The Order of the Set of Idempotent Elements of Semigroup of Partial Isometries of a Finite Chain

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Abstract: ***This paper lists, investigates and establishes the order of the set of idempotent elements of the semigroup of Partial Isometries of a finite chain.***

Keyword: Idempotent elements, Partial Isometries, Chain, Semigroup.

1. Introduction

Let $X_n = \{1, 2, 3, \dots, n\}$ and I_n be the partial one-to-one transformation semigroup on X_n under composition of mappings. Then I_n is an **inverse** semigroup (that is, for all $\alpha \in I_n$ there exist a unique $\alpha' \in I_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). The importance of I_n (more commonly known as the symmetric inverse semigroup or monoid) to inverse semigroup theory may be likened to that of the symmetric group S_n to group theory. Every finite semigroup S is embeddable in I_n. Let $X_n = \{1, 2, \dots, n\}$. A (partial) transformation $\alpha: Dom\,\alpha \subseteq X_n \rightarrow Im\,\alpha$ is said to be **full** or **total** if $Dom\,\alpha = X_n$; otherwise it is **strictly partial.** The height of α is $h(\alpha) = |Im\,\alpha|$, the width or breadth of α is $b(\alpha) = |Dom\,\alpha|$, the right(left) waist of α is $w^+(\alpha) = \max(Im\,\alpha)\,[w^-(\alpha) = \min(Im\,\alpha)]$, the collapse and fix of α are denoted by $c(\alpha)$ and $f(\alpha)$ and defined by $c(\alpha) = |C(\alpha)| = |\cup_{t\in Im\,\alpha}\{t\alpha^{-1}: |t\alpha^{-1}| \geq 2|, f(\alpha) = |F(\alpha)| = |\{x \in X_n: x\alpha = x\}|$ respectively, where $Im\,\alpha$ is the image of α and $Dom\,\alpha$ is the domain of α. A transformation $\alpha \in I_n$ is said to be an isometry or distance-preserving if $(\forall\, x, y \in Dom\,\alpha)$ $|x - y| = |x\alpha - y\alpha|$. It is well known that a partial transformation ε is idempotent $(\varepsilon^2 = \varepsilon)$ if and only if $Im\,\varepsilon = F(\varepsilon)$.

2. Methodology

The methodology is: (i) listing the idempotent elements of the semigroup in Domain/Image of α and (ii) investigating and establishing its order as follows: Let $DI_n = \{\alpha \in I_n: (\forall\, x, y \in X_n)|x - y| = |x\alpha - y\alpha|\}$ be the subsemigroup of I_n consisting of all partial isometries of X_n for $n = 1, 2, 3, 4, \dots$.. then

$E(DI_1)$ on $X_1 = \{1\}$ has 2 elements i.e $\binom{1}{1}$ and $\emptyset$.

$E(DI_2)$ on $X_2 = \{1, 2\}$ has 4 elements i.e

$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \binom{1}{1}, \binom{2}{2}, \emptyset$ with $|Im\,\alpha| = 2$

$Dom\,\alpha \mid Im\,\alpha$	{1, 2}
{1, 2}	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$

$|Im\,\alpha| = 1$

$Dom\,\alpha \mid Im\,\alpha$	{1}	{2}
{1}	$\binom{1}{1}$	
{2}		$\binom{2}{2}$

and $\emptyset$. $E(DI_3)$ on $X_3 = \{1, 2, 3\}$ has 8 elements with:

$|Im\,\alpha| = 3$,

$Dom\,\alpha \mid Im\,\alpha$	{1, 2, 3}
{1, 2, 3}	$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$

$|Im\,\alpha| = 2$,

$Dom\,\alpha \mid Im\,\alpha$	{1, 2}	{2, 3}	{1, 3}
{1, 2}	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$		
{2, 3}		$\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}$	
{1, 3}			$\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$

$|Im\,\alpha| = 1$,

$Dom\,\alpha \mid Im\,\alpha$	{1}	{2}	{3}
{1}	$\binom{1}{1}$		
{2}		$\binom{2}{2}$	
{3}			$\binom{3}{3}$

and $\emptyset$. $E(DI_4)$ on $X_4 = \{1, 2, 3, 4\}$ has 16 elements with:

$|Im\,\alpha| = 4$,

$Dom\,\alpha \mid Im\,\alpha$	{1, 2, 3, 4}
{1, 2, 3, 4}	$\begin{pmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{pmatrix}$

$|Im\,\alpha| = 3$,

$Dom\,\alpha \mid Im\,\alpha$	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}
{1, 2, 3}	$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$			
{1, 2, 4}		$\begin{pmatrix}1 & 2 & 4\\1 & 2 & 4\end{pmatrix}$		
{1, 3, 4}			$\begin{pmatrix}1 & 3 & 4\\1 & 3 & 4\end{pmatrix}$	
{2, 3, 4}				$\begin{pmatrix}2 & 3 & 4\\2 & 3 & 4\end{pmatrix}$

$|Im\ \alpha| = 2,$

$Dom\ \alpha \| Im\ \alpha$	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}
{1, 2}	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$					
{1, 3}		$\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$				
{1, 4}			$\begin{pmatrix}1 & 4\\1 & 4\end{pmatrix}$			
{2, 3}				$\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}$		
{2, 4}					$\begin{pmatrix}2 & 4\\2 & 4\end{pmatrix}$	
{3, 4}						$\begin{pmatrix}3 & 4\\3 & 4\end{pmatrix}$

$|Im\ \alpha| = 1,$

$Dom\ \alpha \| Im\ \alpha$	{1}	{2}	{3}	{4}
{1}	$\binom{1}{1}$			
{2}		$\binom{2}{2}$		
{3}			$\binom{3}{3}$	
{4}				$\binom{4}{4}$

and $\emptyset$. $E(DI_5)$ on $X_5 = \{1, 2, 3, 4, 5\}$ has 32 elements, $E(DI_6)$ on $X_6 = \{1, 2, 3, 4, 5, 6\}$ has 64 elements and $E(DI_7)$ on $X_7 = \{1, 2, 3, 4, 5, 6, 7\}$ has 128 elements. The tables of elements of $E(DI_5)$, $E(DI_6)$ and $E(DI_7)$ were constructed the same way.

3. Results

The results are shown in the triangle of numbers below and we prove a theorem that establishes the order of the set of Idempotent elements of the semi group.

Triangle of numbers F(n;Im α)

n/Im α	0	1	2	3	4	5	6	7	$\sum F(n; Im\ \alpha) = \|E(DI_n)\|$
0	1								1
1	1	1							2
2	1	2	1						4
3	1	3	3	1					8
4	1	4	6	4	1				16
5	1	5	10	10	5	1			32
6	1	6	15	20	15	6	1		64
7	1	7	21	35	35	21	7	1	128

Theorem: Let $E(DI_n)$ be the idempotent elements of DI_n. Then $|E(DI_n)| = 2^n$. Proof. $E(DI_n)$ is a subsemigroup of I_n because $|E(DI_n)| = |E(I_n)|$. In I_n, idempotents are partial identities i.e $id = \begin{pmatrix}1 & 2 & \dots n\\1 & 2 & \dots n\end{pmatrix}$. Also an element α is idempotent i.e $\alpha^2 = \alpha$ if $F(\alpha) = Im\ \alpha (= Dom\ \alpha)$. Idempotents are like the power set which are subset of a set e.g if $|A| = n$ then $|P(A)| = 2^n$. It is also obvious and without loss of generality that idempotents are special case of binomial theorem which says $\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = (x + y)^n = 2^n$ if $x = y = 1$. i.e $\sum_{r=0}^{n} \binom{n}{r} = 2^n. =>$ $|E(DI_n)| = 2^n$.

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