The Order of the Set of Idempotent Elements of Semigroup of Partial Isometries of a Finite Chain

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Abstract: This paper lists, investigates and establishes the order of the set of idempotent elements of the semigroup of Partial Isometries of a finite chain.

Keyword: Idempotent elements, Partial Isometries, Chain, Semigroup.

1. Introduction

Let $X_n = \{1, 2, 3, ..., n\}$ and $I_n$ be the partial one-to-one transformation semigroup on $X_n$ under composition of mappings. Then $I_n$ is an inverse semigroup (that is, for all $\alpha \in I_n$ there exist a unique $\alpha' \in I_n$ such that $\alpha = \alpha \alpha'$ and $\alpha' = \alpha' \alpha$). The importance of $I_n$ (more commonly known as the symmetric inverse semigroup or monoid) to inverse semigroup theory may be likened to that of the symmetric group $S_n$ to group theory. Every finite semigroup $S$ is embeddable in $I_n$. Let $X_n = \{1, 2, ..., n\}$. A (partial) transformation $\alpha: X_n \subseteq \rightarrow X_n$ is said to be full or total if $\alpha(X_n) = X_n$; otherwise it is strictly partial. The height of $\alpha$ is $h(\alpha) = |\alpha(X_n)|$, the width or breadth of $\alpha$ is $\alpha(\alpha) = |\alpha(X_n)|$, the right(left) waist of $\alpha$ is $\alpha(\alpha) = \max(\alpha(X_n))$ [$\alpha(\alpha) = \min(\alpha(X_n))$], the collapse and fix of $\alpha$ are denoted by $\alpha(\alpha)$ and $\alpha(\alpha)$ and defined by $\alpha(\alpha) = |\{\alpha \in \alpha(X_n) | |x - y| = |\alpha(x) - \alpha(y)|\}$ respectively, where $\alpha(X_n)$ is the image of $\alpha$ and $\alpha(X_n)$ is the domain of $\alpha$. A transformation $\alpha \in I_n$ is said to be an isometry or distance-preserving if $(\forall \alpha, \beta \in I_n)$ $|\alpha - \beta| = |\alpha(\alpha) - \beta(\beta)|$.

2. Methodology

The methodology is: (i) listing the idempotent elements of the semigroup in Domain/Image of $\alpha$ and (ii) investigating and establishing its order as follows: Let $DL_n = \{\alpha \in I_n | (\forall \alpha, \beta \in \alpha(X_n)) | |x - y| = |\alpha(x) - \beta(\beta)|\}$ be the subsemigroup of $I_n$ consisting of all partial isometries of $X_n$ for $n = 1, 2, 3, 4, ...$ then $E(DL_1)$ on $X_1 = \{1\}$ has 2 elements i.e $\{1\}$ and $\emptyset$. $E(DL_2)$ on $X_2 = \{1, 2\}$ has 4 elements i.e $\{1 \ 2\}, \{1 \ 2\}, \emptyset$ with $|Im \alpha| = 2$.
\(|\text{Im } \alpha| = 2\),

\begin{array}{c|c|c|c|c|c|c}
\text{Dom } \alpha & \text{Im } \alpha & \{1, 2\} & \{1, 3\} & \{1, 4\} & \{2, 3\} & \{2, 4\} & \{3, 4\} \\
\hline
1 & \{1\} & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
2 & \{2\} & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
3 & \{3\} & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
4 & \{4\} & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}

\(|\text{Im } \alpha| = 1\),

\begin{array}{c|c|c|c|c|c|c}
\text{Dom } \alpha & \text{Im } \alpha & \{1\} & \{2\} & \{3\} & \{4\} & \{5\} \\
\hline
1 & \{1\} & 1 & 1 & 1 & 1 & 1 \\
\hline
2 & \{2\} & 2 & 2 & 2 & 2 & 2 \\
\hline
3 & \{3\} & 3 & 3 & 3 & 3 & 3 \\
\hline
4 & \{4\} & 4 & 4 & 4 & 4 & 4 \\
\hline
5 & \{5\} & 5 & 5 & 5 & 5 & 5 \\
\end{array}

and \(\emptyset, E(D(S))\) on \(X_5 = \{1, 2, 3, 4, 5\}\) has 32 elements, \(E(D(S))\) on \(X_6 = \{1, 2, 3, 4, 5, 6\}\) has 64 elements and \(E(D(S))\) on \(X_7 = \{1, 2, 3, 4, 5, 6, 7\}\) has 128 elements. The tables of elements of \(E(D(S))\), \(E(D(S))\) and \(E(D(S))\) were constructed the same way.

3. Results

The results are shown in the triangle of numbers below and we prove a theorem that establishes the order of the set of Idempotent elements of the semi group.

| Triangle of numbers \(F(n, \text{Im } \alpha)\) |
|---|---|---|---|---|---|---|
| \(n/\text{Im } \alpha\) | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | \(\sum F(n, \text{Im } \alpha)\) | 1 | 2 | 4 | 8 | 16 |
| 6 | | 1 | 2 | 3 | 5 | 10 |
| 5 | | 1 | 2 | 3 | 5 | 7 |
| 4 | | 1 | 2 | 3 | 5 | 7 |
| 3 | | 1 | 2 | 3 | 5 | 7 |
| 2 | | 1 | 2 | 3 | 5 | 7 |
| 1 | | 1 | 2 | 3 | 5 | 7 |
| 0 | | 1 | 2 | 3 | 5 | 7 |

\textbf{Theorem:} Let \(E(D(S))\) be the idempotent elements of \(D(S)\). Then \(|E(D(S))| = 2^n\). Proof. \(E(D(S))\) is a subsemigroup of \(I_n\) because \(|E(D(S))| = |E(I_n)|\). In \(I_n\), idempotents are partial identities i.e \(id = \begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{pmatrix}\). Also an element \(\alpha\) is idempotent i.e \(\alpha^2 = \alpha\) if \(F(\alpha) = \text{Im } \alpha(= \text{Dom } \alpha)\). Idempotents are like the power set which are subset of a set e.g if \(|A| = n\) then \(|P(A)| = 2^n\). It is also obvious and without loss of generality that idempotents are special case of binomial theorem which says \(\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = (x + y)^n = 2^n\) if \(x = y = 1\). i.e \(\sum_{r=0}^{n} \binom{n}{r} = 2^n, \Rightarrow |E(D(S))| = 2^n\).

References