On Bc-open sets

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Abstract: In this paper, we introduce a new class of open sets, called Bc-open sets, it is denoted and studied. Also, we have studied of definition Bc-paracompact spaces and nearly Bc-paracompact spaces and have provide some properties of this concepts.

Keywords: θ-open, Bc-open

1. Introduction

In [5] H. Z. Ibrahim introduced the concept of Bc-open set in topological spaces. This paper consist of two sections. In section one, we give similar definition by using of Bc-open sets and also we proof some properties about it. In section two we obtain new a characterization and preserving theorems of Bc-paracompact spaces, nearly Bc-paracompact spaces and the product of space $X \times Y$ where $X$ is Bc-paracompact space and $Y$ is θ-paracompact space.

Definition (1.1)[3]:
Let $X$ be a topological space and $A \subseteq X$. Then $A$ is called b-open set in $X$ if $A \in \mathcal{B}(X)$. The family of all b-open subset of a topological space $(X, \tau)$ is denoted by $BO(X, \tau)$ or (Briefly $BO(X)$).

Definition (1.2)[5]:
Let $X$ be a topological space and $A \subseteq X$. Then $A$ is called Bc-open set in $X$ if for each $x \in A$, there exists a closed set $F$ such that $x \in F \subseteq A$. The family of all Bc-open subset of a topological space $(X, \tau)$ is denoted by $BcO(X, \tau)$ or (Briefly $BcO(X)$). A subset of $X$ is Bc-closed set if $A'$ is Bc-open set. The family of all Bc-closed subset of a topological space $(X, \tau)$ is denoted by $BcC(X, \tau)$ or (Briefly $BcC(X)$).

Remark (1.3):
It is clear from the definition that every Bc-open set is b-open, but the converse is not true in general as the following example:

Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Then the closed set are: $X, \phi, \{2,3\}, \{1,3\}, \{3\}$. Hence $BO(X) = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$ and $BcO(X) = \{\phi, X, \{1,3\}, \{2,3\}\}$. Then $\{1\}$ is b-open but $\{1\}$ is not Bc-open.

Definition (1.4)[10]:
1) Let $X$ be a topological space and $A \subseteq X$. Then $A$ is called $\theta$-open set in $X$ if for each $x \in A$, there exists an open set $G$ such that $x \in G \subseteq G \subseteq A$. The family of all $\theta$-open subset of a topological space $(X, \tau)$ is denoted by $\theta O(X, \tau)$ or (Briefly $\theta O(X)$).
2) Let $X$ be a topological space and $A \subseteq X$. A point $x \in X$ is said to $\theta$-interior point of $A$ if there exist an $\theta$-open set $U$ such that $x \in U \subseteq A$. The set of all $\theta$-interior points of $A$ is called $\theta$-interior of $A$ and is denoted by $A^{\theta}$.

3) Let $X$ be a topological space and $A \subseteq X$. The $\theta$-closure of $A$ is defined by the intersection of all Bc-closed sets in $X$ containing $A$, and is denoted by $A^{\theta}$.

Remark (1.5)[5]:
1) Every $\theta$-open is Bc-open.
2) Every $\theta$-closed is Bc-closed.

Example (1.6):
The intersection of two Bc-open sets is not Bc-open in general. Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Then $\{1,3\}, \{2,3\}$ is Bc-open set where as $\{1,3\} \cap \{2,3\} = \{3\}$ is not Bc-open set.

Remark (1.7)[2]:
The intersection of an b-open set and an open set is b-open set.

Proposition (1.8):
Let $X$ be a topological space and $A,B \subseteq X$. If $A$ is Bc-open set and $B$ is an $\theta$-open set, then $A \cap B$ is Bc-open set.

Proof:
Let $A$ be a Bc-open set and $B$ is an $\theta$-open set, then $A$ is b-open set and $B$ is an open set since every $\theta$-open is open. Then $A \cap B$ is b-open set by (Remark (1.7)). Now, let $x \in A \cap B$, $x \in A$ and $x \in B$. Then there exists a closed set $F$ such that $x \in F \subseteq A$, and there exists an open set $E$ such that $x \in E \subseteq E \subseteq B$. Therefore, $E \cap F$ is closed since the intersection of closed sets is closed. Thus $x \in E \cap F \subseteq A \cap B$. Then $A \cap B$ is Bc-open set.

Proposition (1.9)[5]:
Let $X$ be a topological space and $A \subseteq X$. Then $A$ is Bc-open if and only if there is a $\alpha$-open set $F_{\alpha}$, where $A \subseteq F_{\alpha}$, and $F_{\alpha}$ is closed sets for each $\alpha$.

Proposition (1.10)[5]:
Let $\{A_{\alpha}; \alpha \in \Lambda\}$ be a collection of Bc-open sets in a topological space $X$. Then $\cap \{A_{\alpha}; \alpha \in \Lambda\}$ is Bc-open.

Lemma (1.11)[4]:
Let $X$ be a topological space and $Y \subseteq X$. If $G$ is an $\theta$-open in $X$, then $G \cap Y$ is an $\theta$-open in $Y$.

Proposition (1.12)[5]:
Let $X$ be a topological space and $Y \subseteq X$. If $G$ is an $\theta$-open in $X$ and $Y$ is an open in $X$, then $G \cap Y$ is $\theta$-open in $Y$. 


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Proposition(1.13):
Let $X$ be a topological space and $Y \subset X$. If $G$ is a $Bc$-open set in $X$ and $Y$ is a $\theta$-open in $X$, then $G \cap Y$ is $Bc$-open in $Y$.

Proof:
Let $x \in G \cap Y$, $x \in G$ and $x \in Y$. Since $G$ is a $Bc$-open set in $X$, then for each $x \in G \in BO(X)$, there exists $F$ is closed set in $X$ such that $x \in F \subset G$ and since $Y$ is an $\theta$-open in $X$, then there exists $U$ is open set in $X$ such that $x \in U \subset \bar{U} \subset Y$. Since $G$ is $Bc$-open, then $G$ is $b$-open and since $Y$ is an $\theta$-open, then $Y$ is an open by proposition(1.12). Therefore, $G \cap Y$ is $b$-open in $Y$. Since $F \cap \bar{U}$ is closed set in $Y$. Thus $x \in F \cap \bar{U} \subset G \cap Y$. Hence $G \cap Y$ is $Bc$-open in $Y$.

Proposition(1.14):
Let $X$ be a topological space and $Y$ is a $\theta$-open subset of $X$. If $G$ is a $Bc$-open in $Y$, then $G$ is $Bc$-open in $X$.

Proof:
Suppose that $Y$ is an $\theta$-open subset of $X$ and $G \subset Y$, since $G$ is a $Bc$-open set in $Y$, then for each $x \in G \in BO(Y)$, there exists $F$ is closed set in $Y$ such that $x \in F \subset G$. Let $G = Y \cup U$, $U \subset X$, and $F = E \cap Y, E \subset C$. Then $x \in E \subset C$. Hence $G$ is $Bc$-open in $X$.

Lemma(1.15)[6]:
Let $X$ and $Y$ be a topological spaces and let $A \subset X, B \subset Y$ be two non empty subset:
1) If $A$ is an open set in $X$ and $B$ is an open set in $Y$, then $A \times B$ is an open set in $X \times Y$.
2) If $A$ is a closed set in $X$ and $B$ is a closed set in $Y$, then $A \times B$ is a closed set in $X \times Y$.
3) $(A \times B) = A \cap B$.

Theorem(1.16):
Let $X$ and $Y$ be a topological spaces and let $A \subset X, B \subset Y$ such that $A$ is an $\theta$-open set of $X$, $B$ is an $\theta$-open set of $Y$, then $A \times B$ is an $\theta$-open set of $X \times Y$.

Proof:
Let $A$ be an $\theta$-open set of $X$ and $B$ be an $\theta$-open set of $Y$, then for each $x \in A$, there exists $G \subset X$ such that $x \in G \subset \bar{G} \subset A$ and for each $y \in B$, there exists $U$ open set in $X$ such that $y \in U \subset \bar{U} \subset B$. By lemma(1.15)(1), then $G \times U$ is an open set in $X \times Y$. Since $G \times U$ is closed set, then $G \times U$ is a closed set in $X \times Y$ by lemma(1.15)(2). Since $G \times U = \bar{G} \times \bar{U}$ by lemma(1.15)(3), then $x \in \bar{G} \times \bar{U} \subset \bar{X} \times \bar{Y} \subset A \times B$. Hence $A \times B$ is an $\theta$-open subset of $X \times Y$.

Theorem(1.17)[8]:
Let $X$ and $Y$ be topological spaces and let $A \subset X, B \subset Y$ such that $A$ is a $b$-open set of $X$, $B$ is an open set of $Y$, then $A \times B$ is a $b$-open subset of $X \times Y$.

Theorem(1.18):
Let $X$ and $Y$ be topological spaces and let $A \subset X, B \subset Y$ such that $A$ is a $Bc$-open set of $X$, $B$ is an $\theta$-open set of $Y$, then $A \times B$ is a $Bc$-open subset of $X \times Y$.

Proof:
Let $A$ be a $Bc$-open set of $X$ and $B$ be an $\theta$-open set of $Y$, then for each $x \in A \in BO(X)$, there exists $F$ closed set in $X$ such that $x \in F \subset A$ and for each $y \in B$, there exists $U$ open set in $Y$ such that $y \in U \subset \bar{U} \subset B$. Since $A$ is a $Bc$-open in $X$ and $B$ is an $\theta$-open in $Y$, then $A$ is a $b$-open in $X$ and $B$ be an open in $Y$. Thus $A \times B$ is a $b$-open subset of $X \times Y$ by proposition(1.17), $x \in A$ and $y \in B$, then $(x, y) \in A \times B \in BO(X)$. Since $x \in F \subset A$ and $y \in U \subset \bar{U} \subset B$ such that $F$ is closed set in $X$ and $U$ is closed set in $Y$, then $F \times U$ is closed set in $X \times Y$. Therefore, $(x, y) \in F \times U \subset A \times B$. Hence $A \times B$ is a $Bc$-open subset in $X \times Y$.

Definition(1.19)[1]:
Let $X$ be a topological space and $x \in X$. Then a subset $N$ of $x$ is said to be a $\theta$-neighborhood of $x$, if there exists $\theta$-open set $U$ in $X$ such that $x \in U \subset N$.

Definition(1.20)[5]:
Let $X$ be a topological space and $A \subset X$. A point $x \in X$ is said to $Bc$-interior point of $A$, if there exist a $Bc$-open set $U$ such that $x \in U \subset A$. The set of all $Bc$-interior points of $A$ is called $Bc$-interior of $A$ and is denoted by $A^{Bc}$.

Theorem(1.21)[5]:
Let $X$ be a topological space and $A, B \subset X$, then the following statements are true:
1) $A^{Bc}$ is the union of all $Bc$-open set which are contained in $A$.
2) $A^{Bc}$ is $Bc$-open set in $X$.
3) $A^{Bc} \subset A$.
4) $A$ is $Bc$-open if and only if $A = A^{Bc}$.
5) $(A^{Bc})^{Bc} = A^{Bc}$.
6) If $A \subset B$, then $A^{Bc} \subset B^{Bc}$.
7) $A^{Bc} \cup B^{Bc} \subset (A \cup B)^{Bc}$.
8) $(A \cap B)^{Bc} \subset A^{Bc} \cap B^{Bc}$.

Definition(1.1.122)[5]:
Let $X$ be a topological space and $A \subset X$. The $Bc$-closure of $A$ is defined by the intersection of all $Bc$-closed sets in $X$ containing $A$, and is denoted by $A^{Bc}$.

Theorem(1.23)[5]:
Let $X$ be a topological space and $A, B \subset X$. Then the following statements are true:
1) $A^{Bc}$ is the intersection of all $Bc$-closed sets containing $A$.
2) $A^{Bc}$ is $Bc$-closed set in $X$.
3) $A \subset A^{Bc}$.
4) $A$ is $Bc$-closed set if and only if $A = A^{Bc}$.
5) $(A^{Bc})^{Bc} = A^{Bc}$.
6) If $A \subset B$, then $A^{Bc} \subset B^{Bc}$.
7) $A^{Bc} \cup B^{Bc} \subset (A \cup B)^{Bc}$.
8) $(A \cap B)^{Bc} \subset A^{Bc} \cap B^{Bc}$.

Proposition(1.24)[5]:
Let $X$ be a topological space and $A \subset X$. Then $x \in A^{Bc}$ if and only if $A \cap U \neq \phi$ for every $Bc$-open set $U$ containing $x$. 

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Definition(1.25)[5]:
Let $X$ be a topological space and $A \subset X$. A point $x$ is said to be Bc-limit point of $A$, if for each Bc-open set $U$ containing $x$, $U \cap (A - \{x\}) \neq \emptyset$. The set of all Bc-limit points of $A$ is called a Bc-derived set of $A$ and is denoted by $A^{Bc}$.

Proposition(1.26)[5]:
Let $X$ be a topological space and $A \subset X$. Then $A^{Bc} = A \cup A^{Bc}$.

Proposition(1.27):
Let $X$ be a topological space and $A \subset X$, then $A^{Bc}$ is the smallest Bc-closed set containing $A$.

Proposition(1.28)[5]:
Let $X$ be a topological space and $A \subset X$, then the following statements are true:
1) $(A^{Bc})^{c} = (A^{c})^{Bc}$.
2) $(A^{+Bc})^{c} = (A^{c})^{Bc}$.
3) $A^{Bc} = (A^{c})^{Bc}$.
4) $A^{Bc} = \left(\overline{A}^{Bc}\right)^{c}$.

Definition(1.29):
Let $X$ be a topological space and $A \subset X$, $A$ is called Bc-regular open set in $X$ iff $A = A^{Bc\theta}$. The complement of Bc-regular open set is called Bc-regular closed.

Definition(1.30):
Let $X$ be a topological space and $A \subset X$, $A$ is called Bc-regular open set in $X$ iff $A = A^{Bc\theta}$. The complement of Bc-regular open set is called Bc-regular closed.

Remark(1.31):
Let $X$ be a topological space and $A \subset X$, $A$ is Bc-regular open set, then $\overline{A}^{Bc\theta}$ is Bc-regular open set.

Proof:
To prove $\overline{A}^{Bc\theta}$ is Bc-regular open we must prove that $\overline{A}^{Bc\theta} = \overline{A}^{Bc\theta} \cap A^{Bc\theta}$, since $A \subset \overline{A}^{Bc\theta}$, then $A^{Bc\theta} \subset \overline{A}^{Bc\theta}$ and since $A$ is Bc-open set, hence $A \subset \overline{A}^{Bc\theta} \cap A^{Bc\theta} \subset \overline{A}^{Bc\theta}$ ... (1) Since $A^{Bc\theta} \subset \overline{A}^{Bc\theta}$, then $\overline{A}^{Bc\theta} \subset \overline{A}^{Bc\theta}$ ... (2) From (1) and (2) we get $\overline{A}^{Bc\theta} = \overline{A}^{Bc\theta} \cap A^{Bc\theta}$. Hence $\overline{A}^{Bc\theta}$ is Bc-regular open.

2. Separation Axiom

Definition(2.1)[7]:
A space $X$ is called $\partial T_{2} - space$ iff for each $x \neq y$ in $X$ there exist disjoint $\theta$-open sets $U, V$ such that $x \in U, y \in V$.

Definition(2.2):
A space $X$ is called Bc-regular space iff for each $x$ in $X$ and $\theta$-closed set such that $x \notin C$, there exist disjoint Bc-open sets $U, V$ such that $x \in U, C \subseteq V$.

Proposition(2.3): A space $X$ is Bc-regular space iff for every $x \in X$ and each $\theta$-open set $U$ in $X$ such that $x \in U$ there exists an Bc-open set $W$ such that $W \subseteq \overline{U}^{Bc} \subseteq U$.

Proof:
Let $X$ be a Bc-regular space and $x \in X$, $U$ is $\theta$-open in $X$ such that $x \in U$. Thus $U^{c}$ is $\theta$-closed set, $x \notin U^{c}$. Then there exist disjoint Bc-open set $W, V$ such that $x \in W, U^{c} \subseteq V$. Conversely let $F$ be an $\theta$-open set such that $x \notin F$. Then $F^{c}$ is $\theta$-open set and $x \notin F^{c}$. Thus there exist $W$ is Bc-open set such that $x \in W \subseteq \overline{W}^{Bc} \subseteq F^{c}$. Then $x \in W \subseteq \overline{W}^{Bc} \subseteq F^{c}$. Hence $X$ is Bc-regular space.

Definition(2.4):
A space $X$ is called Bc*-regular space iff for each $x$ in $X$ and Bc-closed set $C$ such that $x \notin C$, there exist disjoint sets $U, V$ such that $x \in U$, $V$ is a $\theta$-open and $x \in U, C \subseteq V$.

Proposition(2.5):
A space $X$ is Bc*$\theta$-regular space iff for every $x \in X$ and each Bc-open set $U$ in $X$ such that $x \in U$, there exists an $\theta$-open set $W$ such that $x \in W \subseteq \overline{U}^{Bc} \subseteq U$.

Proof:
Let $X$ be a Bc*$\theta$-regular space and $x \in X$, $U$ is Bc-closed in $X$ such that $x \in U$. Thus $U^{c}$ is Bc-$\theta$-closed set, $x \notin U^{c}$. Then there exist disjoint $\theta$-open set $W$, $V$ such that $W$ is $\theta$-open, $V$ is a Bc-open and $x \in W, U^{c} \subseteq V$. Hence $x \in W \subseteq \overline{W}^{BC} \subseteq \overline{V}^{BC} \subseteq V^{c} \subseteq U$. Conversely, let $F$ be a Bc-$\theta$-closed set such that $x \notin F$. Then $F^{c}$ is an $\theta$-open set and $x \notin F^{c}$. Thus there exist $W$ is $\theta$-open set such that $x \in W \subseteq \overline{W}^{BC} \subseteq F^{c}$. Then $x \in W \subseteq \overline{W}^{BC} \subseteq F^{c}$. Hence $X$ is Bc*$\theta$-regular space.

Definition(2.7):
A space $X$ is called almost Bc-regular space iff for each $x$ in $X$ and $C$ is $\theta$-regular closed set such that $x \notin C$, there exist disjoint Bc-closed sets $U, V$ such that $x \in U$, $C \subseteq V$.

Definition(2.8):
A space $X$ is called almost Bc*$\theta$-regular space iff for each $x$ in $X$ and $C$ is Bc-$\theta$-regular closed set such that $x \notin C$, there exist disjoint sets $U, V$ such that $x \in U$, $V$ is Bc-$\theta$-open and $x \in U, C \subseteq V$.

Proposition(2.9):
A space $X$ is almost Bc-regular space iff for every $x \in X$ and each $\theta$-open set $U$ in $X$ such that $x \in U$ there exists an Bc-open set $W$ such that $x \in W \subseteq \overline{W}^{Bc} \subseteq U$.

Proof:
Let $X$ be almost Bc-regular space and $x \in X$, $U$ is $\theta$-open in $X$ such that $x \in U$. Thus $U^{c}$ is $\theta$-closed set, $x \notin U^{c}$. Then there exist disjoint Bc-open set $W, V$ such that $x \in W, U^{c} \subseteq V$. Hence $x \in W \subseteq \overline{W}^{Bc} \subseteq \overline{V}^{Bc} \subseteq U$. Conversely, Let $F$ be an $\theta$-open set such that $x \notin F$. Then $F^{c}$ is an $\theta$-open set and
x ∈ Fc. Thus there exist W is Bc-open set such that x ∈ W ⊆ \( W^{Bc} \subseteq Fc \). Then \( x \in W, F \subseteq (W^{Bc})^c \) and \( W, (W^{Bc})^c \) are disjoint Bc-open set. Hence X is almost Bc-regular space.

**Proposition (2.10):**
A space X is almost Bc*-regular space iff for every x ∈ X and each Bc-regular open set U in X such that x ∈ U there exists an 0-open set W such that x ∈ W \( \subseteq W^{Bc} \subseteq U \).

**Proof:**
Let X be an almost Bc*-regular space and \( x, U \) be Bc-regular open in X such that x ∈ U. Thus U^c is Bc-regular closed set, x \( \subseteq U^c \). Then there exist disjoint set \( W, V \) such that W is 0-open, V is a Bc-open set and \( x \subseteq W \subseteq U^c \subseteq V \). Hence \( x \subseteq W \subseteq W^{Bc} \subseteq V^{Bc} \subseteq U \). Conversely, let F be an Bc-regular closed set such that \( x \in F \). Then F^c is an Bc-open regular set and \( x \subseteq F^c \). Thus there exist W is 0-open set such that \( x \subseteq W \subseteq W^{Bc} \subseteq F^c \). Then \( x \subseteq W, F \subseteq (W^{Bc})^c \) and \( (W^{Bc})^c \) is Bc-open set, \( W \cap (W^{Bc})^c = \phi \). Hence X is almost Bc-regular space.

**Definition (2.11):**
A space X is called Bc-normal space iff for every disjoint 0-closed set \( F_1, F_2 \) there exist disjoint Bc-open sets \( V_1, V_2 \) such that \( F_1 \subseteq V_1, F_2 \subseteq V_2 \).

**Proposition (2.12):**
A space X is called Bc-normal space iff for every 0-open set \( F \subseteq X \) and each 0-open set \( U \) in X such that \( F \subseteq U \) there exists an Bc-open set W such that \( F \subseteq W \subseteq W^{Bc} \subseteq U \).

**Proof:**
Let X be a Bc-normal space and let \( F \) be an 0-closed set in \( X, U \) is an 0-open set such that \( F \subseteq U \). Thus U^c is 0-closed set \( U^c, F \) are disjoint 0-open set, then there exists Bc-open sets \( W, V \) such that \( F \subseteq W, U^c \subseteq V, W \cap V = \phi \). Hence \( F \subseteq W \subseteq W^{Bc} \subseteq (W^{Bc})^c \subseteq U \). Conversely, let \( F_1, F_2 \) be a disjoint 0-closed set, then \( F_1^c, F_2^c \) is an 0-open set and \( F_1 \subseteq F_1^c, F_2 \subseteq F_2^c \). Thus there exist W is Bc-open set such that \( F_1 \subseteq W \subseteq W^{Bc} \subseteq F_2^c \). Then \( F_1 \subseteq W, F_2 \subseteq (W^{Bc})^c \) and \( (W^{Bc})^c \) is disjoint Bc-open set. Hence X is Bc-normal space.

**Proposition (2.13):**
If X is both Bc-normal and \( \partial l_2 \) - space , then X is Bc-regular.

**Proof:**
Let x ∈ X and U be an 0-open set such that x ∈ U. Thus \( \partial l_2 \) is 0-closed subset of X. Thus there exists a Bc-open set \( W \subseteq W^{Bc} \subseteq U \). By proposition (2.12), so that \( x \subseteq W \subseteq W^{Bc} \subseteq U \). Hence by proposition (2.3) X is Bc-regular space.

### 3.Bc-paracompact Spaces

**Definition (3.1)[9]:**
A covering of a topological space X is the family \( \{ A_\alpha : \alpha \in \Lambda \} \) of subsets such that \( \bigcup_{\alpha \in \Lambda} A_\alpha = X \). If each \( A_\alpha \) is open, then \( \{ A_\alpha : \alpha \in \Lambda \} \) is called an open covering, and if each \( A_\alpha \) is closed, then \( \{ A_\alpha : \alpha \in \Lambda \} \) is called a closed covering. A covering \( \{ B_\gamma : \gamma \in \Gamma \} \) is said to be refinement of a covering \( \{ A_\alpha : \alpha \in \Lambda \} \) if for each \( \gamma \) in \( \Gamma \) there exists some \( \alpha \in \Lambda \) such that \( B_\gamma \subseteq A_\alpha \).

**Definition (3.2):**
The family \( \{ B_\alpha : \alpha \in \Lambda \} \) of a subset of a space X is said to be an \( \theta \)-locally finite if for each \( x \in X \) there exist an \( \theta \)-neighborhood \( N_x \) of x such that the set \( \{ \alpha \in \Lambda : N_x \cap B_\alpha \neq \phi \} \) is finite.

**Proposition (3.3):**
If \( \{ B_\alpha : \alpha \in \Lambda \} \) is an \( \theta \)-locally finite family of subset of a space X, then there exist \( \{ A_\alpha : \alpha \in \Lambda \} \) of \( \{ C_\alpha : \alpha \in \Lambda \} \) such that \( \{ A_\alpha : \alpha \in \Lambda \} \subseteq \{ C_\alpha : \alpha \in \Lambda \} \subseteq \{ B_\alpha : \alpha \in \Lambda \} \) for each \( \alpha \), then \( \{ C_\alpha : \alpha \in \Lambda \} \) is an \( \theta \)-locally finite.

**Proof:**
Let \( \{ B_\alpha : \alpha \in \Lambda \} \) is an \( \theta \)-locally finite, for each \( x \in X \), then there exist \( \{ C_\alpha : \alpha \in \Lambda \} \) of \( \{ B_\alpha : \alpha \in \Lambda \} \) such that \( \{ C_\alpha : \alpha \in \Lambda \} \subseteq \{ B_\alpha : \alpha \in \Lambda \} \) for each \( \alpha \), then \( \{ C_\alpha : \alpha \in \Lambda \} \) is an \( \theta \)-locally finite.
then \( V \cap A_\alpha = \phi \), for \( \alpha \in \Lambda \). Now, we have \( \cap (U_{\alpha \in \Lambda} A_\alpha) = \phi \), so that since \( x \in V \), then \( x \notin \cup_{\alpha \in \Lambda} A_\alpha \), by proposition (1.24) which is a contradiction. Thus \( \cup_{\alpha \in \Lambda} A_\alpha = \cup_{\alpha \in \Lambda} \overline{A_\alpha} \), so that \( \cup_{\alpha \in \Lambda} A_\alpha - \cup_{\alpha \in \Lambda} \overline{A_\alpha} \), then \( \cup_{\alpha \in \Lambda} A_\alpha = \cup_{\alpha \in \Lambda} \overline{A_\alpha} \).

Proposition (3.6): The union of member of \( \theta \)-locally finite \( B_c \)-closed sets is \( B_c \)-closed.

**Proof:** Let \( \{A_\alpha\}_{\alpha \in \Lambda} \) be a family of \( \theta \)-locally finite \( B_c \)-closed sets. Then \( \cup_{\alpha \in \Lambda} A_\alpha = \cup_{\alpha \in \Lambda} \overline{A_\alpha} = \cup_{\alpha \in \Lambda} A_\alpha \), by theorem (3.4) and hence \( \cup_{\alpha \in \Lambda} A_\alpha \) is \( B_c \)-closed by theorem (1.23).

Theorem (3.7): Let \( \{A_\alpha\}_{\alpha \in \Lambda} \) be a family of \( \theta \)-open subsets of a space \( X \) and let \( \{B_\gamma\}_{\gamma \in \Gamma} \) be an \( \theta \)-locally finite \( B_c \)-closed covering of \( X \) such that for each \( \gamma \in \Gamma \) the set \( \{\alpha \in \Lambda: B_\gamma \cap A_\alpha \neq \phi\} \) is a finite. Then there exists \( \theta \)-locally finite family \( \{G_\alpha\}_{\alpha \in \Lambda} \) of \( B_c \)-open set of \( X \) such that \( A_\alpha \subseteq G_\alpha \) for each \( \alpha \in \Lambda \).

**Proof:** For each \( \alpha \), let \( G_\alpha = \left((F_\alpha - B_\gamma \cap A_\alpha = \phi)\right)^c \). Clearly \( G_\alpha \subseteq G_\alpha \) and since \( \{B_\gamma\}_{\gamma \in \Gamma} \) is an \( \theta \)-locally finite, it follow that \( G_\alpha \) is \( B_c \)-open by proposition (3.6). Let \( x \) be a point of \( X \), there exists an \( \theta \)-neighborhood \( N \) of \( x \), and a finite subset \( k \) of \( \Gamma \) such that \( N \cap \cap_{\gamma \in \Gamma} F_\gamma = \phi \) if \( \gamma \notin k \). Hence \( \cup_{\gamma \in \Gamma} F_\gamma \). Now \( F_\gamma \cap A_\alpha \neq \phi \) iff \( F_\gamma \cap A_\alpha \neq \phi \). For each \( \alpha \) in \( k \) the set \( \{\alpha \in \Lambda: F_\gamma \cap A_\alpha \neq \phi\} \) is a finite. Hence \( \{\alpha \in \Lambda: N \cap \cap A_\alpha = \phi\} \) is a finite.

Lemma(3.8): If every \( \theta \)-open cover of a topological space \( X \) has an \( \theta \)-locally finite \( B_c \)-closed refinement, then every \( \theta \)-open cover of \( X \) has a \( \theta \)-locally finite \( B_c \)-closed refinement.

**Proof:** Let \( U \) be a \( \theta \)-open cover of \( X \), and \( A = \{A_s: s \in S\} \) an \( \theta \)-locally finite of \( U \) and for each \( x \in X \) choose an \( \theta \)-neighborhood \( V_x \) of \( x \) which meets only finitely many members of \( A \). Let \( F \) be an \( \theta \)-locally finite \( B_c \)-closed refinement of the \( \theta \)-open cover \( \{V_x: x \in X\} \) and for each \( s \in S \), let \( V_s = ((F \cap \cap A_s) \cup F) \). Then \( V_s \) is a \( B_c \)-open and contain \( A_s \), for each \( s \in S \) and \( F \in F \), we have \( V_s \cap F \neq \phi \) iff \( A_s \cap F \neq \phi \). For each \( s \in S \) take a \( U_s \in U \) such that \( A_s \subseteq U_s \) and let \( V_s = W_s \cap U_s \). The family \( \{V_s\}_{s \in S} \) is a \( B_c \)-open refinement of \( U \). Since for each \( x \in X \) has an \( \theta \)-neighborhood such that meets only finitely many members of \( F \) and every members of \( F \) meets only finitely many members of \( A \). Therefore, \( \{V_s\}_{s \in S} \) is an \( \theta \)-locally finite \( B_c \)-closed refinement of \( U \).

Theorem(3.9): If every \( \theta \)-open covering of a space \( X \) has an \( \theta \)-locally finite \( B_c \)-closed refinement, then \( X \) is \( B_c \)-normal space.

**Proof:** Let \( X \) be a topological space such that each \( \theta \)-open covering of \( X \) which has an \( \theta \)-locally finite \( B_c \)-closed refinement and let \( A \), \( B \) be a disjoint \( \theta \)-open set of \( X \). The \( \theta \)-open covering \( \{A^c, B^c\} \) of \( X \) has an \( \theta \)-locally finite \( B_c \)-closed refinement \( W \). Let \( E \) be the union of the members of \( W \) disjoint from \( A \) and let \( S \) be the union of the members of \( W \) disjoint from \( B \). Then \( E \) and \( S \) are \( B_c \)-closed sets and \( E \cap W = \phi \). Thus if \( G = (E)^c \) and \( U = (S)^c \), then \( G, U \) are disjoint \( B_c \)-open sets such that \( A \subseteq G \), \( B \subseteq U \). Hence \( X \) is \( B_c \)-normal space.

Definition (3.10): A topological space \( X \) is said to be \( B_c \)-paracompact if every \( \theta \)-open covering of \( X \) has an \( \theta \)-locally finite \( B_c \)-open refinement.

Proposition (3.11): Let \( X \) be a \( B_c \)-paracompact space, let \( A \) be an \( \theta \)-open subset of \( X \) and let \( B \) be an \( \theta \)-closed set of \( X \) which is disjoint from \( A \). If for every \( x \in B \) there exist \( \theta \)-open sets \( U_x \) such that \( A \subseteq U_x \), \( \cap V \subseteq \phi \) and \( \cup V = \phi \).

**Proof:** Let \( X \) be a \( B_c \)-paracompact space, let \( A \) be an \( \theta \)-open subset of \( X \) and let \( B \) be an \( \theta \)-closed set of \( X \) which is disjoint from \( A \). If \( x \in B \) there exist \( \theta \)-open sets \( U_x \) such that \( A \subseteq U_x \), \( \cap V \subseteq \phi \) and \( \cup V = \phi \).

**Proof:** If \( X \) is a \( B_c \)-paracompact \( \theta \)T2 space, then \( X \) is \( B_c \)-regular.

**Theorem (3.13):** Let \( X \) be a topological space. If each \( \theta \)-open covering of \( X \) has an \( \theta \)-locally finite \( B_c \)-closed refinement, then \( X \) is \( B_c \)-paracompact \( B_c \)-normal Space.

**Proof:** Let \( U \) be an \( \theta \)-open covering of \( X \) and let \( \{A_\alpha\}_{\alpha \in \Lambda} \) be an \( \theta \)-locally finite \( B_c \)-closed refinement of \( X \). Since \( \{A_\alpha\}_{\alpha \in \Lambda} \) is an \( \theta \)-locally finite, for each point \( x \) of \( X \) has an \( \theta \)-neighborhood \( G_x \) such that \( \{\alpha \in \Lambda: G_x \cap A_\alpha = \phi\} \) is a finite. If \( \{B_\gamma\}_{\gamma \in \Gamma} \) is an \( \theta \)-locally finite \( B_c \)-closed refinement of the \( \theta \)-open covering \( \{G_\gamma\}_{\gamma \in \Gamma} \) of \( X \), then for each \( \gamma \in \Gamma \) the set \( \{\alpha \in \Lambda: B_\gamma \cap A_\alpha = \phi\} \) is a finite. It follows from theorem (3.9), that there exist an \( \theta \)-locally finite family \( \{V_\alpha\}_{\alpha \in \Lambda} \) of \( \theta \)-open sets such that \( A_\alpha \subseteq V_\alpha \) for each \( \alpha \). Let \( V_\alpha \) be a member of \( U \) such that \( A_\alpha \subseteq V_\alpha \) for each \( \alpha \in \Lambda \).
Then \((V_a \cap U_a)_{a \in A}\) is an 0-locally finite Bc-open refinement of \(X\). Thus \(X\) is Bc-paracompact, so that \(X\) is Bc-normal space by theorem(3.9).

**Theorem(3.14):**

Bc*-regular space is Bc-paracompact Bc-normal if and only if each \(0\)-open covering has an 0-locally finite Bc-closed refinement.

**Proof:**

Suppose that \(X\) is Bc-paracompact Bc-normal space and let \((A_\alpha)_{\alpha \in A}\) be an \(0\)-open covering of \(X\). Since \(X\) is Bc*-regular, there exists an \(0\)-open set \(V_\alpha\) such that \(x \in V_\alpha \subseteq \overline{U}^{bc}_{\alpha} \subseteq A_\alpha\) for some \(\alpha\). The family \((A_\alpha; x \in X)\) is an \(0\)-open cover of \(X\) and since \(X\) is Bc-paracompact, then there exists an 0-locally finite Bc-open refinement \(W = \{W_\beta^bc: x \in X\}\) of \((A_\alpha; x \in X)\). Hence \(\overline{W}^{bc}_{x} \subseteq \overline{W}^{bc}_{\alpha} \subseteq A_\alpha\). Then \(\{W_\beta^bc: x \in X\}\) is an 0-locally finite Bc-open refinement of \((A_\alpha)_{\alpha \in A}\). Conversely , from theorem(3.13).

**Theorem(3.15):**

Let \(X\) be any Bc*-regular space , the following condition are equivalent:
1) \(X\) is Bc-paracompact.
2) Every \(0\)-open cover of \(X\) has an 0-locally finite refinement.
3) Every \(0\)-open cover of \(X\) has a Bc-closed 0-locally finite refinement.

**Proof:**

1→2
Let \(X\) be a Bc-paracompact , then every \(0\)-open cover of \(X\) has an 0-locally finite refinement.

2→3
Let \(U\) be an \(0\)-open covering of \(X\). Since \(X\) is Bc*-regular, there exists an \(0\)-open set \(V_x\) such that \(x \in V_x \subseteq \overline{U}^{bc}_{x} \subseteq U_x\). The family \(V = \{V_x: x \in X\}\) is an \(0\)-open cover of \(X\), by (2) \(V\) has an \(0\)-locally finite refinement. Hence \(\{\overline{V}^{bc}_{x}: x \in X\}\) is an 0-locally finite Bc-open refinement of \(U\).

3→1
By lemma(3.14).

**Lemma(3.16):**

Let \(X\) be any Bc*-regular Bc-paracompact space. Then every Bc- open cover \(\{G_s: s \in S\}\) has an 0-locally finite Bc-open refinement \(\{U_s: s \in S\}\) such that \(\overline{U}^{bc}_{s} \subseteq G_s\) for each \(s \in S\).

**Proof:**

Let \(\{G_s: s \in S\}\) be any Bc-open cover of \(X\). For \(x \in X\), \(x \in G_s\), for some \(s \in S\) and since \(X\) is Bc*-regular, hence by proposition(1.36), there exists an \(0\)-open cover \(W = \{W_s: x \in X\}\) and \(\overline{W}^{bc}_{s} \subseteq G_s\). Since \(X\) is Bc-paracompact, then \(W\) has an 0-locally finite Bc-open refinement \(\{A_h: h \in H\}\) for each \(h \in H\) choose \(s(h) \in S\) such that \(A_{s(h)}^{bc} \subseteq G_{s(h)}\) and let \(U_s = \{A_{s(h)}: s(h) \in H\}\). Since \(\bigcup_{h \in H} A_{s(h)} \subseteq \overline{W}^{bc}_{s} = \bigcup_{s(h) \in H} A_{s(h)}^{bc} \subseteq G_s\), then \(\{U_s: s \in S\}\) is an 0-locally finite Bc-open refinement of \(\{G_s: s \in S\}\) such that \(\overline{U}^{bc}_{s} \subseteq G_s\) for each \(s \in S\).

**Definition(3.17):**

Let \(X\) be a topological space and \(\subseteq X\). \(A\) is said to be Bc-dense set if \(A^{bc} = X\).

**Definition(3.18):**

A topological space \(X\) is said to be Bc-Lindelof if every Bc-open cover of \(X\) has a countable sub cover.

**Theorem(3.19):**

Let \(X\) be any Bc*-regular Bc-paracompact space such that there exists an 0-open Bc-dense Bc-Lindelof set \(A\), then \(X\) is a Bc-Lindelof.

**Proof:**

Let \(U = \{U_s: s \in S\}\) be any Bc-open cover of \(X\). For each \(x \in X\), \(x \in U_s\), for some \(s \in S\). By lemma(3.16), there exists a Bc-0 locally finite Bc-dense Bc-Lindelof set \(\{A_s: s \in S\}\) such that \(\overline{A_s^{bc}} \subseteq U_s\) for each \(s \in S\). Then \(\bigcup_{s \in S} A_s\) is Bc-open cover of \(A\), by proposition(1.13). Since \(A\) is Bc-Lindelof, there exists a countable sub cover \(S_a \subseteq S\) such that \(A = \bigcup_{s \in S_a} A_s\). So \(X = A^{bc} = \bigcup_{s \in S_a} \overline{A_s^{bc}} = \bigcup_{s \in S_a} \overline{A_s^{bc}} \subseteq \bigcup_{s \in S_a} U_s\), hence \(X\) is Bc-Lindelof.

**Lemma(3.20):**

If \(U\) is an \(0\)-open covering of a topological space product \(X \times Y\) of a Bc-paracompact space \(X\) and an \(0\)-compact space and let \(U\) has a refinement of the form \(\{V_{a,i} \times G_{a,i}: i = 1, \ldots, n_a\}\), where \(V_{a,i} : a \in A\) is an \(0\)-locally finite Bc-open covering of \(X\), and for each \(a_i\), \(G_{a,i} = G_{a,i}: i = 1, \ldots, n_a\) is a finite \(0\)-open covering of \(Y\).

**Proof:**

Let \(x\) be a point of \(X\). Since \(Y\) is an \(0\)-compact there exists an \(0\)-open neighborhood \(W_{x}\) of \(x\) and a finite \(0\)-open covering \(\{V_{a,i}\}_{a \in A}\) of \(Y\) such that \(W_{x} \times V_{a,i} \subseteq U_{s}\) for each \(s \in S\). Then \(\{V_{a,i}\}_{a \in A}\) is an \(0\)-locally finite Bc-open refinement of \(W_{x}\) of the Bc-paracompact space \(X\). For \(x \in A\), choose \(x \in X\) such that \(V_{a,i} \subseteq W_{x}\) and let \(G_{x,a} = \{G_{a,i}: i = 1, \ldots, n_a\}\). Then \(\{G_{x,a}\}\) is a Bc-open refinement of \(U\).

**Proposition(3.21):**

The product of a Bc-paracompact space and an \(0\)-compact space is a Bc-paracompact space.

**Proof:**

Let \(X\) be a Bc-paracompact space and \(Y\) be an \(0\)-compact space and let \(U\) be an \(0\)-open covering of the topological product \(X \times Y\). Then by lemma(3.20) \(U\) has a Bc-open refinement of the form \(\{V_{a,i} \times G_{a,i}: i = 1, \ldots, n_a\}\), where \(V_{a,i} : a \in A\) is an \(0\)-locally finite Bc-open refinement and \(G_{a,i} = G_{a,i}: i = 1, \ldots, n_a\) is a finite \(0\)-open covering for \(Y\) and \(a_i \in A\). Therefore, \(X \times Y\) is a Bc-paracompact space.

**Definition(3.22):**

A space \(X\) is said to be nearly Bc-paracompact space if each \(0\)-regular open covering of \(X\) has a \(0\)-locally finite Bc-open refinement.
Lemma (3.23): Let $X$ be any almost Bc*-regular nearly Bc-paracompact space. Then every Bc-regular open cover $\{G_s : s \in S\}$ has an $\theta$-locally finite Bc-regular open refinement $\{V_s : s \in S\}$ such that $V_s \subseteq G_s$ for each $s \in S$.

Proof: Let $\{G_s : s \in S\}$ be any Bc-regular open cover of $X$. For $x \in X, x \in G_s$, for some $s \in S$ and since $X$ is almost Bc*-regular, hence by proposition (2.10), there exists an $\theta$-regular open cover $W = \{W_x : x \in X\}$ and $W_s^{BC} \subseteq G_s$. Since $X$ is nearly Bc-paracompact, then $W$ has an $\theta$-locally finite Bc-open refinement $\{A_h : h \in H\}$ for each $h \in H$ choose $s(h) \in S$ such that $A_{s(h)}^{BC} \subseteq G_{s(h)}$, and let $U_s = \bigcup_{s(h)=s} A_h$. Since $\bigcup_{s(h)=s} A_h \subseteq \bigcup_{s(h)=s} A_{s(h)}^{BC} = \bigcup_{s(h)=s} A_h^{BC} \subseteq G_s$, then $U_s \subseteq \bigcup_{s(h)=s} A_h^{BC} \subseteq G_s$. Hence $U_s \subseteq \bigcup_{s(h)=s} A_h^{BC} \subseteq G_s$. Let $V_s = U_s^{BC}$, then $\{V_s : s \in S\}$ is an $\theta$-locally finite Bc-regular open refinement of $\{G_s : s \in S\}$ such that $V_s^{BC} \subseteq G_s$ for each $s \in S$.

Theorem (3.24): For any space $X$, the following are equivalent:
1) $X$ is nearly Bc-paracompact.
2) Every $\theta$-regular open cover of $X$ has a Bc-regular open $\theta$-locally finite refinement.
3) Every $\theta$-regular open cover of $X$ has a Bc-regular closed $\theta$-locally finite refinement.

Proof: 1$\Rightarrow$2
Let $U$ be any $\theta$-regular open cover of $X$, then $U$ has an $\theta$-locally finite Bc-open refinement $V$. Consider the family $W = \{V^{BC} : V \in \mathcal{V}\}$ is an $\theta$-locally finite Bc-regular open refinement of $U$.

2$\Rightarrow$3
It is clear since every Bc-regular open set is Bc-regular closed set.

3$\Rightarrow$1
From lemma (3.8).

References