On Bc-open sets

Raad Aziz Al-Abdulla[1], Ruaa Muslim Abed[2]

[1, 2]Department of Mathematics, College of Computer Sciences and Mathematics, University of AL-Qadisiyah

Abstract: ***In this paper, we introduce a new class of open sets, called Bc-open sets, it is denoted and studied. Also, we have studied of definition Bc-paracompact spaces and nearly Bc-paracompact spaces and have provide some properties of this concepts.***

Keywords: θ-open, Bc-open

1.Introduction

In [5] H. Z. Ibrahim introduced the concept of Bc-open set in topological spaces. This paper consist of two sections. In section one, we give similar definition by using of Bc-open sets and also we proof some properties about it. In section two we obtain new a characterization and preserving theorems of Bc-paracompact spaces, nearly Bc-paracompact spaces and the product of space $X \times Y$ where X is Bc-paracompact space and Y is θ-compact space.

Definition(1.1)[3]:
Let X be a topological space and $A \subset X$. Then A is called b-open set in X if $A \subseteq \overline{A^{\circ}} \cup \overset{\circ}{\overline{A}}$. The family of all b-open subset of a topological space (X, τ) is denoted by $BO(X, \tau)$ or (Briefly $BO(X)$).

Definition(1.2)[5]:
Let X be a topological space and $A \subset X$. Then A is called Bc-open set in X if for each $x \in A \in BO(X, \tau)$, there exists a closed set F such that $x \in F \subset A$. The family of all Bc-open subset of a topological space (X, τ) is denoted by $BcO(X, \tau)$ or (Briefly $BcO(X)$), A is Bc-closed set if A^c is Bc-open set. The family of all Bc-closed subset of a topological space (X, τ) is denoted by $BcC(X, \tau)$ or (Briefly $BcC(X)$).

Remark(1.3):
It is clear from the definition that every Bc-open set is b-open, but the converse is not true in general as the following example:

Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Then the closed set are: $X, \phi, \{2,3\}, \{1,3\}, \{3\}$. Hence $BO(X) = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ and $BcO(X) = \{\phi, X, \{1,3\}, \{2,3\}\}$. Then $\{1\}$ is b-open but $\{1\}$ is not Bc-open.

Definition (1.4)[10]:
1) Let X be a topological space and $A \subset X$. Then A is called θ-open set in X if for each $x \in A$, there exists an open set G such that $x \in G \subset \bar{G} \subset A$. The family of all θ-open subset of a topological space (X, τ) is denoted by $\theta O(X, \tau)$ or (Briefly $\theta O(X)$).
2) Let X be a topological space and $A \subset X$. A point $x \in X$ is said to θ-interior point of A, if there exist an θ-open set U such that $x \in U \subset A$. The set of all θ-interior points of A is called θ-interior of A and is denoted by $A^{\circ\theta}$.
3) Let X be a topological space and $A \subset X$. The θ-closure of A is defined by the intersection of all Bc-closed sets in X containing A, and is denoted by $\bar{A}^{\theta}$.

Remark (1.5)[5]:
1) Every θ-open is Bc-open.
2) Every θ-closed is Bc-closed.

Example (1.6):
The intersection of two Bc-open sets is not Bc-open in general. Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Then $\{1,3\}, \{2,3\}$ is Bc-open set where as $\{1,3\} \cap \{2,3\} = \{3\}$ is not Bc-open set.

Remark (1.7)[2]:
The intersection of an b-open set and an open set is b-open set.

Proposition (1.8):
Let X be a topological space and $A, B \subset X$. If A is Bc-open set and B is an θ-open set , then $A \cap B$ is Bc-open set.

Proof:
Let A be a Bc-open set and B is an θ-open set, then A is b-open set and B is an open set since every θ-open is open. Then $A \cap B$ is b-open set by (Remark(1.7)). Now, let $x \in A \cap B$, $x \in A$ and $x \in B$, then there exists a closed set F such that $x \in F \subset A$, and there exists an open set E such that $x \in E \subset \bar{E} \subset B$. Therefore, $E \cap \bar{E}$ is closed since the intersection of closed sets is closed. Thus $x \in E \cap \bar{E} \subset A \cap B$ .Then $A \cap B$ is Bc-open set.

Proposition(1.9)[5]:
Let X be a topological space and $A \subset X$. Then A is Bc-open set if and only if A is b-open set and it is a union of closed sets. That is $A = \cup F_{\alpha}$ where A is b-open set and F_{α} is closed sets for each α.

Proposition(1.10)[5]:
Let $\{A_{\alpha} : \alpha \in \Lambda\}$ be a collection of Bc-open sets in a topological space X. Then $\cup\{A_{\alpha} : \alpha \in \Lambda\}$ is Bc-open.

Lemma(1.11)[4]:
Let X be a topological space and $Y \subset X$. If G is an θ-open in X, then $G \cap Y$ is an θ-open in Y.

Proposition(1.12)[5]:
Let X be a topological space and $Y \subset X$. If G is an b-open in X and Y is an open in X, then $G \cap Y$ is b-open in Y.

Proposition(1.13):
Let X be a topological space and $Y \subset X$. If G is an Bc-open in X and Y is an θ-open in X, then $G \cap Y$ is Bc-open in Y.

Proof:
Let $x \in G \cap Y$, $x \in G$ and $x \in Y$, Since G is a Bc-open set in X, then for each $x \in G \in BO(X)$, there exists F is closed set in X such that $x \in F \subset G$ and since Y is an θ-open in X, then there exists U is open set in X such that $x \in U \subset \bar{U} \subset Y$. Since G is Bc-open , then G is b-open and since Y is an θ-open, then Y is an open by proposition(1.12). Therefore, $G \cap Y$ is b-open in Y. Since $F, \bar{U}$ are closed set in X and $Y \subset X$, then $F \cap \bar{U}$ is closed set in Y. Thus $x \in F \cap \bar{U} \subset G \cap Y$. Hence $G \cap Y$ is Bc-open in Y.

Proposition(1.14):
Let X be a topological space and Y is an θ-open subset of X. If G is an Bc-open in Y, then G is Bc-open in X.

Proof:
Suppose that Y is an θ-open subset of X and $G \subset Y$, since G is a Bc-open set in Y, then for each $x \in G \in BO(Y)$, there exists F is closed set in Y such that $x \in F \subset G$. Let $G = Y \cap U$, $U \subset X$, and $F = E \cap Y$, $E \subset X$. Then $x \in E \subset X$. Hence G is Bc-open in X.

Lemma(1.15)[6]:
Let X and Y be a topological spaces and let $A \subset X, B \subset Y$ be two non empty subset:
1) If A is an open set in X and B is an open set in Y, then $A \times B$ is an open subset in $X \times Y$.
2) If A is a closed set in X and B is a closed set in Y, then $A \times B$ is a closed subset in $X \times Y$.
3) $\overline{(A \times B)} = \bar{A} \times \bar{B}$.

Theorem(1.16):
Let X and Y be a topological spaces and let $A \subset X, B \subset Y$ such that A is an θ-open set of X, B is an θ-open set of Y, then $A \times B$ is an θ-open subset of $X \times Y$.

Proof:
Let A be an θ-open set of X and B be an θ-open set of Y, then for each $x \in A$, there exists G open set in X such that $x \in G \subset \bar{G} \subset A$ and for each $y \in B$, there exists U open set in X such that $y \in U \subset \bar{U} \subset B$. By lemma(1.15)(1), then $G \times U$ is an open set in $X \times Y$. Since $\bar{G}$, $\bar{U}$ is closed set, then $\bar{G} \times \bar{U}$ is a closed set in $X \times Y$ by lemma (1.15)(2). Since $\bar{G} \times \bar{U} = \overline{G \times U}$ by lemma(1.15)(3), then $x \in G \times U \subset \overline{G \times U} \subset A \times B$. Hence $A \times B$ is an θ-open subset of $X \times Y$.

Proposition(1.17)[8]:
Let X and Y be a topological spaces and let $A \subset X, B \subset Y$ such that A is a b-open set of X, B is an open set of Y, then $A \times B$ is a b-open subset of $X \times Y$.

Proposition(1.18):
Let X and Y be a topological spaces and let $A \subset X, B \subset Y$ such that A is a Bc-open set of X, B is an θ-open set of Y, then $A \times B$ is a Bc-open subset of $X \times Y$.

Proof:
Let A be a Bc-open set of X and B be an θ-open set of Y, then for each $x \in A \in BO(X)$, there exists F closed set in X such that $x \in F \subset A$ and for each $y \in B$, there exists U open set in Y such that $y \in U \subset \bar{U} \subset B$. Since A is a Bc-open in X and B is an θ-open in Y, then A is a b-open in X and B be an open in Y. Thus $A \times B$ is a b-open subset of $X \times Y$ by proposition(1.17), $x \in A$ and $y \in B$, then $(x, y) \in A \times B \in BO(X)$. Since $x \in F \subset A$ and $y \in U \subset \bar{U} \subset B$ such that F is closed set in X and $\bar{U}$ is closed set in Y, then $F \times \bar{U}$ is closed set in $X \times Y$. Therefore, $(x, y) \in F \times \bar{U} \subset A \times B$. Hence $A \times B$ is a Bc-open subset in $X \times Y$.

Definition(1.19)[1]:
Let X be a topological space and $x \in X$. Then a subset N of x is said to be a θ-neighborhood of x, if there exists θ-open set U in X such that $x \in U \subset N$.

Definition(1.20)[5]:
Let X be a topological space and $A \subset X$. A point $x \in X$ is said to Bc-interior point of A, if there exist a Bc-open set U such that $x \in U \subset A$. The set of all Bc-interior points of A is called Bc-interior of A and is denoted by $A^{\circ Bc}$.

Theorem(1.21)[5]:
Let X be a topological space and $A, B \subset X$, then the following statements are true:
1) $A^{\circ Bc}$ is the union of all Bc-open set which are contained in A.
2) $A^{\circ Bc}$ is Bc-open set in X.
3) $A^{\circ Bc} \subset A$.
4) A is Bc-open if and only if $A = A^{\circ Bc}$.
5) $(A^{\circ Bc})^{\circ Bc} = A^{\circ Bc}$.
6) If $A \subset B$, then $A^{\circ Bc} \subset B^{\circ Bc}$.
7) $A^{\circ Bc} \cup B^{\circ Bc} \subset (A \cup B)^{\circ Bc}$.
8) $(A \cap B)^{\circ Bc} \subset A^{\circ Bc} \cap B^{\circ Bc}$.

Definition(1.1.22)[5]:
Let X be a topological space and $A \subset X$. The Bc-closure of A is defined by the intersection of all Bc-closed sets in X containing A, and is denoted by $\bar{A}^{Bc}$.

Theorem(1.23)[5]:
Let X be a topological space and $A, B \subset X$. Then the following statements are true:
1) $\bar{A}^{Bc}$ is the intersection of all Bc-closed sets containing A.
2) $\bar{A}^{Bc}$is Bc-closed set in X.
3) $A \subset \bar{A}^{Bc}$.
4) A is Bc-closed set if and only if $A = \bar{A}^{Bc}$.
5) $\overline{(\bar{A}^{Bc})}^{Bc} = \bar{A}^{Bc}$.
6) If $A \subset B$. then $\bar{A}^{Bc} \subset \bar{B}^{Bc}$.
7) $\bar{A}^{Bc} \cup \bar{B}^{Bc} \subset \overline{(A \cup B)}^{Bc}$.
8) $\overline{(A \cap B)}^{Bc} \subset \bar{A}^{Bc} \cap \bar{B}^{Bc}$.

Proposition(1.24)[5]:
Let X be a topological space and $A \subset X$. Then $x \in \bar{A}^{Bc}$ if and only if $A \cap U \neq \phi$ for every Bc-open set U containing x.

Definition(1.25)[5]:
Let X be a topological space and $A \subset X$. A point x is said to be Bc-limit point of A, if for each Bc-open set U containing x, $U \cap (A - \{x\}) \neq \phi$. The set of all Bc-limit points of A is called a Bc-derived set of A and is denoted by $\acute{A}^{Bc}$.

Proposition(1.26)[5]:
Let X be a topological space and $A \subset X$. Then $\bar{A}^{Bc} = A \cup \acute{A}^{Bc}$

Proposition(1.27):
Let X be a topological space and $A \subset X$, then $\bar{A}^{Bc}$ is the smallest Bc-closed set containing A.

Proposition(1.28)[5]:
Let X be a topological space and $A \subset X$, then the following statements are true:
1) $(\bar{A}^{Bc})^c = (A^c)^{\circ Bc}$.
2) $(A^{\circ Bc})^c = \overline{(A^c)}^{Bc}$.
3) $\bar{A}^{Bc} = \left(A^{c \circ Bc}\right)^c$.
4) $A^{\circ Bc} = \left(\overline{A^c}^{Bc}\right)^c$.

Definition(1.29):
Let X be a topological space and $A \subset X$, A is called θ-regular open set in X iff $A = \bar{A}^{\theta \circ \theta}$. The complement of θ-regular open set is called θ-regular closed.

Definition(1.30):
Let X be a topological space and $A \subset X$, A is called Bc-regular open set in X iff $A = \bar{A}^{Bc \circ Bc}$. The complement of Bc-regular open set is called Bc-regular closed.

Remark(1.31):
Let X be a topological space and $A \subset X$, A is Bc-regular open set, then $\bar{A}^{Bc \circ Bc}$ is Bc-regular open set.

Proof:
To prove $\bar{A}^{Bc \circ Bc}$ is Bc-regular open we must prove that $\bar{A}^{Bc \circ Bc} = \overline{\bar{A}^{Bc \circ Bc}}^{Bc \circ Bc}$, since $A \subset \bar{A}^{Bc}$, then $A^{\circ Bc} \subset \bar{A}^{Bc \circ Bc}$ and since A is Bc-open set, hence $A \subset \bar{A}^{Bc \circ Bc}$ $\bar{A}^{Bc \circ Bc} \subset \overline{\bar{A}^{Bc \circ Bc}}^{Bc \circ Bc}$ … (1) Since $\bar{A}^{Bc \circ Bc} \subset \bar{A}^{Bc}$, then $\overline{\bar{A}^{Bc \circ Bc}}^{Bc} \subset \bar{A}^{Bc \circ Bc} = \bar{A}^{Bc}$, hence $\overline{\bar{A}^{Bc \circ Bc}}^{Bc \circ Bc} \subset \bar{A}^{Bc \circ Bc}$ … (2) From (1) and (2) we get $\bar{A}^{Bc \circ Bc} = \overline{\bar{A}^{Bc \circ Bc}}^{Bc \circ Bc}$. Hence $\bar{A}^{Bc \circ Bc}$ is Bc-regular open.

2.Separation Axiom

Definition(2.1)[7]:
A space X is called $\theta T_2 - space$ iff for each $x \neq y$ in X there exist disjoint θ-open sets U, V such that $x \in U, y \in V$.

Definition(2.2):
A space X is called Bc-regular space iff for each x in X and C θ-closed set such that $x \notin C$, there exist disjoint Bc-open sets U, V such that $x \in U, C \subseteq V$.

Proposition(2.3):
A space X is Bc-regular space iff for every $x \in X$ and each θ-open set U in X such that $x \in U$ there exists an Bc-open set W such that $\in W \subseteq \bar{W}^{Bc} \subseteq U$.

Proof:
Let X be a Bc-regular space and $x \in X$, U is θ-open in X such that $x \in U$. Thus U^c is θ-closed set, $x \notin U^c$. Then there exist disjoint Bc-open set W, V such that $x \in W$, $U^c \subseteq V$. Hence $x \in W \subseteq \bar{W}^{Bc} \subseteq \bar{V^c}^{Bc} \subseteq V^c \subseteq U$. Conversely let F be an θ-closed set such that $x \notin F$. Then F^c is an θ-open set and $x \in F^c$. Thus there exist W is Bc-open set such that $x \in W \subseteq \bar{W}^{Bc} \subseteq F^c$. Then $x \in W$, $F \subseteq (\bar{W}^{Bc})^c$ and W, $(\bar{W}^{Bc})^c$ are disjoint Bc-open set. Hence X is Bc-regular space.

Definition(2.4):
A space X is called Bc*-regular space iff for each x in X and Bc-closed set C such that $x \notin C$, there exist disjoint sets U, V such that U is an θ-open, V is a Bc-open and $x \in U, C \subseteq V$.

Proposition(2.5):
A space X is Bc*-regular space iff for every $x \in X$ and each Bc-open set U in X such that $x \in U$ there exists an θ-open set W such that $x \in W \subseteq \bar{W}^{Bc} \subseteq U$.

Proof:
Let X be a Bc*-regular space and $x \in X$, U is Bc-open in X such that $x \in U$. Thus U^c is Bc-closed set, $x \notin U^c$. Then there exist disjoint set W, V such that W is an θ-open, V is a Bc-open and $x \in W, U^c \subseteq V$. Hence $x \in W \subseteq \bar{W}^{Bc} \subseteq \bar{V^c}^{Bc} \subseteq V^c \subseteq U$. Conversely, let F be an Bc-closed set such that $x \notin F$. Then F^c is an Bc-open set and $x \in F^c$. Thus there exist W is θ-open set such that $x \in W \subseteq \bar{W}^{Bc} \subseteq F^c$. Then $x \in W$, $F \subseteq (\bar{W}^{Bc})^c$ and $(\bar{W}^{Bc})^c$ is Bc-open set, $W \cap (\bar{W}^{Bc})^c = \phi$. Hence X is Bc*-regular space.

Definition(2.7):
A space X is called almost Bc-regular space iff for each x in X and C is θ-regular closed set such that $x \notin C$, there exist disjoint Bc-open sets U, V such that $x \in U$, $C \subseteq V$.

Definition(2.8):
A space X is called almost Bc*-regular space iff for each x in X and C is Bc-regular closed set such that $x \notin C$, there exist disjoint sets U, V such that U is θ-open, V is Bc-open and $x \in U, C \subseteq V$.

Proposition(2.9):
A space X is almost Bc-regular space iff for every $x \in X$ and each θ-regular open set U in X such that $x \in U$ there exists an Bc-open set W such that $x \in W \subseteq \bar{W}^{Bc} \subseteq U$.

Proof:
Let X be a almost Bc-regular space and $x \in X$, U is θ-regular open set in X such that $x \in U$. Thus U^c is θ-regular closed set, $x \notin U^c$. Then there exist disjoint Bc-open set W, V such that $x \in W, U^c \subseteq V$. Hence $x \in W \subseteq \bar{W}^{Bc} \subseteq \bar{V^c}^{Bc} \subseteq V^c \subseteq U$. Conversely, Let F be an θ-regular closed set such that $x \notin F$. Then F^c is an θ-regular-open set and

$x \in F^c$. Thus there exist W is Bc-open set such that $x \in W \subseteq \overline{W}^{Bc} \subseteq F^c$. Then $x \in W$, $F \subseteq (\overline{W}^{Bc})^c$ and $W, (\overline{W}^{Bc})^c$ are disjoint Bc-open set. Hence X is almost Bc-regular space.

Proposition(2.10):
A space X is almost Bc*-regular space iff for every $x \in X$ and each Bc-regular open set U in X such that $x \in U$ there exists an θ-open set W such that $x \in W \subseteq \overline{W}^{Bc} \subseteq U$.

Proof:
Let X be a almost Bc*-regular space and $x \in X$, U is Bc-regular open in X such that $x \in U$. Thus U^c is Bc-regular closed set, $x \notin U^c$. Then there exist disjoint set W, V such that W is an θ-open, V is a Bc-open and $x \in W, U^c \subseteq V$. Hence $x \in W \subseteq \overline{W}^{Bc} \subseteq \overline{V^c}^{Bc} \subseteq V^c \subseteq U$. Conversely, let F be an Bc-regular closed set such that $x \notin F$. Then F^c is an Bc-regular open set and $x \in F^c$. Thus there exist W is θ-open set such that $x \in W \subseteq \overline{W}^{Bc} \subseteq F^c$. Then $x \in W$, $F \subseteq (\overline{W}^{Bc})^c$ and $(\overline{W}^{Bc})^c$ is Bc-open set, $W \cap (\overline{W}^{Bc})^c = \phi$. Hence X is almost Bc*-regular space.

Definition(2.11):
A space X is called Bc-normal space iff for every disjoint θ-closed set F_1, F_2 there exist disjoint Bc-open sets V_1, V_2 such that $F_1 \subseteq V_1, F_2 \subseteq V_2$.

Proposition(2.12):
A space X is called Bc-normal space iff for every θ-closed set $F \subseteq X$ and each θ-open set U in X such that $F \subseteq U$ there exists an Bc-open set W such that $F \subseteq W \subseteq \overline{W}^{Bc} \subseteq U$.

Proof:
Let X be a Bc-normal space and let F is an θ-closed set in X, U is an θ-open set such that $F \subseteq U$. Thus U^c is θ-closed set U^c, F are disjoint θ-open set, then there exists Bc-open sets W, V such that $F \subseteq W$, $U^c \subseteq V, W \cap V = \phi$. Hence $F \subseteq W \subseteq \overline{W}^{Bc} \subseteq \overline{V^c}^{Bc} = V^c \subseteq U$. Conversely, let F_1, F_2 be a disjoint θ-closed set. Then ${F_1}^c$ is an θ-open set and $F_1 \subseteq {F_2}^c$. Thus there exist W is Bc-open set such that $F_1 \subseteq W \subseteq \overline{W}^{Bc} \subseteq {F_2}^c$. Then $F_1 \subseteq W$, $F_2 \subseteq (\overline{W}^{Bc})^c$ and $W, (\overline{W}^{Bc})^c$ are disjoint Bc-open set. Hence X is Bc-normal space.

Proposition(2.13):
If X is both Bc-normal and $\theta T_2 - space$, then X is Bc-regular.

Proof:
Let $x \in X$ and U be an θ-open set such that $x \in U$. Then $\{x\}$ is θ-closed subset of X. Thus there exists a Bc-open set W such that $\{x\} \subseteq W \subseteq \overline{W}^{Bc} \subseteq U$. By proposition (2.12). So that $x \in W \subseteq \overline{W}^{Bc} \subseteq U$ and hence by proposition (2.3) X is Bc-regular space.

3.Bc-paracompact Spaces

Definition(3.1)[9]:
A covering of a topological space X is the family $\{A_\alpha : \alpha \in \Lambda\}$ of subsets such that $\bigcup_{\alpha\in\Lambda} A_\alpha = X$. If each $A_\alpha$is open, then $\{A_\alpha : \alpha \in \Lambda\}$is called an open covering, and if each set $A_\alpha$is closed, then $\{A_\alpha : \alpha \in \Lambda\}$ is called a closed covering. A covering $\{B_\gamma : \gamma \in \Gamma\}$ is said to be refinement of a covering $\{A_\alpha : \alpha \in \Lambda\}$ if for each γ in Γ there exists some α in Λ such that $B_\gamma \subset A_\alpha$.

Definition(3.2):
The family $\{B_\alpha\}_{\alpha\in\Lambda}$ of a subset of a space X is said to be an θ-locally finite if for each $x \in X$ there exist an θ-neighborhood N_x of x such that the set $\{\alpha \in \Lambda : N_x \cap B_\alpha \neq \phi\}$ is finite.

Proposition(3.3):
If $\{B_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite family of subset of a space X, there exist a family $\{C_\alpha\}_{\alpha\in\Lambda}$, $C_\alpha \subset B_\alpha$for each α, then $\{C_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite.

Proof:
Let $\{B_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite, for each $x \in X$, then there exist G_x θ-open set containing x such that $G_x \cap B_{\alpha i} \neq \phi, i = 1, \dots, n$, hence $G_x \cap B_{\alpha j} = \phi, j = n+1, n+2, \dots$. Since G_x is an θ-open set, then G_x is Bc-open set, and hence ${G_x}^c$ is Bc-closed set. Therefore, for $B_{\alpha i} \subset {G_x}^c$, $i = n+1, n+2, \dots$. Hence $C_{\alpha i} \subset B_{\alpha i} \subset {G_x}^c, i = n+1, n+2, \dots$ .This implies $G_x \cap C_{\alpha i} = \phi, i = n+1, n+2, \dots$. Hence $\{G_x \cap C_{\alpha i} \neq \phi, j = 1, \dots, n\}$.Therefore, $\{C_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite.

Proposition(3.4):
Let (X, τ) be a topological space and $A \subset X$. If A_α is an θ-locally finite, then $\overline{A_\alpha}^{Bc}$is an θ-locally finite.

Proof:
Let $\{A_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite, for each $x \in X$, then there exist θ-open set G_x containing x such that $G_x \cap A_{\alpha i} \neq \phi, i = 1, \dots, n$, hence $G_x \cap A_{\alpha j} = \phi, j = n+1, n+2, \dots$. Since G_x is an θ-open set, then G_x is Bc-open set , and hence ${G_x}^c$ is Bc-closed set. Therefore, $A_{\alpha i} \subset {G_x}^c, i = n+1, n+2, \dots$. Hence $\overline{A_{\alpha i}}^{Bc} \subset \overline{{G_x}^c}^{Bc} = {G_x}^c, i = n+1, n+2, \dots$,then $\overline{A_{\alpha i}}^{Bc} \subset {G_x}^c, i = n+1, n+2, \dots$. This implies $G_x \cap \overline{A_{\alpha i}}^{Bc} = \phi, i = n+1, n+2, \dots$. Hence $\{G_x \cap \overline{A_{\alpha i}}^{Bc} \neq \phi, i = 1, \dots, n\}$. Therefore , $\overline{A_\alpha}^{Bc}$ is an θ-locally finite .

Proposition(3.5):
Let $\{A_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite Bc-closed family of a space X then $\bigcup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc} = \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}^{Bc}$.

Proof:
Since $A_\alpha \subset \bigcup_{\alpha\in\Lambda} A_\alpha$, then $\overline{A_\alpha}^{Bc} \subset \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}^{Bc}$by theorem(1.23) and hence$\bigcup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc} \subset \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}^{Bc}$. To prove that $\overline{\bigcup_{\alpha\in\Lambda} A_\alpha}^{Bc} \subset \bigcup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc}$. Let $x \in \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}^{Bc}$ such that $x \notin \bigcup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc}$, then $x \notin \overline{A_\alpha}^{Bc}$, for each $\alpha \in \Lambda$. Since $\{A_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite, then there exists an θ-open set G_x containing x such that $G_x \cap A_\alpha \neq \phi$ for only a finite member of α say $\alpha_1, \dots, \alpha_n$. Since $x \notin \overline{A_\alpha}^{Bc}$ for each $\in \Lambda$, then $x \notin A_\alpha$ and $x \notin \acute{A_\alpha}^{Bc}$ for each $\alpha \in \Lambda$ by proposition(1.27) . Thus there exists an Bc-open set U_x which contain x such that $U_x \cap A_\alpha = \phi$ for each $\alpha \neq \alpha_1, \dots, \alpha_n$. Let $x \in U_x \cap G_x = V$ is a Bc-open and since $U_x \cap A_\alpha = \phi$,for each $\alpha = \alpha_1, \dots, \alpha_n$, Since $V \subseteq U_x$ then $V \cap A_{\alpha 1} = \phi, \dots, V \cap A_{\alpha n} = \phi$. Since $G_x \cap A_\alpha = \phi$, for $\alpha = \alpha_1, \dots, \alpha_n$, then $V \cap A_\alpha = \phi$ for each $\alpha \neq \alpha_1, \dots, \alpha_n$,

then $V \cap A_\alpha = \phi$, for $\alpha \in \Lambda$. Now, we have $\cap(\cup_{\alpha\in\Lambda} A_\alpha) = \phi$, so that since $x \in V$, then $x \notin \overline{\cup_{\alpha\in\Lambda} A_\alpha}^{Bc}$, by proposition(1.24) which is a contradiction . Thus $\in \overline{\cup_{\alpha\in\Lambda} A_\alpha}^{Bc}$, so that $\overline{\cup_{\alpha\in\Lambda} A_\alpha}^{Bc} \subset \cup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc}$, then $\overline{\cup_{\alpha\in\Lambda} A_\alpha}^{Bc} = \cup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc}$.

Proposition (3.6):
The union of member of θ-locally finite Bc-closed sets is Bc-closed. **Proof:**
Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of θ-locally finite Bc-closed sets. Then $\overline{\cup_{\alpha\in\Lambda} A_\alpha}^{Bc} = \cup_{\alpha\in\Lambda} \overline{A_\alpha}^{Bc} = \cup_{\alpha\in\Lambda} A_\alpha$, by theorem (3.4) and hence $\cup_{\alpha\in\Lambda} A_\alpha$ is Bc-closed set by theorem (1.23).

Theorem (3.7):
Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of Bc-open subsets of a space X and let $\{B_\gamma\}_{\gamma\in\Gamma}$ be an θ-locally finite Bc-closed covering of X such that for each $\gamma \in \Gamma$ the set $\{\alpha \in \Lambda: B_\gamma \cap A_\alpha \neq \phi\}$ is a finite. Then there exists θ-locally finite family $\{G_\alpha\}_{\alpha\in\Lambda}$ of Bc-open set of X such that $A_\alpha \subset G_\alpha$ for each $\alpha \in \Lambda$.

Proof:
For each α, let $G_\alpha = (\{F_\gamma - B_\gamma \cap A_\alpha = \phi\})^c$. Clearly $A_\alpha \subset G_\alpha$ and since $\{B_\gamma\}_{\gamma\in\Gamma}$ is an θ-locally finite, it follow that G_α is Bc-open by proposition (3.6). Let x be a point of X, there exists an θ-neighborhood N of x , and a finite subset k of Γ such that $N \cap F_\gamma = \phi$ if $\gamma \notin k$. Hence $\subset \cup_{\gamma\in k} F_\gamma$. Now $F_\gamma \cap G_\alpha \neq \phi$ iff $F_\gamma \cap A_\alpha \neq \phi$. For each $\alpha \in k$ the set $\{\alpha \in \Lambda: F_\gamma \cap A_\alpha \neq \phi\}$ is a finite. Hence $\{\alpha \in \Lambda: N \cap G_\alpha \neq \phi\}$ is a finite.

Lemma(3.8):
If every θ-open cover of a topological space X has an θ-locally finite Bc-closed refinement, then every θ-open cover of X has an θ-locally finite Bc-open refinement.

Proof:
Let $\mathcal{U}$ be θ-open cover of X, and $\mathcal{A} = \{A_s: s \in S\}$ an θ-locally finite of $\mathcal{U}$ and for each $x \in X$ choose an θ-neighborhood V_x of x which meets only finitely many members of $\mathcal{A}$. Let $\mathcal{F}$ be an θ-locally finite Bc-closed refinement of the θ-open cover $\{V_x: x \in X\}$ and for each $s \in S$, let $W_s = (\{F \in \mathcal{F}: F \cap A_s\})^c$, then W_s is a Bc-open and contain A_s , for each $s \in S$ and $F \in \mathcal{F}$, we have $W_s \cap F \neq \phi$ iff $A_s \cap F \neq \phi$. For each $s \in S$ take a $U_s \in \mathcal{U}$ such that $A_s \in U_s$ and let $V_s = W_s \cap U_s$. The family $\{V_s\}_{s\in S}$ is a Bc-open refinement of $\mathcal{U}$. Since for each $x \in X$ has an θ-neighborhood such that meets only finitely many members of $\mathcal{F}$ and every members of $\mathcal{F}$ meets only finitely many members of $\mathcal{A}$. Therefore, $\{V_s\}_{s\in S}$ is an θ-locally finite Bc-open refinement of $\mathcal{U}$.

Theorem(3.9):
If every finite θ-open covering of a space X has an θ-locally finite Bc- closed refinement, then X is Bc-normal space.

Proof:
Let X be a topological space such that each finite θ-open covering of X which has an θ-locally finite Bc-closed refinement and let A, B be a disjoint θ-closed set of X. The θ-open covering $\{A^c, B^c\}$ of X has an θ-locally finite Bc-closed refinement W. Let E be the union of the members of W disjoint from A and let S be the union of the members of W disjoint from B.Then E and S are Bc-closed sets and $E \cup S = X$. Thus if $G = (E)^c$and $U = (S)^c$, then G, U are disjoint Bc-open sets such that $A \subseteq G$,$B \subseteq U$. Hence X is Bc-normal space.

Definition (3.10):
A topological space X is said to be Bc-paracompact if every θ-open covering of X has an θ-locally finite Bc-open refinement.

Proposition (3.11):
Let X be a Bc paracompact space, let A be an θ-open subset of X and let B be an θ-closed set of X which is disjoint from A. If for every $x \in B$ there exist θ-open sets U_x ,V_x such that $A \subset U_x$, $x \in V_x$ and $U_x \cap V_x = \phi$, then also there exist Bc-open sets U ,V such that $A \subset U$, $x \in V$ and $U \cap V = \phi$.

Proof:
The family $\{V_x: x \in B\} \cup \{(B)^c\}$ is an θ-open cover of Bc-paracompact , so that it has an θ-locally finite Bc-open refinement $\{W_\gamma\}_{\gamma\in\Gamma}$. Let $\Gamma_1 = \{\gamma \in \Gamma: W_\gamma \subset V_x \text{ for some } x \in B\}$. If $\in \Gamma_1$, then $U_x \cap W_\gamma = \phi$ for some x by proposition (3.4) , then $\overline{W_\gamma}^{Bc}$ is an θ-locally finite Bc-closed. Therefore,$A \cap \overline{W_\gamma}^{Bc} = \phi$. Now, let $U = \left(\cup_{\gamma\in\Gamma_1} \overline{W_\gamma}^{Bc}\right)^c$ and $V = \cup_{\gamma\in\Gamma_1} W_\gamma$. Then $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

Proposition(3.12):
If X is a Bc-paracompact θT_2 −space , then X is Bc-regular.

Proof:
Let $x \in X$and F be an θ-closed set in X such that $x \notin F$. Then for each $y \in F$ there exists θ-open sets U_y ,V_y such that $x \in U_y$, $y \in V_y$. It follow from proposition(3.11) there exists Bc-open sets U and V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$. Thus X is
Bc-regular.

Proposition(3.13):
Let X be a topological space . If each θ-open covering of X has an θ-locally finite Bc-closed refinement, then X is Bc-paracompact Bc-normal Space.

Proof:
Let $\mathcal{U}$ be an θ-open covering of X and let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an θ-locally finite Bc-closed refinement of . Since $\{A_\alpha\}_{\alpha\in\Lambda}$ is an θ-locally finite, for each point x of X has an θ-neighborhood G_x such that $\{\alpha \in \Lambda: G_x \cap A_\alpha \neq \phi\}$ is a finite. If $\{B_\gamma\}_{\gamma\in\Gamma}$ is an θ-locally finite Bc-closed refinement of the θ-open covering $\{G_x\}_{x\in X}$ of , then for each $\gamma \in \Gamma$ the set $\{\alpha \in \Lambda: B_\gamma \cap A_\alpha \neq \phi\}$ is a finite . It follows from theorem (3.9), that there exist an θ-locally finite family $\{V_\alpha\}_{\alpha\in\Lambda}$of Bc-open sets, such that $A_\alpha \subseteq V_\alpha$ for each . Let U_α be a member of $\mathcal{U}$ such that $A_\alpha \subseteq U_\alpha$, for each $\alpha \in \Lambda$.

Then $(V_\alpha \cap U_\alpha)_{\alpha\in\Lambda}$ is an θ-locally finite Bc-open refinement of . Thus X is Bc-paracompact, so that X is Bc-normal space by theorem(3.9).

Theorem(3.14):
Bc*-regular space is Bc-paracompact Bc-normal if and only if each θ-open covering has an θ-locally finite Bc-closed refinement.

Proof:
Suppose that X is Bc-paracompact Bc-normal space and let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an θ- open covering of . Since X is Bc*-regular, there exists an θ-open set V_x such that $x \in V_x \subseteq \overline{V_x}^{Bc} \subseteq A_\alpha$ for some . The family $\{A_x : x \in X\}$ is an θ-open cover of X and since X is Bc-paracompact, then there exists an θ-locally finite Bc-open refinement $\mathcal{W} = \{W_x : x \in X\}$ of $\{A_x : x \in X\}$. Hence $\overline{W_x}^{Bc} \subset \overline{V_x}^{Bc} \subset A_\alpha$, then $\left\{\overline{W_x}^{Bc} : x \in X\right\}$ is an θ-locally finite Bc-open refinement of $\{A_\alpha\}_{\alpha\in\Lambda}$. Conversely, from theorem(3.13).

Theorem(3.15):
Let X be any Bc*-regular space , the following condition are equivalent:
1) X is Bc-paracompact.
2) Every θ-open cover of X has an θ-locally finite refinement.
3) Every θ-open cover of X has a Bc-closed θ-locally finite refinement.

Proof:
1→2
Let X be a Bc-paracompact , then every θ-open cover of X has an θ-locally finite refinement.
2→3
Let $\mathcal{U}$ be an θ-open covering of X. Since X is Bc*-regular, there exists θ-open set V_x such that $\in V_x \subseteq \overline{V_x}^{Bc} \subseteq U_x$. The family $\mathcal{V} = \{V_x : x \in X\}$ is an θ-open cover of X, by (2) $\mathcal{V}$ has an θ-locally finite refinement. Hence $\left\{\overline{V_x}^{Bc} : x \in X\right\}$ is an θ-locally finite Bc-open refinement of $\mathcal{U}$.
3→1
By lemma(3.14).

Lemma(3.16):
Let X be any Bc*-regular Bc-paracompact space. Then every Bc- open cover $\{G_s : s \in S\}$ has an θ-locally finite Bc-open refinement $\{U_s : s \in S\}$ such that $\overline{U_s}^{Bc} \subseteq G_s$ for each $\in S$.

Proof:
Let $\{G_s : s \in S\}$ be any Bc-open cover of . For $x \in X$,$x \in G_s$,for some $s \in S$ and since X is Bc*-regular , hence by proposition(1.36), there exists an θ-open cover $\mathcal{W} = \{W_x : x \in X\}$ and $\overline{W_x}^{Bc} \subseteq G_s$. Since X is Bc-paracompact , then $\mathcal{W}$ has an θ-locally finite Bc-open refinement $\{A_h : h \in H\}$ for each $h \in H$ choose $s(h) \in S$ such that $\overline{A_h}^{Bc} \subseteq G_{s(h)}$,and let $U_s = \cup_{s(h)=s} A_h$. Since $\cup_{s(h)=s} A_h \subset \overline{\cup_{s(h)=s} A_h}^{Bc} = \cup_{s(h)=s} \overline{A_h}^{Bc} \subset G_s$,then $\{U_s : s \in S\}$ is an θ-locally finite Bc-open refinement of $\{G_s : s \in S\}$ such that $\overline{U_s}^{Bc} \subseteq G_s$ for each $s \in S$.

Definition(3.17):
Let X be a topological space and $\subseteq X$. A is said to be Bc-dence set if $\bar{A}^{Bc} = X$.

Definition(3.18):
A topological space X is said to be Bc-Lindelof if every Bc-open cover of X has a countable sub cover.

Theorem(3.19):
Let X be any Bc*-regular Bc-paracompact space such that there exists an θ-open Bc-dense Bc-Lindelof set , then X is a Bc-Lindelof .

Proof:
Let $\mathcal{U} = \{U_s : s \in S\}$ be any Bc-open cover of X. For each $x \in X$, $x \in U_s$, for some $s \in S$. By lemma (3.16), there exists a Bc-open θ-locally finite refinement $\{V_s : s \in S\}$ of $\mathcal{U}$ such that $\overline{V_s}^{Bc} \subseteq U_s$, for each $s \in S$. Then $\{V_s \cap A : s \in S\}$ is Bc-open cover of A, by proposition(1.13). Since A is Bc-Lindelof , there exists a countable set $S_\circ \subset S$ such that $A = \cup\{V_s \cap A : s \in S_\circ\}$. So $X = \bar{A}^{Bc} = \overline{\cup_{s\in S_\circ} V_s \cap A}^{Bc} = \cup_{s\in S_\circ} \overline{V_s \cap A}^{Bc} \subset \cup_{s\in S_\circ} \overline{V_s}^{Bc} \subset \cup_{s\in S_\circ} U_s$, hence X is Bc-Lindelof.

Lemma(3.20):
If $\mathcal{U}$ is an θ-open covering of a topological space product $X \times Y$ of a Bc-paracompact space X and an θ-compact space , then $\mathcal{U}$ has a refinement of the form $\{V_\alpha \times G_{i\alpha} : i = 1, \dots, n_\alpha\}$. Where $\{V_\alpha : \alpha \in \Lambda\}$ is an θ-locally finite Bc-open covering of X, and for each α,$\{G_{i\alpha} : i = 1, \dots, n_\alpha\}$ is a finite θ-open covering of Y.

Proof:
Let x be a point of . Since Y is an θ-compact there exists an θ-open neighborhood W_x of x and a finite θ-open covering $\mathcal{G}_x$ of Y such that $W_x \times G$ is contained in some member of $\mathcal{U}$ if $G \in \mathcal{G}_x$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an θ-locally finite Bc-open refinement of open covering $\{W_x : x \in \mathrm{X}\}$ of the Bc-paracompact space X. For α in Λ choose x in X such that $V_\alpha \subset W_x$ and let $\mathcal{G}_x = \{G_{i\alpha} : i = 1, \dots, n_\alpha\}$. Then $\{V_\alpha \times G_{i\alpha}\}$ is a Bc-open refinement of $\mathcal{U}$.

Proposition(3.21):
The product of a Bc-paracompact space and an θ-compact space is a Bc-paracompact space.

Proof:
Let X be a Bc-paracompact space and Y be an θ-compact space and let $\mathcal{U}$ be an θ-open covering of the topological product $X \times Y$. Then by lemma(3.20) $\mathcal{U}$ has a Bc-open refinement of the form $\{V_\alpha \times G_{i\alpha} : i = 1, \dots, n_\alpha\}$, where $\{V_\alpha : \alpha \in \Lambda\}$ is an θ-locally finite Bc-open refinement and $\{G_{i\alpha} : i = 1, \dots, n_\alpha\}$ is a finite θ-open covering of Y for $\alpha \in \Lambda$. Therefore, $X \times Y$ is a Bc-paracompact space.

Definition (3.22):
A space X is said to be nearly Bc-paracompact space if each θ-regular open covering of X has an θ-locally finite Bc-open refinement.

Lemma(3.23):
Let X be any almost Bc*-regular nearly Bc-paracompact space. Then every Bc-regular open cover $\{G_s: s \in S\}$ has an θ-locally finite Bc-regular open refinement $\{V_s: s \in S\}$ such that $\overline{V_s}^{Bc} \subseteq G_s$ for each $\in S$.

Proof:
Let $\{G_s: s \in S\}$ be any Bc-regular open cover of X. For $x \in X, x \in G_s$, for some $s \in S$ and since X is almost Bc*-regular, hence by proposition(2.10), there exists an θ-regular open cover $\mathcal{W} = \{W_x: x \in X\}$ and $\overline{W_x}^{Bc} \subseteq G_s$. Since X is nearly Bc-paracompact, then $\mathcal{W}$ has an θ-locally finite Bc-open refinement $\{A_h: h \in H\}$ for each $h \in H$ choose $s(h) \in S$ such that $\overline{A_h}^{Bc} \subseteq G_{s(h)}$, and let $U_s = \cup_{s(h)=s} A_h$. Since $\cup_{s(h)=s} A_h \subset \overline{\cup_{s(h)=s} A_h}^{Bc} = \cup_{s(h)=s} \overline{A_h}^{Bc} \subset G_s$, then $U_s \subset \overline{U_s}^{Bc} \subset G_s$, hence $U_s \subset \overline{U_s}^{Bc \circ Bc} \subset \overline{U_s}^{Bc} \subset G_s$. Let $V_s = \overline{U_s}^{Bc \circ Bc}$, then $\{V_s: s \in S\}$ is an θ-locally finite Bc-regular open refinement of $\{G_s: s \in S\}$ such that $\overline{V_s}^{Bc} \subseteq G_s$ for each $s \in S$.

Theorem (3.24):
For any space , the following are equivalent:
1) X is nearly Bc-paracompact.
2) Every θ-regular open cover of X has a Bc-regular open θ-locally finite refinement.
3) Every θ-regular open cover of X has a Bc-regular closed θ-locally finite refinement.

Proof:
1→2
Let $\mathcal{U}$ be any θ-regular open cover of X, then $\mathcal{U}$ has an θ-locally finite Bc-open refinement $\mathcal{V}$. Consider the family $\mathcal{W} = \left\{\overline{V}^{Bc \circ Bc}: V \in \mathcal{V}\right\}$ is an θ-locally finite Bc-regular open refinement of $\mathcal{U}$.
2→3
It is clear since every Bc-regular open set is Bc-regular closed set.
3→1
From lemma(3.8).

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