\forall a,b \in M)\}}$
- ii)  $\lambda(-a) = \lambda(a), \forall a \in M$
- iii)  $\lambda(0) = 1$
- iv)  $\lambda(ra) = \lambda(a) \ (\forall a \in M, r \in R)$

We write (M, $\lambda$ ) by  $\lambda_M$ 

#### Definition1.2

Let  $\lambda_M$  and  $\eta_N$  be arbitrary fuzzy left R-modules. A fuzzy R-map

 $f: \lambda_{\rm M} \rightarrow \eta_{\rm N}$  should satisfy the following conditions.

i)  $f: M \rightarrow N$  is an R-map,

ii)  $\eta(f(a)) \ge \lambda(a), \forall a \in M$ 

#### **Definition 1.3**

Let  $f: M \rightarrow N$  and  $\mu$  be a fuzzy subset of N. The fuzzy subset  $f^{1}(\mu)$  of M defined as follows; for all  $x \in M$ ,  $f^{1}(\mu)(x) = \mu(f(x))$  is called fuzzy preimage of  $\mu$  under f.

#### **Definition 1.4**

A **fuzzy submodules** of M is a fuzzy subset of M such that  $i)\mu(0) = 1$ 

ii) $\mu(rx) \ge \mu(x), \forall r \in R \text{ and } \forall x \in M$ iii) $\mu(x+y) \ge \min(\mu(x), \mu(y)), \forall x, y \in M$ 

#### **Definition 1.5**

Let **A** and **B** be topologically enriched categories. A functor  $F : \mathbf{A} \rightarrow \mathbf{B}$  is called **continuous** if

 $F : A(A,B) \rightarrow B(FA, FB)$  is continuous for all A and B in A.

A natural transformation  $\alpha : F \to G$  of continuous functors F, G :  $A \to B$  is called continuous if



is a commutative diagram of continuous maps for all A and B in A. A collection of morphisms { $\beta(A)$ : FA  $\rightarrow$  GA; A  $\in$  ob.A} is called a continuous natural transformation up to homotopy if the above diagram is homotopy commutative.

#### **Definition 1.6**

A pair  $(X, \varepsilon)$  consisting of a Hausdorff space X and a celldecomposition  $\varepsilon$  of X is called a **CW-complex** if the following 3 axioms are satisfied:

Axiom 1: (Characteristic Maps) For each n-cell  $e \in \varepsilon$ there is a map  $\Phi e: D^n \to X$  restricting to a homeomorphism  $\Phi e|int(D^n): int(D_n) \to e$  and taking  $S^{n-1}$ into  $X^{n-1}$ .

Volume 3 Issue 10, October 2014 www.ijsr.net Axiom 2: (Closure Finiteness) For any cell  $e \in \varepsilon$  the closure  $\overline{e}$  intersects only a finite number of other cells in  $\varepsilon$ .

Axiom 3: (Weak Topology) A subset  $A \subseteq X$  is closed iff  $A \cap \overline{e}$  is closed in X for each  $e \in \varepsilon$ .

#### **Definition 1.7**

Two spaces X and Y are said to have the same homotopy type if there exist maps  $f:X \rightarrow Y$  and  $g:Y \rightarrow X$  such that fg:  $Y \rightarrow Y$  is homotopic to the identity, and gf:  $X \rightarrow X$  is homotopic to the identity

#### Lemma1.8

Let  $Hom_R(\lambda_M, \eta_N)$  denotes the set of all fuzzy R-maps from  $\lambda_M$  to  $\eta_N$ , then  $Hom_R(\lambda_M, \eta_N)$  is an additive group. Moreover, if R is a commutative ring, then  $Hom_R(\lambda_M, \eta_N)$  is a left R-modules.

Proof: Using [9], it follows.

## Lemma1.9

For a commutative ring R, R-modules M and N,  $Hom_R(M,N)$  is an R-module Proof: Using [2], it follows.

## Lemma1.10

Let  $\operatorname{Hom}_R(M,N)$  denotes the set of all fuzzy R-maps from R-modules M to R-modules N, then  $\operatorname{Hom}_R(M,N)$  is an R-module, if R is a commutative ring Proof

Using Definition 1.1 and [6], it follows

## Lemma 1.11

Given a fixed R-module M, the R-homomorphism  $f{:}N{\rightarrow}P$  induces

a) an R-homomorphism  $f_*:Hom_R(M,N) \rightarrow Hom_R(M,P)$  defined by

 $f_*(\alpha) = f_\circ \alpha$ ,  $\forall \alpha \in Hom_R(M,N)$ 

b)an R-homomorphism  $f^*$  :Hom<sub>R</sub>(P,M)  $\rightarrow$ Hom<sub>R</sub>(N,M) defined by

 $f^*(\beta) = \beta \circ f, \forall \beta \in Hom_R(P,M)$ 

Proof: Using [6,7], it follows

## Lemma1.12

Then for any R- module A

i)  $(g \circ f)_* : \operatorname{Hom}_R(A,M) \to \operatorname{Hom}_R(A,P)$  is an R-homomorphism such that  $(g \circ f)_* = g_* \circ f_*;$ ii)  $(g \circ f)^* : \operatorname{Hom}_R(P,A) \to \operatorname{Hom}_R(M,A)$  is an R-homomorphism such that  $(\mathbf{g} \circ \mathbf{f})^* = \mathbf{f}^* \circ \mathbf{g}^*;$ 

Proof: Using [6,7], it follows

## 2. Functor with Fuzzy Modules

In this section we construct a homotopy type invariant functor with fuzzy modules.

In [10], Proposition 2.1 and 2.2, shown that '**M**' is a category of R-modules and R-homomrphisms and '**F**' is a category of fuzzy left R-module and fuzzy R-map.To find functor we recall the following Lemmas:

## Lemma 2.1

Let R be a ring and M be a fixed R- module, then  $Hom_R(M,N)$  is a fuzzy R-module, for any R-module N. Proof Using Definition 1.1 and [6], it follows.

#### Lemma 2.2

Let R be a ring and M be a fixed R- module, the R-homomorphism  $f{:}N{\rightarrow}P$  induces

 $i) \quad \mbox{an fuzzy } R\mbox{-homomorphism } f_* : \mbox{Hom}_R(M,N) \\ \rightarrow \mbox{Hom}_R(M,P) \mbox{ and } \\ ii) \quad \mbox{an fuzzy } R\mbox{-homomorphism } f^* : \mbox{Hom}_R(P,M) \\ \rightarrow \mbox{Hom}_R(N,M)$ 

Proof: Using Definition 1.1 and Lemma1.12, it follows.

## Corollary 2.3

For any fixed R-module M, the fuzzy R module  $\operatorname{Hom}_R(M,N)$  and their fuzzy R-homomorphisms forms a category, for any R-module N; this category is denoted by 'F'

## Proof

Using ref.[10] and Lemma1.12(a), it follows

## **Proposition 2.4**

'Hom<sub>R</sub>' is a invariant functor in the sense that it is both a covariant and a contravariant functor

 $\operatorname{Hom}_R: M \to F$  is an invariant functor, for any fixed R-module M.

## **Proof:**

Define  $\operatorname{Hom}_{R} : \mathbf{M} \to \mathbf{F}$  by  $\operatorname{Hom}_{R}(N) = \operatorname{Hom}_{R}(M,N)$ , for any fixed R-module M, which is the object of  $\mathbf{F}$ . Let N,P are two R-modules in  $\mathbf{M}$  and f:N $\to$  P be Rhomomorphisms in  $\mathbf{M}$ , then  $\operatorname{Hom}(\mathbf{M}, \mathbf{N}) \to \operatorname{Hom}(\mathbf{M}, \mathbf{R})$  in  $\mathbf{F}$  and

 $\operatorname{Hom}_{R}(f) = f_{*}:\operatorname{Hom}_{R}(M,N) \to \operatorname{Hom}_{R}(M,P)$  in **F** and  $f^{*}:\operatorname{Hom}_{R}(P,M) \to \operatorname{Hom}_{R}(N,M)$  are well defined mapping and so by **Definition 1.1, Lemma 1.12** and **Lemma 2.2**, the theorem follows.

#### **Proposition 2.5**

'Hom\_R' is a Homotopy type functor in the sense that if f is a homotopy equivalence for any two R-modules M and N , then Hom\_R(f) is a isomorphism

Proof:

Since f is a homotopy equivalence for any two R-modules M and N, there exists  $f:M \to N$  and g:  $N \to M$  such that g.f  $\cong I_M$  and f.g  $\cong I_N$ , then  $Hom_R$  (f):  $Hom_R(P,M) \to Hom_R(P,N)$  and  $Hom_R$  (f):  $Hom_R(N,P) \to Hom_R(M,P)$  are fuzzy R- homomorphisms, then  $Hom_R$  satisfies the following conditions:

i)  $\mathbf{f} \cong \mathbf{g} \Rightarrow \operatorname{Hom}_{R}(f) = \operatorname{Hom}_{R}(g)$ ii)  $g.f \cong I_{M}$   $\Rightarrow \operatorname{Hom}_{R}(gf) = \operatorname{Hom}_{R}(I_{M}) = \operatorname{Id}.$   $\Rightarrow \operatorname{Hom}_{R}(g).\operatorname{Hom}_{R}(f) = \operatorname{Id}.$ iii)  $f.g \cong I_{N}$   $\Rightarrow \operatorname{Hom}_{R}(fg) = \operatorname{Hom}_{R}(I_{N}) = \operatorname{Id}.$  $\Rightarrow \operatorname{Hom}_{R}(f).\operatorname{Hom}_{R}(g) = \operatorname{Id}.$ 

Thus  $Hom_R(f)$  is isomorphic to  $Hom_R(g)$ 

#### **Corollary 2.6**

Hom<sub>R</sub> is also a Homotopy type invariant functor.

#### **Proposition 2.7**

All homotopy type invariant continuous functors form a function spaces from category M to the category F, it is denoted by  $F^{M}$ .

Proof:

Using the **Defination1.8**, **Proposition2.4**, **Proposition2.5** and **Corollary 2.6**, it follows.

In the next section we study generalize these results:

3. In this section we study the some properties of the class  $\Omega$ , that can generalized the above result (Proposition2.7), where  $\Omega$  denotes the class of all homotopy type with fuzzy modules.

#### Proposition3.1

Let F denotes the category of fuzzy left R-module and fuzzy R-map, then F is a countable CW-Complex.

Proof

Using the **Definition1.1**, **Definition1.2** and **Definition1.6**, the theorem follows.

## Proposition3.2

Let 'M' denotes the category of left R-modules and R-homomorphisms, then 'M' is a compact metric space.

Proof

Using the Lemma1.11, Lemma1.12, ref.[10] and since **M** is a finite category, the theorem follows.

#### **Proposition 3.3**

The following statements are equivalent:

**a**) F belongs to the class  $\Omega$ .

**b**) F is dominated by a countable CW-Complex

c) F has the homotopy type of a countable locally finite simplicial complex of fuzzy modules;

**d**) F has the homotopy type of an absolute neighbourhood retract

Proof:

Using the ref.[11], (a)  $\Leftrightarrow$ (b)  $\Leftrightarrow$ (c) it follows, but (c)  $\Rightarrow$ (a)  $\Rightarrow$ (b) are trivial.

In the sense of path connected (b)  $\Rightarrow$  (c). But if F is dominated by a countable CW-Complex, then each path component of F is an open set; and the collection of path components is countable.

Also (c) $\Rightarrow$ (d) $\Rightarrow$ (b) follows from O.Hanner [10]

#### Corollary3.4

Every separable manifold belongs to the class  $\Omega$ .

Proof:

Using [11], it follows.

## **Proposition 3.5**

If 'F' belongs to  $\Omega$  and M is compact metric, then the function spaces  $F^M$  belongs to  $\Omega$ .

Proof:

Using **Proposition 3.3**, we may assume that  $\mathbf{F}$  is absolute neighborhoods retract and by **Kuratowski lemma** this implies that  $\mathbf{F}^{\mathbf{M}}$  is also a absolute neighborhood retract, this completes the proof.

From these results we have

## **Corollary 3.6**

All Homotopy type invariant functors of fuzzy modules form **Function Spaces.** 

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