Application of Laplace Transform to Newtonian Fluid Problems

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Abstract: ***Laplace transform is employed to solve the following three problems of Newtonian fluid flow on an infinite plate: (i) Stokes' first problem for suddenly started plate and suddenly stopped plate, (ii) flow on an infinite plate, (iii) Ekman layer problem. Solutions are compared with those of Laplace transform and similarity methods. The results reveal that the method is very effective and simple.***

Keywords: Stokes' first problem; Ekman layer; Laplace transform; Exact solutions

1. Introduction

He Laplace transform is a widely used integral transform with many applications in physics and engineering. Denoted $L(f(t))$, it is a linear operator of a function $f(t)$ with a real argument $t(t \geq 0)$ that transforms $f(t)$ to a function $F(s)$ with complex argument s. This transformation is bijective for the majority of practical uses; the most-common pairs of $f(t)$ and $F(s)$ are often given in tables for easy reference. The Laplace transform has the useful property that many relationships and operations over the original $f(t)$ correspond to simpler relationships and operations over its image (s) .

The Laplace transform is a mathematical tool based on integration that has a number of applications. It particular, it can simplify the solving of many differential equations. We will find it particularly useful when dealing with non-homogeneous equations in which the forcing functions are not continuous. This makes it a valuable tool for engineers and scientists dealing with "real-world" applications. By the way, the Laplace transform is just one of many "integral transforms" in general use. Conceptually and computationally, it is probably the simplest. If you understand the Laplace transform, then you will find it much easier to pick up the other transforms as needed.

Inverse Laplace transform methods have a long history in the development of time-domain fluid line models. This paper presents a study combining the new Laplace-domain input/output (I/O) model derived from the network admittance matrix with the Fourier series expansion numerical inverse Laplace transform (NILT) to serve as a time-domain simulation model. A series of theorems are presented demonstrating the stability of the I/O model, which is important for the construction of the NILT method.

The Laplace Transform is defined by

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt \quad ; \quad t \geq 0 \qquad (1)\ (1)$$

In this work, we apply Laplace Transform to solve some Newtonian fluid flow problems. Solutions are compared

with those of Sumudu Transforms and similarity methods. The results reveal that the proposed method is very effective and simple.

2. Stokes' first problem

Consider a Cartesian coordinate system with the x-axis along an infinitely long flat plate, and an incompressible viscous fluid occupying the half-space $x \geq 0$. Since the fluid is viscous, we expect that the plate's effect diffuses into the fluid. If the motion of the boundary is in the x-direction, it may be reasonably assumed that the motion of the fluid will also be in that direction. Thus the only non-zero velocity component will be U and this velocity component will be a function of y and t only. Therefore

$$U = U(x,t)\ ,\ V = 0\ , W = 0 \ (2)$$

Then the pressure will be independent of , since U is independent of x , so will p be independent of x . That is, the pressure will be constant everywhere in the fluid. Using these properties of the flow field, the governing equations reduce to the following linear partial differential equation [7,12]

$$\frac{\partial U(x,t)}{\partial t} = \upsilon \frac{\partial^2 U(x,t)}{\partial x^2} \ (3)$$

2.1 Suddenly started plate

Initially, both the plate and the fluid are at rest. Suddenly, the plate is jerked into motion in its own plane with a constant velocity $U_0 i$ and continues to translate with this velocity for $t > 0$. Since the fluid is viscous, we expect that with the passage of time, the motion of the plate will be communicated to fluid. Thus, the boundary conditions for the problem under consideration are

$$U(0,t) = \begin{cases} 0, & t \leq 0 \\ U_0, & t > 0 \end{cases} \quad , \quad U(\infty,t) = 0 \tag{4}$$

We have the governing equation, initial condition, boundary conditions; therefore, the problem is well posed. We utilize the Laplace Transform method reducing the two variables into single variable, i.e. transferring partial differential equation into ordinary differential equation. This procedure will greatly reduce the difficulties of treating the original differential equation.

A whole set of governing equation, initial conditions and boundary conditions are prescribed, and this problem can be solved by Laplace Transform technique. The Eq. (3) and the boundary conditions take the following forms

$$\frac{d^2\overline{U}(x,s)}{dy^2} - \frac{s}{v}\overline{U}(x,s) = 0 \quad and \quad \overline{U}(0,s) = U_0 \quad ; \quad \overline{U}(\infty,s) = 0 \tag{5}$$

The general solution to Eq. (5) is

$$\overline{U}(x,s) = Ae^{-\left(\sqrt{\frac{s}{v}}\right)x} + Be^{\left(\sqrt{\frac{s}{v}}\right)x} \tag{6}$$

Using boundary conditions to solve the arbitrary constants *A* and *B* , then substitute values of these constants into Eq. (6) we get

$$\overline{U}(x,s) = U_0 e^{-\left(\sqrt{\frac{s}{v}}\right)x} \tag{7}$$

Taking inverse Laplace Transform, the velocity profile is

$$U(x,t) = L^{-1}\left[U_0 e^{-\left(\sqrt{\frac{s}{v}}\right)x}\right]$$

$$U(x,t) = U_0\left[1 - erf\left(\frac{x}{2}\sqrt{\frac{t}{v}}\right)\right] = U_0 erfc\left(\frac{x}{2}\sqrt{\frac{t}{v}}\right) \tag{8}$$

Where $erf(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-s^2}\,d\alpha$ (9)

is the complementary error function. The values of the complementary error function are tabulated in Table 2.

2.2 Suddenly stopped plate

In this case, initially, the fluid and plate are moving with uniform speed U_0 and the plate is suddenly decelerated to zero velocity. Therefore, the boundary conditions are

$$U(0,t) = \begin{cases} 0, & t > 0 \\ U_0, & t \leq 0 \end{cases} \quad , \quad U(\infty,t) = 0 \tag{10}$$

Taking Laplace Transform of Eq. (3) and the boundary conditions and using initial condition, we get

$$\frac{d^2\overline{U}(x,s)}{dy^2} - \frac{s}{v}\overline{U}(x,s) = -U_0\frac{s}{v} \quad and \quad \overline{U}(0,s) = U_0 \quad ; \quad \overline{U}(\infty,s) = 0 \tag{11}$$

The general solution of the non-homogeneous differential Eq. (11) is

$$\overline{U}(x,s) = Ce^{-\left(\sqrt{\frac{s}{v}}\right)x} + De^{\left(\sqrt{\frac{s}{v}}\right)x} + U_0 \tag{12}$$

Using boundary conditions, we obtain

$$\overline{U}(x,s) = U_0\left(1 - e^{-\left(\sqrt{\frac{s}{v}}\right)x}\right) \tag{13}$$

Taking inverse Laplace Transform to obtain

$$U(x,t) = U_0 erf\left(\frac{x}{2}\sqrt{\frac{t}{v}}\right) \tag{14}$$

The solutions given by Eqs. (8) and (14) are identical to those given by Sumudu Transforms and similarity methods [6,7,12].

It is obvious from Eqs. (8) and (14) that in both cases, the plate's effect diffuses into the fluid at a rate proportional to the square root of the kinematics viscosity. It is customary to define the shear layer thickness as the point where the wall effect on the fluid has dropped to 1 percent: in the first case (suddenly started plate), where $\frac{U}{U_0} = 0.01$; in the second case (suddenly stopped plate), where $\frac{U}{U_0} = 0.01$; these both correspond to $erf(\eta) = 0.01,\ or\ erf(\eta) = 1.82$. Then the shear layer thickness δ in these flows is, approximately,

$$\delta \approx 3.64\sqrt{\frac{t}{v}} \tag{15}$$

For example, for air at $20°C$ with $v = 1.5E-5\,\frac{m^2}{s}$, $\delta \approx 11cm$ after 1 min.

Table 1: Numerical Values of the complementary error function

(η)	$erf(\eta)$	(η)	$erf(\eta)$
0	1.0	1.1	0.11980
0.05	0.94363	1.2	0.08969
0.1	0.88754	1.3	0.06599
0.15	0.83200	1.4	0.04772
0.2	0.77730	1.5	0.03390
0.25	0.72367	1.6	0.02365
0.3	0.67137	1.7	0.01621
0.35	0.62062	1.8	0.01091
0.4	0.57161	1.9	0.00721
0.5	0.47950	2.0	0.00468
0.6	0.39615	2.5	0.000407
0.7	0.32220	3.0	0.0000221
0.8	0.25790	3.5	0.00000074
0.9	0.20309	4.0	0.00000001
1.0	0.15730	∞	0.0

3. Flow on an Infinite Plate

Consider the flow of a viscous fluid flow on an infinite plate under a constant pressure. If $x-axis$ is the direction of main flow then $U \neq 0, V = 0, W = 0$. The continuity equation is simply

$$\frac{\partial U}{\partial x} = 0$$

If $y - axis$ is taken normal to the plate, then we conclude that $U = U(y)$. The equations of motion reduce to one equation which is [8]

$$\frac{\partial^2 U}{\partial x^2} = 0, p = constant \quad (16)$$

Taking no-slip condition

$$U = 0 \; at \; y = 0 \quad (17)$$

and denoting the wall shear as

$$\tau_w = \mu \left(\frac{\partial U}{\partial y}\right)_{y=0} \quad (18)$$

The Laplace Transform of Eq. (16) and use of conditions (17) and (18) yields

$$\overline{U}(u) = \frac{\tau_w}{\mu} u \quad (19)$$

and the inverse Laplace Transform of the above equation gives

$$U(y) = \frac{\tau_w}{\mu} y \quad (20)$$

The Eq. (20) shows that the distribution of velocity is linear in . This result is identical to the analytical solution [8].

4. Ekman Layer Problem

Consider a viscous fluid on a surface when the surface is rotating at a constant angular velocity Ω, e.g., the Earth. We consider the surface to be almost flat and fluid to have a horizontal free surface. We introduce rectangular Cartesian coordinate system on the free surface with $z = 0$ at the free surface and $z - axis$ along the normal to the free surface. Let the free surface be subjected to a constant shearing force μS along the $x - axis$. Since the motion is steady, the velocity distribution is

$$U = U(z), \; V = V(z), \quad W = 0 \quad (21)$$

Further, by absorbing the centripetal and the body forces, a modified pressure P [8] is defined as

$$P = p + \rho\chi - \frac{1}{2}\rho\Omega^2 R^2 \quad (22)$$

where χ is the body force potential $f = -grad\,\chi$ and R is the perpendicular distance of a point from the axis of rotation. Since the velocity distribution is a function of z only, we have

$$\frac{\partial P}{\partial x} = 0, \qquad \frac{\partial P}{\partial y} = 0 \quad (23)$$

Thus, the governing equations are [8]

$$-2\Omega_3 V = \nu\frac{d^2 U}{dz^2}; \qquad 2\Omega_3 U = \nu\frac{d^2 V}{dz^2} \quad (24)$$

The pressure gradient in the z-direction is balanced by the Coriolis force [8], i.e.,

$$2(\Omega_1 V - \Omega_2 U) = -\frac{1}{\rho}\frac{dP}{dz} \quad (25)$$

Note that $\Omega_3 = \Omega\cos\theta$ where θ is the angle between the vector Ω and the unit vector k along the $z - axis$.

Taking Laplace Transform of Eq. (24) and using the following conditions

$$U = \frac{S}{2k}; \quad V = -\frac{S}{2k}; \quad at \; z = 0, \quad (26)$$

And $\mu\frac{dU}{dz} = \mu S, \quad \mu\frac{dV}{dz} = 0 \quad at \; z = 0, \quad (27)$

Where $k = \left(\frac{\Omega_3}{\nu}\right)^{\left(\frac{1}{2}\right)}$,

We obtain

$$-2\Omega_3\overline{V}(u) = \nu\left[\frac{\overline{U}(u)}{u^2} - \frac{S}{2ku^2} - \frac{S}{u}\right]; \quad 2\Omega_3\overline{U}(u) = \nu\left[\frac{\overline{V}(u)}{u^2} + \frac{S}{2ku^2}\right] \quad (28)$$

Coupling Eqs. (28), we get

$$(2i\Omega_3 u^2 - v)\lambda(u) = -v\left(\frac{S}{2k}(1-i) + Su\right), \quad where \ \ \lambda(u) = \overline{U} + i\overline{V} \tag{29}$$

Separating variables, we get from (29) that

$$\overline{U}(u) = \frac{S}{2k(1+4k^4u^4)} + \frac{Su}{(1+4k^4u^4)} + \frac{kSu^2}{(1+4k^4u^4)} \tag{30}$$

$$\overline{V}(u) = -\frac{S}{2k(1+4k^4u^4)} + \frac{kSu^2}{(1+4k^4u^4)} + \frac{2k^2Su^3}{(1+4k^4u^4)} \tag{31}$$

Taking inverse Laplace Transform of the Eqs. (30) and (31) and simplifying the resulting equations to obtain

$$U(z) = \frac{S}{\sqrt{2k}} e^{kz} \cos\left(kz - \frac{\pi}{4}\right), \quad V(z) = \frac{S}{\sqrt{2k}} e^{kz} \sin\left(kz - \frac{\pi}{4}\right) \tag{32}$$

It is worth mentioning again that these solutions are identical to analytical solutions [8]. At a depth $z = \frac{\pi}{k}\left(or\ z = -\frac{\pi}{k}\right)$ the velocity vector is

$$V = -\frac{S}{2k} e^{-\pi}(i - j) \tag{33}$$

Thus, at a depth $\frac{\pi}{k}$ the velocity vector has decreased by a factor of $e^{-\pi}$ and its direction has become opposite to that at the free surface. The depth

$$\frac{\pi}{k} = \pi\left(\frac{v}{\Omega_3}\right)^{\frac{1}{2}}, \tag{34}$$

is a measure of the Ekman layer thickness.

5. Conclusion

In this communication, we successfully applied Laplace Transform to solve three Newtonian fluid problems. The results are identical to those given in the literature. It gives a simple and a powerful mathematical tool. The results reveal that the method is very effective and simple.

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