Implementation of 1D NN in Signal Processing Application to Get Relevant Recurrence Coefficient Value in Discrete Orthogonal Polynomial

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Abstract: ***In the implementation of algorithm we are evaluating the indexes of 1-D NN to the alpha by which we will get the relevant recurrence coefficient value in discrete orthogonal polynomial. In mathematics, a recurrence coefficient is an equation that recursively defines a sequence, once one or more initial terms are given and each further term of the sequence is defined as a function of the preceding terms.***

Keywords: recurrence coefficient, difference equation, linear recurrence sequence, infinite impulse response, PSNR

1. Introduction

1.1 Recurrence Coefficient:

In mathematics, the term difference equation sometimes and for the purposes of this research article refers to a specific type of recurrence coefficient. However, "difference equation" is frequently used to refer to *any* recurrence coefficient. An example of a recurrence coefficient is the logistic map:

$$x_{n+1} = rx_n(1 - x_n),$$

with a given constant r; given the initial term x0 each subsequent term is determined by this coefficient. Some simply defined recurrence coefficients can have very complex (chaotic) behaviours, and they are a part of the field of mathematics known as nonlinear analysis. Solving a recurrence coefficient means obtaining a closed-form solution: a non-recursive function of n.

1.2 Fibonacci numbers

The Fibonacci numbers are the archetype of a linear, homogeneous recurrence coefficient with constant coefficients (see below). They are defined using the linear recurrence coefficient

$$F_n = F_{n-1} + F_{n-2}$$

with seed values:

$$F_0 = 0$$
$$F_1 = 1$$

Explicitly, recurrence yields the equations:

$$F_2 = F_1 + F_0$$
$$F_3 = F_2 + F_1$$
$$F_4 = F_3 + F_2$$

etc.

We obtain the sequence of Fibonacci numbers which begins:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...

It can be solved by methods described below yielding the closed-form expression which involve powers of the two roots of the characteristic polynomial $t^2 = t + 1$; the generating function of the sequence is the rational function

$$\frac{t}{1 - t - t^2}.$$

2. Experimentation And Evaluation

2.1 Structure

Linear homogeneous recurrence coefficients with constant coefficients. An order *d* linear homogeneous recurrence coefficient with constant coefficients is an equation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d},$$

where the *d* coefficients c_i (for all *i*) are constants.

More precisely, this is an infinite list of simultaneous linear equations, one for each $n>d-1$. A sequence which satisfies a coefficient of this form is called a linear recurrence sequence or LRS. There are *d* degrees of freedom for LRS, i.e., the initial values $a_0, \ldots, a_{d-1}$ can be taken to be any values but then the linear recurrence determines the sequence uniquely.

The same coefficients yield the characteristic polynomial (also "auxiliary polynomial")

$$p(t) = t^d - c_1 t^{d-1} - c_2 t^{d-2} - \cdots - c_d$$

whose *d* roots play a crucial role in finding and understanding the sequences satisfying the recurrence. If the roots r_1, r_2, ... are all distinct, then the solution to the recurrence takes the form

$$a_n = k_1 r_1^n + k_2 r_2^n + \cdots + k_d r_d^n,$$

where the coefficients k_i are determined in order to fit the initial conditions of the recurrence. When the same roots occur multiple times, the terms in this formula corresponding to the second and later occurrences of the same root are multiplied by increasing powers of n. For instance, if the characteristic polynomial can be factored as $(x - r)^3$, with the same root r occurring three times, then the solution would take the form

$$a_n = k_1 r^n + k_2 n r^n + k_3 n^2 r^n.$$ [1]

2.2 Rational Generating Function

Linear recursive sequences are precisely the sequences whose generating function is a rational function: the denominator is the polynomial obtained from the auxiliary polynomial by reversing the order of the coefficients, and the numerator is determined by the initial values of the sequence.

The simplest cases are periodic sequences, $a_n = a_{n-d}, n \geq d$, which have sequence $a_0, a_1, \ldots, a_{d-1}, a_0, \ldots$ and generating function a sum of geometric series:

$$\frac{a_0 + a_1 x^1 + \cdots + a_{d-1} x^{d-1}}{1 - x^d}$$
$$= \left(a_0 + a_1 x^1 - \cdots + a_{d-1} x^{d-1}\right)$$
$$+ \left(a_0 + a_1 x^1 + \cdots + a_{d-1} x^{d-1}\right) x^d$$
$$+ \left(a_0 + a_1 x^1 + \cdots + a_{d-1} x^{d-1}\right) x^{2d} + \cdots.$$

More generally, given the recurrence coefficient:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

with generating function

$$a_0 + a_1 x^1 + a_2 x^2 + \cdots,$$

the series is annihilated at a_d and above by the polynomial:

$$1 - c_1 x^1 - c_2 x^2 - \cdots - c_d x^d.$$

That is, multiplying the generating function by the polynomial yields

$$b_n = a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_d a_{n-d}$$

as the coefficient on x^n, which vanishes (by the recurrence coefficient) for $n \geq d$. Thus

$$(a_0 + a_1 x^1 + a_2 x^2 + \cdots)(1 - c_1 x^1 - c_2 x^2 - \cdots - c_d x^d) = (b_0 + b_1 x^1 + b_2 x^2 + \cdots + b_{d-1} x^{d-1})$$

so dividing yields

$$a_0 + a_1 x^1 + a_2 x^2 + \cdots = \frac{b_0 + b_1 x^1 + b_2 x^2 + \cdots + b_{d-1} x^{d-1}}{1 - c_1 x^1 - c_2 x^2 - \cdots - c_d x^d},$$

Expressing the generating function as a rational function.

The denominator is $x^d p\left(x^{-1}\right)$, a transform of the auxiliary polynomial (equivalently, reversing the order of coefficients); one could also use any multiple of this, but this normalization is chosen both because of the simple coefficient to the auxiliary polynomial, and so that $b_0 = a_0$.

Coefficientship to difference equations narrowly defined. Given an ordered sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers: the first difference $\Delta(a_n)$ is defined as

$$\Delta(a_n) = a_{n+1} - a_n.$$

The second difference $\Delta^2(a_n)$ is defined as

$$\Delta^2(a_n) = \Delta(a_{n+1}) - \Delta(a_n),$$

which can be simplified to

$$\Delta^2(a_n) = a_{n+2} - 2a_{n+1} + a_n.$$

More generally: the k^{th} difference of the sequence a_n is written as $\Delta^k(a_n)$ is defined recursively as

$$\Delta^k(a_n) = \Delta^{k-1}(a_{n+1}) - \Delta^{k-1}(a_n) = \sum_{t=0}^{k} \binom{k}{t} (-1)^t a_{n+k-t}.$$

The sequence and its differences are related by a binomial transform. The more restrictive definition of difference equation is an equation composed of a_n and its k^{th} differences. (A widely used broader definition treats "difference equation" as synonymous with "recurrence coefficient". See for example rational difference equation and matrix difference equation.). Linear recurrence coefficients are difference equations, and conversely; since this is a simple and common form of recurrence, some authors use the two terms interchangeably. For example, the difference equation

$$3\Delta^2(a_n) + 2\Delta(a_n) + 7a_n = 0$$

is equivalent to the recurrence coefficient

$$3a_{n+2} = 4a_{n+1} - 8a_n$$

Thus one can solve many recurrence coefficients by rephrasing them as difference equations, and then solving the difference equation, analogously to how one solves ordinary differential equations. However, the Ackermann numbers are an example of a recurrence coefficient that do not map to a difference equation, much less points on the solution to a differential equation. The time scale calculus provides for a unification of the theory of difference equations with that of differential equations. Summation equations relate to difference equations as integral equations relate to differential equations. From sequences to grids Single-variable or one-dimensional recurrence coefficients are about sequences (i.e. functions defined on one-dimensional grids). Multi-variable or n-dimensional recurrence coefficients are about n-dimensional grids. Functions defined on n-grids can also be studied with partial difference equations [2].

2.3 Solving General Methods

For order 1, the recurrence

$$a_n = r a_{n-1}$$

has the solution $a_n = r^n$ with $a_0 = 1$ and the most general solution is $a_n = k r^n$ with $a_0 = k$. The characteristic polynomial equated to zero (the characteristic equation) is simply $t - r = 0$.

Solutions to such recurrence coefficients of higher order are found by systematic means, often using the fact that $a_n = r^n$

is a solution for the recurrence exactly when $t = r$ is a root of the characteristic polynomial. This can be approached directly or using generating functions (formal power series) or matrices.

Consider, for example, a recurrence coefficient of the form

$$a_n = Aa_{n-1} + Ba_{n-2}.$$

When does it have a solution of the same general form as $a_n = r^n$? Substituting this guess (ansatz) in the recurrence coefficient, we find that

$r^n = Ar^{n-1} + Br^{n-2}$ must be true for all $n > 1$.

Dividing through by r^{n-2}, we get that all these equations reduce to the same thing:

$$r^2 = Ar + B,$$

$$r^2 - Ar - B = 0,$$

which is the characteristic equation of the recurrence coefficient. Solve for r to obtain the two roots λ_1, λ_2: these roots are known as the characteristic roots or eigenvalues of the characteristic equation. Different solutions are obtained depending on the nature of the roots: If these roots are distinct, we have the general solution

$$a_n = C\lambda_1^n + D\lambda_2^n$$

while if they are identical (when $A^2 + 4B = 0$), we have

$$a_n = C\lambda^n + Dn\lambda^n$$

This is the most general solution; the two constants C and D can be chosen based on two given initial conditions a_0 and a_1 to produce a specific solution.

In the case of complex eigenvalues (which also gives rise to complex values for the solution parameters C and D), the use of complex numbers can be eliminated by rewriting the solution in trigonometric form. In this case we can write the eigen values as $\lambda_1, \lambda_2 = \alpha \pm \beta i$. Then it can be shown that $a_n = C\lambda_1^n + D\lambda_2^n$ can be rewritten as [3]:576-585

$$a_n = 2M^n\left(E\cos(\theta n) + F\sin(\theta n)\right) = 2GM^n\cos(\theta n - \delta),$$

where

$$M = \sqrt{\alpha^2 + \beta^2} \quad \cos\theta = \tfrac{\alpha}{M} \quad \sin\theta = \tfrac{\beta}{M}$$

$$C, D = E \mp Fi$$

$$G = \sqrt{E^2 + F^2} \quad \cos\delta = \tfrac{E}{G} \quad \sin\delta = \tfrac{F}{G}$$

Here E and F (or equivalently, G and δ) are real constants which depend on the initial conditions.

Using the facts that $\lambda_1 - \lambda_2 = 2\alpha = A$ and $\lambda_1 \lambda_2 = \alpha^2 + \beta^2 = -B$, one may simplify the solution given above as

$$a_n = (-B)^{n/2}\left(E\cos(\theta n) + F\sin(\theta n)\right),$$

where a_1 and a_2 are the initial conditions and

$$E = \frac{-Aa_1 + a_2}{B} \quad F = -i\frac{A^2a_1 - Aa_2 + 2a_1B}{B\sqrt{A^2+4B}} \quad \theta = \arccos\frac{A}{2\sqrt{-B}}$$

In this way there is no need to solve for λ_1 and λ_2.

In all cases—real distinct eigenvalues, real duplicated eigenvalues, and complex conjugate eigenvalues—the equation is stable (that is, the variable a converges to a fixed value (specifically, zero)); if and only if *both* eigenvalues are smaller than one in absolute value. In this second-order case, this condition on the eigenvalues can be shown[4] to be equivalent to $|A| < 1 - B < 2$, which is equivalent to $|B| < 1$ and $|A| < 1 - B$.

The equation in the above example was homogeneous, in that there was no constant term. If one starts with the non-homogeneous recurrence

$$b_n = Ab_{n-1} + Bb_{n-2} + K$$

with constant term K, this can be converted into homogeneous form as follows: The steady state is found by setting $b_n = b_{n-1} = b_{n-2} = b^*$ to obtain

$$b^* = \frac{K}{1 - A - B}.$$

Then the non-homogeneous recurrence can be rewritten in homogeneous form as

$$[b_n - b^*] = A[b_{n-1} - b^*] + B[b_{n-2} - b^*],$$

which can be solved as above.

The stability condition stated above in terms of eigenvalues for the second-order case remains valid for the general n^{th}-order case: the equation is stable if and only if all eigenvalues of the characteristic equation are less than one in absolute value.

Solving via linear algebra

Given a linearly recursive sequence, let C be the transpose of the companion matrix of its characteristic polynomial, that is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{d-1} \end{bmatrix}$$

where $T_n + c_{d-1}T_{n-1} + c_{d-2}T_{n-2} + \cdots + c_0T_{n-d} = 0$. Call this matrix C. Observe that

$$\begin{bmatrix} a_n \\ \vdots \\ a_{n+(d-1)} \end{bmatrix} = C^n \begin{bmatrix} a_0 \\ \vdots \\ a_{d-1} \end{bmatrix}$$

Determine an eigenbasis $v_1, \ldots, v_d$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_d$. Then express the seed (the initial conditions of the LRS) as a linear combination of the eigenbasis vectors:

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{d-1} \end{bmatrix} = b_1v_1 + \cdots + b_dv_d$$

Then it conveniently works out that:

$$\begin{bmatrix} a_n \\ \vdots \\ a_{n+(d-1)} \end{bmatrix} = C^n \begin{bmatrix} a_0 \\ \vdots \\ a_{d-1} \end{bmatrix} = C^n(b_1v_1 + \cdots + b_dv_d) = \lambda_1^n b_1 v_1 + \cdots + \lambda_d^n b_d v_d$$

This description is really no different from general method above, however it is more succinct. It also works nicely for situations like

$$a_n = a_{n-1} - b_{n-1}.$$

$$b_n = 2a_{n-1} + b_{n-1}.$$

Where there are several linked recurrences.

Solving with z-transforms

Certain difference equations, in particular Linear constant coefficient difference equations, can be solved using z-transforms. The *z*-transforms are a class of integral transforms that lead to more convenient algebraic manipulations and more straightforward solutions. There are cases in which obtaining a direct solution would be all but impossible, yet solving the problem via a thoughtfully chosen integral transform is straightforward.

Theorem

Given a linear homogeneous recurrence coefficient with constant coefficients of order d, let $p(t)$ *be the characteristic polynomial (also "auxiliary polynomial")*

$$t^d - c_1 t^{d-1} - c_2 t^{d-2} - \cdots - c_d = 0$$

such that each c_i corresponds to each c_i in the original recurrence coefficient (see the general form above). Suppose λ is a root of $p(t)$ having multiplicity r. This is to say that $(t-\lambda)^r$ divides $p(t)$. The following two properties hold:

1. Each of the r sequences $\lambda^n, n\lambda^n, n^2\lambda^n, \ldots, n^{r-1}\lambda^n$ satisfies the recurrence coefficient.
2. Any sequence satisfying the recurrence coefficient can be written uniquely as a linear combination of solutions constructed in part 1 as λ varies over all distinct roots of $p(t)$.

As a result of this theorem a linear homogeneous recurrence coefficient with constant coefficients can be solved in the following manner:

1. Find the characteristic polynomial $p(t)$.
2. Find the roots of $p(t)$ counting multiplicity.
3. Write a_n as a linear combination of all the roots (counting multiplicity as shown in the theorem above) with unknown coefficients b_i.

$$a_n = (b_1\lambda_1^n + b_2 n\lambda_1^n + b_3 n^2\lambda_1^n + \cdots + b_r n^{r-1}\lambda_1^n) + \cdots + (b_{d-q+1}\lambda_*^n + \cdots + b_d n^{q-1}\lambda_*^n)$$

This is the general solution to the original recurrence relation.

(q is the multiplicity of λ_*)

4. Equate each $a_0, a_1, a_2, \ldots, a_d$ from part 3 (plugging in $n = 0, \ldots, d$ into the general solution of the recurrence coefficient) with the known values $a_0, a_1, a_2, \ldots, a_d$ from the original recurrence coefficient. However, the values a_n from the original recurrence coefficient used do not have to be contiguous, just d of them are needed (i.e., for an original linear homogeneous recurrence coefficient of order 3 one could use the values a_0, a_1, a_4). This process will produce a linear system of d equations with d unknowns. Solving these equations for the unknown coefficients $b_1, b_2, b_3, \ldots, b_d$ of the general solution and plugging these values back into the general solution will produce the particular solution to the original recurrence coefficient that fits the original recurrence coefficient's initial conditions (as well as all subsequent values $a_0, a_1, a_2, a_3, \ldots$ of the original recurrence coefficient). The method for solving linear differential equations is similar to the method above—the "intelligent guess" (ansatz) for linear differential equations with constant coefficients is $e^{\lambda x}$ where λ is a complex number that is determined by substituting the guess into the differential equation.

This is not a coincidence. Considering the Taylor series of the solution to a linear differential equation:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

it can be seen that the coefficients of the series are given by the n^{th} derivative of $f(x)$ evaluated at the point a. The differential equation provides a linear difference equation relating these coefficients. This equivalence can be used to quickly solve for the recurrence coefficientship for the coefficients in the power series solution of a linear differential equation. The rule of thumb (for equations in which the polynomial multiplying the first term is non-zero at zero) is that:

$$y^{[k]} \to f[n+k]$$

and more generally

$$x^m * y^{[k]} \to n(n-1)(n-m+1)f[n+k-m]$$

Example: The recurrence coefficientship for the Taylor series coefficients of the equation:

$$(x^2+3x-4)y^{[3]} - (3x+1)y^{[2]} + 2y = 0$$

is given by

$$n(n-1)f[n+1] + 3nf[n+2] - 4f[n+3] - 3nf[n+1] - f[n+2] + 2f[n] = 0$$

Or

$$-4f[n+3] + 2nf[n+2] + n(n-4)f[n+1] + 2f[n] = 0.$$

This example shows how problems generally solved using the power series solution method taught in normal differential equation classes can be solved in a much easier way.

Example: The differential equation

$$ay'' + by' + cy = 0$$

has solution

$$y = e^{ax}.$$

The conversion of the differential equation to a difference equation of the Taylor coefficients is $af[n+2] + bf[n+1] + cf[n] = 0$

It is easy to see that the nth derivative of e^{ax} evaluated at 0 is a^n

Solving non-homogeneous recurrence coefficients

If the recurrence is inhomogeneous, a particular solution can be found by the method of undetermined coefficients and the solution is the sum of the solution of the homogeneous and the particular solutions. Another method to solve an inhomogeneous recurrence is the method of *symbolic differentiation*. For example, consider the following recurrence:

$$a_{n+1} = a_n + 1$$

This is an inhomogeneous recurrence. If we substitute $n \mapsto n+1$, we obtain the recurrence

$$a_{n+2} = a_{n+1} + 1$$

Subtracting the original recurrence from this equation yields

$$a_{n+2} - a_{n+1} = a_{n+1} - a_n$$

or equivalently

$$a_{n+2} = 2a_{n+1} - a_n$$

This is a homogeneous recurrence which can be solved by the methods explained above. In general, if a linear recurrence has the form

$$a_{n+k} = \lambda_{k-1}a_{n+k-1} - \lambda_{k-2}a_{n+k-2} + \cdots + \lambda_1 a_{n+1} + \lambda_0 a_n + p(n)$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ are constant coefficients and $p(n)$ is the inhomogeneity, then if $p(n)$ is a polynomial with degree r, then this inhomogeneous recurrence can be reduced to a homogeneous recurrence by applying the method of symbolic differencing r times.

If

$$P(x) = \sum_{n=0}^{\infty} p_n x^n$$

is the generating function of the inhomogeneity, the generating function

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n$$

of the inhomogeneous recurrence

$$a_n = \sum_{i=1}^{s} c_i a_{n-i} + p_n, \quad n \geq n_r,$$

with constant coefficients c_i is derived from

$$(1-\sum_{i=1}^{s} c_i x^i)A(x) - P(x) + \sum_{n=0}^{n_r-1} [a_n - p_n]x^n - \sum_{i=1}^{s} c_i x^i \sum_{n=0}^{n_r-i-1} a_n x^n.$$

If P(x) is a rational generating function, A(x) is also one. The case discussed above, where p_n=K is a constant, emerges as one example of this formula, with P(x)=K/(1-x). The solution of homogeneous recurrences is incorporated as p=P=0.

Moreover, for the general first-order linear inhomogeneous recurrence coefficient with variable coefficient(s) $a_{n+1} = f_n a_n + g_n, \; f_n \neq 0$, there is also a nice method to solve it:[5]

$$a_{n+1} = f_n a_n + g_n$$

$$a_{n+1} - f_n a_n = g_n$$

$$\frac{a_{n+1}}{\prod_{k=0}^{n} f_k} - \frac{f_n a_n}{\prod_{k=0}^{n} f_k} = \frac{g_n}{\prod_{k=0}^{n} f_k}$$

$$\frac{a_{n+1}}{\prod_{k=0}^{n} f_k} - \frac{a_n}{\prod_{k=0}^{n-1} f_k} = \frac{g_n}{\prod_{k=0}^{n} f_k}$$

Let $A_n = \frac{a_n}{\prod_{k=0}^{n-1} f_k}$,

Then $A_{n+1} - A_n = \frac{g_n}{\prod_{k=0}^{n} f_k}$

$$\sum_{m=0}^{n-1}(A_{m+1} - A_m) = A_n - A_0 = \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^{m} f_k}$$

$$\frac{a_n}{\prod_{k=0}^{n-1} f_k} = A_0 + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^{m} f_k}$$

$$a_n = \left(\prod_{k=0}^{n-1} f_k\right)\left(A_0 + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^{m} f_k}\right)$$

General linear homogeneous recurrence coefficients

Many linear homogeneous recurrence coefficients may be solved by means of the generalized hypergeometric series. Special cases of these lead to recurrence coefficients for the orthogonal polynomials, and many special functions. For example, the solution to

$$J_{n+1} = \frac{2n}{z} J_n - J_{n-1}$$

is given by

$$J_n = J_n(z),$$

the Bessel function, while

$$(b-n)M_{n-1} + (2n-b-z)M_n - nM_{n+1} = 0$$

is solved by

$$M_n = M(n, b; z)$$

the confluent hypergeometric series.

Solving a first order rational difference equation

A first order rational difference equation has the form $w_{t+1} = \frac{aw_t+b}{cw_t+d}$. Such an equation can be solved by writing w_t as a nonlinear transformation of another variable x_t which itself evolves linearly. Then standard methods can be used to solve the linear difference equation in x_t.

Stability

Stability of linear higher-order recurrences. The linear recurrence of order *d*,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d},$$

has the characteristic equation

$$\lambda^d - c_1\lambda^{d-1} - c_2\lambda^{d-2} - \dots - c_d\lambda^0 = 0.$$

The recurrence is stable, meaning that the iterates converge asymptotically to a fixed value, if and only if the eigenvalues (i.e., the roots of the characteristic equation), whether real or complex, are all less than unity in absolute value. Stability of linear first-order matrix recurrences

In the first-order matrix difference equation

$$[x_t - x^*] = A[x_{t-1} - x^*]$$

with state vector *x* and transition matrix *A*, *x* converges asymptotically to the steady state vector *x** if and only if all eigenvalues of the transition matrix *A* (whether real or complex) have an absolute value which is less than 1.

Stability of nonlinear first-order recurrences

Consider the nonlinear first-order recurrence

$$x_n = f(x_{n-1}).$$

This recurrence is locally stable, meaning that it converges to a fixed point *x** from points sufficiently close to *x**, if the slope of *f* in the neighborhood of *x** is smaller than unity in absolute value: that is,

$$|f'(x^*)| < 1.$$

A nonlinear recurrence could have multiple fixed points, in which case some fixed points may be locally stable and others locally unstable; for continuous *f* two adjacent fixed points cannot both be locally stable.A nonlinear recurrence coefficient could also have a cycle of period *k* for *k* > 1. Such a cycle is stable, meaning that it attracts a set of initial conditions of positive measure, if the composite function $g(x) := f \circ f \circ \cdots \circ f(x)$ with *f* appearing *k* times is locally stable according to the same criterion:

$$|g'(x^*)| < 1,$$

where *x** is any point on the cycle.

In a chaotic recurrence coefficient, the variable *x* stays in a bounded region but never converges to a fixed point or an attracting cycle; any fixed points or cycles of the equation are unstable.

Coefficientship to differential equations

When solving an ordinary differential equation numerically, one typically encounters a recurrence coefficient. For example, when solving the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

with Euler's method and a step size *h*, one calculates the values

$$y_0 = y(t_0), \quad y_1 = y(t_0 + h), \quad y_2 = y(t_0 + 2h), \dots$$

by the recurrence

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Systems of linear first order differential equations can be discretized exactly analytically using the methods shown in the discretization arena.

3. Results and Discussion

3.1 Applications in Biology

Some of the best-known difference equations have their origins in the attempt to model population dynamics. For example, the Fibonacci numbers were once used as a model for the growth of a rabbit population. The logistic map is used either directly to model population growth, or as a starting point for more detailed models. In this context, coupled difference equations are often used to model the interaction of two or more populations. For example, the

Nicholson-Bailey model for a host-parasite interaction is given by

$$N_{t+1} = \lambda N_t e^{-aP_t}$$
$$P_{t+1} = N_t(1 - e^{-aP_t}),$$

with N_t representing the hosts, and P_t the parasites, at time t.

Integrodifference equations are a form of recurrence coefficient important to spatial ecology. These and other difference equations are particularly suited to modeling univoltine populations.

3.2 Digital signal processing

In digital signal processing, recurrence coefficients can model feedback in a system, where outputs at one time become inputs for future time. They thus arise in infinite impulse response (IIR) digital filters. For example, the equation for a "feedforward" IIR comb filter of delay T is:

$$y_t = (1 - \alpha)x_t + \alpha y_{t-T}$$

Where x_t is the input at time t, y_t is the output at time t, and α controls how much of the delayed signal is fed back into the output. From this we can see that

$$y_t = (1 - \alpha)x_t + \alpha((1 - \alpha)x_{t-T} + \alpha y_{t-2T})$$
$$y_t = (1 - \alpha)x_t + (\alpha - \alpha^2)x_{t-T} + \alpha^2 y_{t-2T} \text{ etc.}$$

3.3 Peak Signal to Noise Ratio

Peak Signal-to-Noise Ratio, often abbreviated PSNR, is an engineering term for the ratio between the maximum possible power of a signal and the power of corrupting noise that affects the fidelity of its representation. Because many signals have a very wide dynamic range, PSNR is usually expressed in terms of the logarithmic decibel scale. PSNR is most commonly used to measure of quality of reconstruction of lossy compression codecs (e.g., for image compression). The signal in this case is the original data, and the noise is the error introduced by compression. When comparing compression codecs, PSNR is an approximation to human perception of reconstruction quality. Although a higher PSNR generally indicates that the reconstruction is of higher quality, in some cases the reverse may be true. One has to be extremely careful with the range of validity of this metric; it is only conclusively valid when it is used to compare results from the same codec (or codec type) and same content [1][2]. PSNR is most easily defined via the mean squared error (MSE). Given a noise-free m×n monochrome image I and its noisy approximation K, MSE is defined as:

$$MSE = \frac{1}{m\,n}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[I(i,j) - K(i,j)]^2$$

The PSNR is defined as:

$$\begin{aligned} PSNR &= 10 \cdot \log_{10}\left(\frac{MAX_I^2}{MSE}\right) \\ &= 20 \cdot \log_{10}\left(\frac{MAX_I}{\sqrt{MSE}}\right) \\ &= 20 \cdot \log_{10}(MAX_I) - 10 \cdot \log_{10}(MSE) \end{aligned}$$

```
[M N] = size(clean_signal);
MSE = sum(sum((clean_signal).^2))/(M*N);
PSNR = 10*log10(255*255/MSE);
```

Here, MAX_I is the maximum possible pixel value of the image. When the pixels are represented using 8 bits per sample, this is 255. More generally, when samples are represented using linear PCM with B bits per sample, MAX_I is 2^B-1. For color images with three RGB values per pixel, the definition of PSNR is the same except the MSE is the sum over all squared value differences divided by image size and by three. Alternately, for color images the image is converted to a different color space and PSNR is reported against each channel of that color space, e.g., YCbCr or HSL. Typical values for the PSNR in lossy image and video compression are between 30 and 50 dB, where higher is better. Acceptable values for wireless transmission quality loss are considered to be about 20 dB to 25 dB.

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