Applications of Meromorphic Multivalent Functions Associated with Differential Subordination

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Abstract: ***In this paper authors introduced subclasses*** $D_p^*(\alpha,\beta)$, $f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\varphi)$, $g_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\varphi)$, ***and*** $h_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\varphi)$, ***as well as*** $F_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\varphi)$, $G_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\varphi)$, ***and*** $H_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\varphi)$ ***of meromorphic multivalent functions in the punctured unit disk*** $D^*=\{z:0<|z|<1\}=D\backslash\{0\}$. ***By using the method of differential subordinations, we derive some certain properties of meromorphically multivalent functions.***

Keywords: Meromorphically multivalent function, Analytic function, Subordination, Differentiation.

1 Introduction

Let S denote the class of functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}a_{p+n-1}z^{p+n-1},\quad (p\in N) \tag{1}$$

Which are analytic and p-valent in the punctured unit disc $D^*=\{z:0<|z|<1\}=D\backslash\{0\}$. Let $f(z)$ and $g(z)$ be analytic in D, then, we say that $f(z)$ is subordinate to $g(z)$ in D, Where $f(z)\prec g(z)$, if there exists an analytic function $h(z)$ in D, such that $|h(z)|\le|z|$ and $f(z)=g[h(z)]$ $(z\in D)$. If $g(z)$ is univalent in D then the subordination

$f(z)\prec g(z)$ (D) $\Leftrightarrow$ $f(0)=g(0)$ and $f(D)\subset g(D)$.

Let $q(z)=1+q_1z+q_2z+\cdots\cdots$ be analytic in D,

Such that

Such that $q(z)\prec\frac{1+2\alpha z}{1+2\beta z}$ $\quad(z\in D)$. (2)

$$\left|q(z)-\frac{1-4\alpha\beta}{1-4\beta^2}\right|<\frac{2(\alpha-\beta)}{1-4\beta^2},\quad\left(-\frac{1}{2}\le\beta<\alpha\le\frac{1}{2}\right). \tag{3}$$

$$Re\,q(z)>\frac{1-2\alpha}{2},\qquad (2\beta=-1,\ z\in D). \tag{4}$$

Recently, several authors proved some interesting properties of meromorphically multivalent functions. In the present topic, we are going to prove some subordination properties for the class S.

When $g(z)=f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}b_{p+n-1}z^{p+n-1}$, $\quad(p\in N)$.

We define the Hadamard product (convolution) of $f(z)$ and $g(z)$ by

$$(f*g)(z)=f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}a_{p+n-1}b_{p+n-1}z^{p+n-1}.$$

Where $(p\in N,m\in N_0=N\cup\{0\},z\in D)$.

We define a linear operator by

$$I_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)$$
$$=\frac{1}{z^p}+\sum_{n=0}^{\infty}\left[\frac{\sigma\varepsilon(\xi+\eta)(p+n)}{(\delta-\upsilon)}+1\right]^m a_{p+n-1}z^{p+n-1}$$
$$=\left(\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{p,m}*f\right)(z).$$ Where

$$\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{p,m}(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}\left[\frac{\sigma\varepsilon(\xi+\eta)(p+n)}{(\delta-\upsilon)}+1\right]^m z^{p+n-1}.$$

Throughout this paper for our convenience we are taking $\frac{(\xi+\eta)}{\sigma\varepsilon(\delta-\upsilon)}=\Im$ and $I_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)=\psi_p^m f(z)$

$\left(p\in N,m\in N_0=N\cup\{0\},z\in D,\xi\ge0,\eta\ge0,0<\sigma\le\frac{1}{2}\right)$.

It is easy to verify that

$$\frac{\sigma\varepsilon(\xi+\eta)}{(\delta-\upsilon)}z\left[I_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)\right]'$$
$$=I_p^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)$$
$$-\left[\frac{\sigma\varepsilon(\xi+\eta)}{(\delta-\upsilon)}p+1\right]I_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z) \tag{5}$$

$$\therefore\ \Im z\left[\psi_p^m f(z)\right]'=\psi_p^{m+1}f(z)-(\Im+1)\psi_p^m f(z)$$

We note that

$\psi_p^0(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)=f(z)$ and

$$I_p^1\left(\frac{1}{2},1,1,1,\frac{1}{2},1\right)f(z)=\frac{\left[z^{p+1}f(z)\right]'}{z^p}$$
$$=(p+1)f(z)+zf'(z).$$

The above operators are analytic in D and satisfy the following condition

$Re\{q(z)\}>0,\quad(z\in D).\ for\ k\in N,\epsilon_k=exp\left(\frac{2\pi i}{k}\right)$,

$$f_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$$
$$=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}\left[\psi_p^m f\right](\epsilon_k^j z)=\frac{1}{z^p}+\cdots\quad(f\in S) \tag{6}$$

$$g_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$$
$$=\frac{1}{2}\left[\psi_p^m f(z)\right]+\overline{\psi_p^m f(\bar z)}=\frac{1}{z^p}+\cdots\quad(g\in S).$$

$$h_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]=\frac{1}{2}\left[\psi_p^m f(z)\right]+\overline{\psi_p^m f(-\bar z)}$$
$$=\frac{1}{z^p}+\cdots\quad(h\in S).$$

For $k=1$ we have,

$f_{p,1}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]=\psi_p^m f(z)$.

We now introduce and investigate the following subclasses of the class S of meromorphically p-valent functions.

1.1 Definitions

Definition 1.1.1 Function $f\in S$ is said to be in the class $F_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies the following subordination condition

$$\frac{-z\left[(1+\propto)\left(\psi_p^m f\right)'(z)+\propto\left(\psi_p^{m+1}f\right)'(z)\right]}{p\left[(1+\propto)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{p,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]}\prec\varphi(z).$$

$(\propto\ge0,z\in D^*,\varphi\in P)$, $f\in F_{p,1}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$f_{p,1}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\ne0$.

For simplicity we can write

$F_{p,k}^m(0;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi) = F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$.

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

Definition 1.1.2 Function $f\in S$ is said to be in the class $G_p^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$ if it satisfies the following subordination condition

$$\frac{-z\left[(1+\propto)(\psi_p^m f)'(z)+\propto(\psi_p^{m+1}f)'(z)\right]}{p[(1+\propto)g_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)+\propto g_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)]}\prec\varphi(z).$$

$(z\in D,\propto\ge 0), g\in G_p^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$,

and $g_p^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z]\neq 0$.

For simplicity we can write

$G_p^m(0;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi) = G_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$.

Definition 1.1.3 Function $f\in S$ is said to be in the class $H_p^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$. If it satisfies the following subordination condition:

$$\frac{-z\left[(1+\propto)(\psi_p^m f)'(z)+\propto(\psi_p^{m+1}f)'(z)\right]}{p[(1+\propto)h_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)+\propto h_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)]}\prec\varphi(z).$$

$z\in D,\propto\ge 0, h\in H_p^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$

and $h_p^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z]\neq 0$.

For simplicity we can write

$H_p^m(0;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi) = H_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$.

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

Definition 1.1.4 Function $f\in S$ is said to be in the class $\mathfrak{I}_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$ if it satisfies the following subordination condition

$$\frac{-z\left[(1+\propto)(\psi_p^m f)'(z)+\propto(\psi_p^{m+1}f)'(z)\right]}{p[(1+\propto)\ell_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)+\propto\ell_{p,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)]}\prec\varphi(z).$$

$z\in D,\propto\ge 0,\ell\in F_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$

$and\ \ell_{p,1}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z]\neq 0$.

For simplicity we can write

$\mathfrak{I}_{p,k}^m(0;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi) = \mathfrak{I}_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$.

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

Definition 1.1.5 Function $f\in S$ is said to be in the class $\mathfrak{U}_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$, if it satisfies the following subordination condition

$$\frac{-z\left[(1+\propto)(\psi_p^m f)'(z)+\propto(\psi_p^{m+1}f)'(z)\right]}{p\left[(1+\propto)u_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)+\propto u_{p,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)\right]}\prec\varphi(z).$$

$z\in D,\propto\ge 0, u\in$

$\mathfrak{U}_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)\ and\ u_{p,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z]\neq 0$.

For simplicity we can write

$\mathfrak{U}_{p,k}^m(0;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi) = \mathfrak{U}_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$.

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

Definition 1.1.6 Function $f\in S$ is said to be in the class $\mathfrak{X}_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$. If it satisfies the following subordination condition:

$$\frac{-z\left[(1+\propto)(\psi_p^m f)'(z)+\propto(\psi_p^{m+1}f)'(z)\right]}{p\left[(1+\propto)x_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)+\propto x_{p,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)\right]}\prec\varphi(z).$$

$z\in D,\propto\ge 0, x\in\mathfrak{X}_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z)$

$and\ x_{p,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;z]\neq 0$.

For simplicity we can write

$\mathfrak{X}_{p,k}^m(0;\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi) = \mathfrak{X}_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$.

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

1.2 Remarks

Remark 1.2.1 put $p=0,\xi=\eta=\delta=1,\varepsilon=\vartheta=\sigma=\frac{1}{2}$ and $\varphi(z)=\frac{1+z}{1-z}$ in definition 1.1.1, we have the class $Re\left\{-\frac{z\left[(1+3\propto)f'(z)+\propto\left(zf'(z)\right)\right]'}{(1+3\propto)T_s f(z)+\propto z[T_s f(z)]'}\right\}>0$, where. $T_s f(z)=\frac{1}{2}[f(z)-f(-z)]$.

Remark 1.2.2 For $\propto=0,\xi=\eta=\delta=1,\varepsilon=\vartheta=\sigma=\frac{1}{2}$ we have the class

$F_{p,k}^m\left(0;\frac{1}{2},1,1,\varepsilon,\delta,\vartheta;\varphi\right)=F_{p,k}^m(\varepsilon,\delta,\vartheta;\varphi)$.

Where the class $F_{p,k}^m(\varepsilon,\delta,\vartheta;\varphi)$ consisting of functions $f(z)\in S$, this satisfies the following subordination condition

$-\frac{z[\psi_p^m f]'(z)}{pf_{p,k}^m(\varepsilon,\delta,\vartheta;z)}\prec\varphi(z)$.Where $\varphi\in p\ and$

$f_{p,k}^m[\varepsilon,\delta,\vartheta;z]=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}[\psi_p^m f](\epsilon_k^j z)\neq 0$.

Such that $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

Remark 1.2.3 For $\propto=0,\xi=\eta=1,\sigma=\frac{1}{2}$ we have the class $F_{p,k}^m\left(0;\frac{1}{2},1,1,\frac{1}{2},1,1;\varphi\right)=F_{p,k}^m(\varepsilon,\delta,\vartheta;\varphi)$, where the class $F_{p,k}^m(\varepsilon,\delta,\vartheta;\varphi)$ consisting of functions $f(z)\in S$ which satisfies the following subordination condition

$-\frac{z[I_p^m(\varepsilon,\delta,\vartheta)f]'(z)}{pf_{p,k}^m(z)}\prec\varphi(z)$. Where $\varphi\in p$

$and\ f_{p,k}^m[\varepsilon,\delta,\vartheta;z]=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}[\psi_p^m f](\epsilon_k^j z)\neq 0$,

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

Remark 1.2.4 Putting $p=1,m=0,k=2,\xi=\eta=\delta=1,\varepsilon=\vartheta=\sigma=\frac{1}{2}$ and $\varphi(z)=\frac{1+z}{1-z}$ in definition 1.1.3, we have the class $Re\left\{-\frac{z\left[(1+3\propto)f'(z)+\propto\left(zf'(z)\right)\right]'}{(1+3\propto)T_{sc}f(z)+\propto z[T_s f(z)]'}\right\}>0$, where $T_{sc}f(z)=\frac{1}{2}[f(z)-\overline{f(-z)}]$

and $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\vartheta\ge 0,\delta>\vartheta,z\in D \end{pmatrix}$.

1.3 Preliminary Lemmas

Lemma 1.3.1: Let $\ell(z)$ be analytic and starlike univalent in D with $h(0)=0$. If $g(z)$ is analytic in $D\ and\ zg'(z)\prec\ell(z)$, then $g(z)\prec g(0)+\int_0^z\frac{h(t)}{t}dt$.

Lemma 1.3.2 Let $q(z)$ be analytic and other than constant $in\ D$ with $q(0)=1$. If $0<|z_0|<1$ and $Req(z_0)=min_{|z|\le|z_0|}Req(z)$. Then $z_0q'(z_0)\le\frac{-|1-q(z_0)|^2}{2[1-Re\,q(z_0)]}$.

Lemma 1.3.3 Let $d,r\in C$; and $\emptyset(z)$ is convex and univalent in D with $\emptyset(0)=1\ and\ Re[d\emptyset(z)+r]>0$. If $q(z)$ is analytic in D with $q(0)=1$, then the following subordination $q(z)+\frac{zq'(z)}{dq(z)+r}\prec\emptyset(z)\Rightarrow q(z)\prec\emptyset(z)$.

Lemma 1.3.4 Let $d, r \in C$; and $\emptyset(z)$ is convex and univalent in D with $\emptyset(0) = 1$ and $Re[d\emptyset(z) + r] > 0$. Also let $q(z) \prec \emptyset(z)$. If $q(z)$ is analytic in D with $q(0) = 1$, then the following subordination

$$q(z) + \frac{zq'(z)}{dq(z)+r} \prec \emptyset(z) \Rightarrow q(z) \prec \emptyset(z) \qquad (z \in D).$$

Lemma 1.3.5 Let $f \in F_{p,k}^{m}(\alpha; \sigma, \xi, \eta, \varepsilon, \delta, \upsilon; \varphi)$ then

$$\frac{-z\left[(1+\propto)\left(f_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(f_p^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{p\left[(1+\propto)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{p,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z)$$

$(z \in D)$. Therefore, $if\ \emptyset \in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right] > 0.$$

Where $\begin{pmatrix} 0 < \sigma \le \frac{1}{2}, \xi > 0, \eta \ge 0, 0 < \varepsilon \le \frac{1}{2}, \\ a > 0, \delta > 0, \upsilon \ge 0, \delta > \upsilon, z \in D \end{pmatrix}$.

Proof: $For\ (j \in \{0, 1, 2, \cdots, k-1\})$ we have obtained

$$f_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] = \frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}\left[\psi_p^m f\right]\left(\epsilon_k^j z\right).$$

Hence $f_{p,k}^m\left[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right]$

$$= \frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{np}\left[\psi_p^m f\right]\left(\epsilon_k^{n+j} z\right)$$

$$= \epsilon_k^{-jp}\frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{(n+j)p}\left[\psi_p^m f\right]\left(\epsilon_k^{n+j} z\right)$$

$$= \epsilon_k^{-jp} f_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \qquad (7)$$

And $\left[f_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\right]'$

$$= \frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{j(p+1)}\left[f_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right]'\left(\epsilon_k^j z\right). \qquad (8)$$

Rep*l*acing m *by* $m+1$ in (A) we get

$$f_{p,k}^{m+1}\left[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right]$$

$$= \epsilon_k^{-jp} f_{p,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \qquad (9)$$

Replacing m *by* $m+1$ in (B) we get

$$\left[f_{p,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\right]'$$

$$= \frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{j(p+1)}\left[f_p^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right]'\left(\epsilon_k^j z\right). \qquad (10)$$

From (7) and (10) we obtained

$$\frac{-z\left[(1+\propto)\left(f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(f_{p,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{p\left[(1+\propto)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{p,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]}$$

$$= \frac{-1}{k}\sum_{j=0}^{k-1}\frac{\epsilon_k^{j(p+1)}z\left[(1+\propto)\left(\psi_p^m f\right)'\left(\epsilon_k^j z\right)+\propto\left(\psi_p^{m+1} f\right)'\left(\epsilon_k^j z\right)\right]}{p\left[(1+\propto)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{p,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]}$$

$$= \frac{-1}{k}\sum_{j=0}^{k-1}\frac{\epsilon_k^j z\left[(1+\propto)\left(\psi_p^m f\right)'\left(\epsilon_k^j z\right)+\propto\left(\psi_p^{m+1} f\right)'\left(\epsilon_k^j z\right)\right]}{p\left[(1+\propto)f_{p,k}^m\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)+\propto f_{p,k}^{m+1}\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)\right]}.$$

It is clear that

$$\frac{-\epsilon_k^j z\left[(1+\propto)\left(\psi_p^m f\right)'\left(\epsilon_k^j z\right)+\propto\left(\psi_p^{m+1} f\right)'\left(\epsilon_k^j z\right)\right]}{p\left[(1+\propto)f_{p,k}^m\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)+\propto f_{p,k}^{m+1}\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)\right]} \prec \emptyset(z).$$

Noting that $\emptyset(z)$ is convex and univalent in D, we conclude that Lemma 1.3.5 holds true.

Using equations (5) and (6) we get

$$z\left(f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)' + \left(p+\frac{1}{\Im}\right)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$$

$$= \frac{1}{\Im k}\sum_{j=0}^{k-1}\epsilon_k^{jp}\left[\psi_p^{m+1} f\right]\left(\epsilon_k^j z\right) = \frac{1}{\Im}f_{p,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)$$

$\forall\ (f \in S)$. Let $f \in F_{p,k}^m(\alpha;\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\emptyset)$ and suppose

$$\Omega(z) = -\frac{z\left[f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right]'}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}$$

Then $\Omega(z)$ is analytic in D and $\Omega(0) = 1$

Hence $p + \frac{1}{\Im} - p\Omega(z) = \frac{1}{\Im}\frac{f_{p,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}{f_{p,k}^m(\sigma,\eta,\xi,\ell;z)}$

$z\left[f_{p,k}^{m+1}(\sigma,\eta,\xi,\ell;z)\right]'$

$$= -p\Im\begin{Bmatrix} z\,\Omega'(z) + p + \frac{1}{\Im} \\ -p\Omega(z) \end{Bmatrix} f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z) \qquad (z \in D^*)$$

From above relations we obtained

$$-\frac{z\left[(1+\propto)\left(f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(f_{p,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{p\left[(1+\propto)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{p,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]}$$

$$= \frac{p(1+\propto)\Omega(z)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\alpha\Im p\left[p+\frac{1}{\Im}-p\Omega(z)\right]\Omega(z)f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}{p(1+\propto)f_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\alpha\Im p\left[p+\frac{1}{\Im}-p\Omega(z)\right]f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}$$

$$= \frac{(1+\propto)\Omega(z)+\alpha\Im\left\{z\Omega'(z)+\left[p+\frac{1}{\Im}-p\Omega(z)\right]\Omega(z)\right\}}{(1+\propto)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]}$$

$$= \frac{\alpha\Im z\Omega'(z)+\left\{(1+\propto)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]\right\}\cdot\Omega(z)}{(1+\propto)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]}$$

$$= \Omega(z) + \frac{z\Omega'(z)}{\frac{1}{\Im\alpha}+2\frac{1}{\Im}+p-p\Omega(z)} \prec \varphi(z) \qquad (z \in D).$$

Since $Re\left(\frac{1}{\Im}\left(2+\frac{1}{a}\right)+p[1-\varphi(z)]\right) > 0$

and by Lemma 3 we get

$$\Omega(z) = -\frac{z\left(f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} \prec \varphi(z) \qquad (z \in D).$$

Lemma 1.3.6: Let $f \in G_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$-\frac{z\left[(1+\propto)\left(g_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(g_p^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{p\left[(1+\propto)g_p^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]+\propto g_p^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\right]}$$

$\prec \varphi(z)$. $\quad(z \in D)$.Therefore,

if $\emptyset \in P, Re\left[\frac{1}{\Im}\left(2+\frac{1}{a}\right)+p(1-\varphi(z))\right] > 0$

And $\begin{pmatrix} 0 < \sigma \le \frac{1}{2}, \xi > 0, \eta \ge 0, 0 < \varepsilon \le \frac{1}{2}, \\ a > 0, \delta > 0, \upsilon \ge 0, \delta > \upsilon, z \in D \end{pmatrix}$.

then $\frac{-z\left(g_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{pg_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} \prec \varphi(z) \qquad (z \in D).$

Lemma 1.3.7 Let $f \in F_{p,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\frac{-z\left[(1+\propto)\left(h_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(h_p^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{p\left[(1+\propto)h_p^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]+\propto h_p^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\right]} \prec \varphi(z)$$

Therefore, $if\ \emptyset \in P$ with $Re\left[\frac{1}{\Im}\left(2+\frac{1}{a}\right)+p(1-\varphi(z))\right] > 0.$

Where $\begin{pmatrix} 0 < \sigma \le \frac{1}{2}, \xi > 0, \eta \ge 0, 0 < \varepsilon \le \frac{1}{2}, \\ a > 0, \delta > 0, \upsilon \ge 0, \delta > \upsilon, z \in D \end{pmatrix}$.

$$-\frac{z\left(h_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{ph_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} \prec \varphi(z) \qquad (z \in D).$$

2 Main Results

Theorem 2.1 Let $0 < a \le 1$ and $0 < b < 1$. If $f(z) \in S$ satisfies $f(z) \neq 0\ z \in D^*$ and

$$\left|\frac{1}{z^p f(z)}\left(\frac{zf'(z)}{f(z)}\right)+p\right| < \delta\ (z \in D) \qquad (11)$$

Where δ is the minimum positive root of the equation

$$\frac{a}{2}sin\left(\pi\frac{b}{2}\right)x^2 - x + \left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right) = 0 \qquad (12)$$

Then $\left|arg\left(z^p f(z)-\frac{a}{2}\right)\right| < \pi\frac{b}{2} \quad (z \in D) \qquad (13)$

The bound b is the best possible for each $0 < a \le 1$

Proof Let

$$g(x) = \frac{a}{2}sin\left(\pi\frac{b}{2}\right)x^2 - x + \left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right) \qquad (14)$$

There exist two roots for the Equation

$$\frac{a}{2}sin\left(\pi\frac{b}{2}\right)x^2 - x + \left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right) = 0$$

$Since\ g(0) = \frac{a}{2}sin\left(\pi\frac{b}{2}\right)\cdot(0)^2 - 0 + \left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right)$

$= \left(1-\frac{a}{2}\right) sin\left(\pi\frac{b}{2}\right) > 0$

$g(1) = \frac{a}{2} sin\left(\pi\frac{b}{2}\right) \cdot (1)^2 - 1 + \left(1-\frac{a}{2}\right) sin\left(\pi\frac{b}{2}\right)$

$= \frac{a}{2} sin\left(\pi\frac{b}{2}\right) + \frac{a}{2} sin\left(\pi\frac{b}{2}\right) = 2\frac{a}{2} sin\left(\pi\frac{b}{2}\right) < 0$

Hence we get $0 < \frac{a}{2-a}\delta \le \delta < 1$ (15)

Put $z^p f(z) = \frac{a}{2} + \left(1-\frac{a}{2}\right) q(z)$ (16)

Then from the assumption of the theorem, we see that $q(z)$ is analytic in D with $q(0) = 1$ and $\frac{a}{2} + \left(1-\frac{a}{2}\right) q(z) \neq 0$ for all $(z \in D)$. Taking the logarithmic differentiations on both sides of $z^p f(z) = \frac{a}{2} + \left(1-\frac{a}{2}\right) q(z)$, we obtained

$\frac{zf'(z)}{f(z)} + p = \frac{(2-a)zq'(z)}{a+(2-a)q(z)}$. (17)

$z^p f(z)\left[\frac{zf'(z)}{f(z)} + p\right] = \frac{(2-a)zq'(z)}{[a+(2-a)q(z)]^2} \quad \forall\, (z \in D).$ (18)

Thus the inequality $\left|\frac{1}{z^p f(z)}\left(\frac{zf'(z)}{f(z)}\right) + p\right| < \delta \qquad (z \in D).$

It is equivalent to $\frac{(2-a)zq'(z)}{[a+(2-a)q(z)]^2} \prec \delta z.$ (19)

By using Lemma 1, above inequality leads to

$\int_0^z \frac{(2-a)q'(t)}{[a+(2-a)q(t)]^2} dt \prec \delta z$. Or to $1 - \frac{2}{a+(2-a)q(z)} \prec \delta z.$ (20)

In view of above results it can be written as

$q(z) \prec \frac{1+\frac{a}{2-a}\delta z}{1-\delta z}.$ (21)

Now by taking $\propto = \frac{a}{2-a} \cdot \frac{\delta}{2}$ and $\beta = -\frac{\delta}{2}$ in (1.2), we have

$\left|arg\left(z^p f(z) - \frac{a}{2}\right)\right| = |arg\ q(z)|$

$< arc\ sin\left(\frac{2\delta}{2-a+a\delta^2}\right) = \pi\frac{b}{2} \qquad \forall\, (z \in D)$ because of

$g(\delta) = 0$. This proves the statement.

Next, we consider the function $f(z)$ defined by

$f(z) = \frac{z^{-p}}{1-\delta z} \qquad (z \in D^*).$

It is easy to see that

$\left|z^p f(z)\left[\frac{zf'(z)}{f(z)} + p\right]\right| = |\delta z| < \delta \qquad (z \in D).$

Since $z^p f(z) - \frac{a}{2} = \frac{1+\frac{a}{2-a}\delta z}{1-\delta z}$.

It follows from (3) that

$sup_{z\in D}\left|arg\left(z^p f(z) - \frac{a}{2}\right)\right| = arc\ sin\left(\frac{2\delta}{2-a+a\delta^2}\right) = \pi\frac{b}{2}.$

Hence, we conclude that the bound b is the best possible for all $a \in (0,1]$. Next, we derive the following.

Theorem 2.2 If $f(z) \in S$ satisfies $f(z) \neq o, (z \in D^*)$ and

$\text{Re}\left[z^p f(z)\left[\frac{zf'(z)}{f(z)} + p\right]\right] < \varepsilon \qquad (z \in D).$ (22)

$0 < \varepsilon < \frac{1}{2log2}.$ (23)

Then $\text{Re}\frac{1}{z^p f(z)} > 1 - 2\varepsilon \cdot log2 \qquad (z \in D).$ (24)

The above inequality holds good.

Proof Let $q(z) = z^p f(z)$ (25)

then $q(z)$ is analytic in D with $q(0) = 1$ and $q(z) \neq 0$ for $z \in D$. In accordance with (17) and (20), we obtained

$1 - \frac{zq'(z)}{\varepsilon q^2(z)} \prec \frac{1+z}{1-z}$

Let $z\left[\frac{1}{q(z)}\right]' \prec \frac{2\varepsilon z}{1-z}$ (26)

Now by Lemma 1, we obtain

$\frac{1}{q(z)} \prec 1 - 2\varepsilon \cdot log(1-z).$

Since the function $1 - 2\varepsilon \cdot log(1-z)$ is convex univalent in D and

$\text{Re}[1 - 2\varepsilon \cdot log(1-z)] > 1 - 2\varepsilon \cdot log\ 2 \qquad (z \in D).$

From $z\left[\frac{1}{q(z)}\right]' \prec \frac{2\varepsilon z}{1-z}$ we obtained the inequality

$Re\frac{1}{z^p f(z)} > 1 - 2\varepsilon \cdot log2.$

To show that the bound $Re\frac{1}{z^p f(z)} > 1 - 2\varepsilon \cdot log2 \quad (z \in D)$, cannot be increased, we consider

$f(z) = \frac{1}{z^p[1-2\varepsilon \cdot log(1-z)]} \qquad (z \in D^*).$

We can verify that the function $f(z)$ satisfies the inequality

$Re\left[z^p f(z)\left[\frac{zf'(z)}{f(z)} + p\right]\right] < \varepsilon \qquad (z \in D).$

On the other hand we have

$Re\ z^p f(z) \to 1 - 2\varepsilon \cdot log2\ \ as\ \ z \to -1.$

Hence the Theorem holds good.

Theorem 2.3 Let $f(z) \in S\ satisfies\ f(z) \neq o,\ (z \in D^*)$.

If $\left|Im\left\{\frac{zf'(z)}{f(z)}[z^p f(z) - \tau]\right\}\right| < \sqrt{\tau(\tau + \tau p)}$ (27)

$(z \in D)$ and $\tau > 0. Then\ \ Re\ z^p f(z) > 0$ (28)

Proof Let $q(z)\ in\ D$ be defined as

$Re\ z^p f(z) = q(z)\ then\ q(0) = 1, q(z) \neq 0,$

And $\frac{zf'(z)}{f(z)}(z^p f(z) - \tau)$

$= [q(z) - \tau] \cdot \left[\frac{zq'(z)}{q(z)} - p\right]$ (29)

$Re\ q(z) > 0$, $|z| < |z_0|$ and $q(z_0) = ib$ (30)

$(z \in D)$, b is real and $b \neq 0$. Then by Lemma 1.2.2 we have

$z_0 \cdot q'(z_0) \le \frac{-(1-b^2)}{2}.$ (31)

Thus it follows from above obtained results that

$J_o = Im\left[\frac{z_0 f'(z_0)}{f(z_0)}(z^p f(z) - \tau)\right]$

$= -pb + \frac{\tau}{b} z_0 \cdot q'(z_0).$ (32)

In accordance with $\tau > 0$ and from the statements (2.2.1) and (2.2.2) we obtained

$J_0 \ge \frac{-[\tau+(\tau+2p)b^2]}{2b} \ge \sqrt{\tau(\tau + 2p)} \qquad (b > 0).$ (33)

$J_0 \le \frac{[\tau+(\tau+2p)b^2]}{2b} \le -\sqrt{\tau(\tau + 2p)} \qquad (b > 0)$ (34)

But both The inequalities obtained above contradict the assumption given below.

$\left|Im\left\{\frac{zf'(z)}{f(z)}[z^p f(z) - \tau]\right\}\right| < \sqrt{\tau(\tau + \tau p)}$

$(z \in D)$ and $\tau > 0$. Therefore we have,

$Re\ q(z) > 0\ for\ all\ \ (z \epsilon D)$. This shows that

$Re\ z^p f(z) > 0 \quad (z \in D)$. Theorem holds true.

2.1 Inclusion relationships

Theorem 2.1.1 Let $\varphi \in P$ with s

$Re\left[\frac{1}{\Im}\left(2 + \frac{1}{\alpha}\right) + p(1 - \varphi(z))\right] > 0.$

Where $\begin{pmatrix} 0 < \sigma \le \frac{1}{2}, \xi > 0, \eta \ge 0, 0 < \varepsilon \le \frac{1}{2}, \\ a > 0, \delta > 0, \upsilon \ge 0, \delta > \upsilon, z \in D \end{pmatrix}.$

Then $F_{p,k}^m(\propto; \sigma, \xi, \eta, \varepsilon, \delta, \upsilon; \varphi) \subset F_{p,k}^m(\sigma, \xi, \eta, \varepsilon, \delta, \upsilon; \varphi).$

Proof Let $f \in F_{p,k}^m(\propto; \sigma, \xi, \eta, \varepsilon, \delta, \upsilon; \varphi)$ and

$q(z) = -\frac{z(\psi_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f)'}{p\psi_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} \qquad (z \in D)$. Then $q(z)$ is analytic in D and $q(0) = 1$ hence

$q(z) f_{p,k}^m(\sigma, \xi, \eta, \varepsilon, \delta, \upsilon; z) = \frac{-1}{\Im p}\psi_p^{m+1} f(z) + \frac{(1+\Im p)}{\Im p}\psi_p^m f(z)$

Differentiating both sides we get

$$zq'(z)+\left(\frac{1}{\Im}+p+\frac{z\left(f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\right)q(z)$$

$$=\frac{-1}{\Im p}\cdot\frac{z\left(\psi_p^{m+1}f(z)\right)'}{f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}.$$

$$\therefore\ \frac{-z\left[(1+\propto)\left(\psi_p^{m}f\right)'(z)+\propto\left(\psi_p^{m+1}f\right)'(z)\right]}{p\left[(1+\propto)f_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{p,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]}$$

$$=\frac{p(1+\propto)q(z)f_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)}{p(1+\propto)f_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}$$

$$+\frac{\alpha\Im p\left[zq'(z)+p+\frac{1}{\Im}-p\Omega(z)\right]q(z)f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}{p(1+\propto)f_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}$$

$$=\frac{(1+\propto)q(z)+\alpha\Im\left\{zq'(z)+\left[p+\frac{1}{\Im}-p\Omega(z)\right]q(z)\right\}}{(1+\propto)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]}$$

$$=\frac{\alpha\Im zq\prime(z)+\left\{(1+\propto)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]\right\}q(z)}{(1+\propto)+\alpha\Im\left[p+\frac{1}{\Im}-p\Omega(z)\right]}$$

$$=q(z)+\frac{zq\prime(z)}{\frac{1}{\Im\alpha}+\frac{2}{\Im}+p-p\Omega(z)}\prec\varphi(z)\qquad(z\in D).$$

Since $Re\left(\frac{1}{\Im\alpha}+\frac{2}{\Im}+p-p\Omega(z)\right)>0$

and by Lemma 3 we get

$$\Omega(z)=-\frac{z\left(f_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{pf_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\prec\varphi(z)\qquad(z\in D).$$

By Lemma 1.2.2 we find that $q(z)\prec\varphi(z)\quad(z\in D)$.

$\therefore\ f\in F_{p,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$

$\Rightarrow F_{p,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset F_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.1 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $G_p^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset G_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.2 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $H_p^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset H_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$

Theorem 2.1.2 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $\Im_{p,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset\Im_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Proof Let $f\in\Im_{p,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$and suppose that

$$q(z)=-\frac{z\left(I_p^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{p\ell_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\qquad(z\in D).$$

Thus $q(z)$ is analytic in D and $q(0)=1$.

$\therefore\ q(z)=\ell_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)$

$$=\frac{-1}{\Im p}I_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon)f(z)+\frac{(1+\Im p)}{\Im p}I_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon)f(z)$$

Differentiating both sides we get

$$zq'(z)+\left(\frac{1}{\Im}+p+\frac{z\left(\ell_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{\ell_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\right)q(z)$$

$$=\frac{-1}{\Im p}\cdot\frac{z\left(\psi_p^{m+1}f(z)\right)'}{\ell_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}$$

$$\varphi(z)=-\frac{z\left(\ell_p^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{p\ell_{p,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\qquad(z\in D).$$

$$\therefore\ q(z)\prec\varphi(z)\qquad(z\in D).$$

This implies that $f\in\Im_{p,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.3 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $\mathfrak{U}_p^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset\mathfrak{U}_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.4: Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $\mathfrak{X}_p^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset\mathfrak{X}_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.5 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $F_{p,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset F_{p,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.6 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $G_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset G_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.7 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $H_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset H_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.8: Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $\Im_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset\Im_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.9 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix}0<\sigma\le\frac{1}{2},\xi>0,\eta\ge0,0<\varepsilon\le\frac{1}{2},\\ a>0,\delta>0,\upsilon\ge0,\delta>\upsilon,z\in D\end{pmatrix}$.

Then $\mathfrak{U}_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset\mathfrak{U}_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Corollary 2.1.10 Let $\varphi\in P$ with

$$Re\left[\frac{1}{\Im}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where $\begin{pmatrix} 0<\sigma\le\frac{1}{2},\xi>0,\eta\ge 0,0<\varepsilon\le\frac{1}{2}, \\ a>0,\delta>0,\upsilon\ge 0,\delta>\upsilon,z\in D \end{pmatrix}$.

Then $\mathfrak{X}_p^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)\subset\mathfrak{X}_p^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

2.2 Integral Representation

In this section we are going to prove integral representations associated with the function classes $F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$, $G_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ and $H_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Theorem 2.2.1 *Let* $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)=z^{-p}exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}d\zeta\right).$$

Where $f_{p,k}^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$$=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}\left[\Omega_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right]\left(\epsilon_k^j z\right)$$

$$=\frac{1}{z^p}+\cdots\qquad (f\in S).$$

Where $w(z)$ is analytic in D with $w(0)=0$.
And $|w(z)|<1$. $(z\in D)$.

Proof Suppose that $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

$$\frac{-z(\psi_p^m f)'\left(\epsilon_k^j z\right)}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}=\varphi[w(z)]\qquad (z\in D).$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1$ $(z\in D)$

Replacing z *by* $\epsilon_k^j z$ $(j=0,1,2,\cdots\cdots)$ above equation holds true. That is

$$-\frac{\epsilon_k^j z(\psi_p^m f)'\left(\epsilon_k^j z\right)}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}=\varphi[w(\epsilon_k^j z)]\qquad (z\in D)$$

We note that

$$f_{p,k}^m\left(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)=\epsilon_k^{-jp}f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)$$

Letting $(j=0,1,2,\cdots\cdots)$, successively and summing the resultant equations we obtained

$$\frac{-z\left(f_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}=\frac{1}{k}\sum_{j=0}^{k-1}\varphi\left(w(\epsilon_k^j z)\right)\qquad (z\in D).$$

$$\therefore\frac{\left(f_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}+\frac{p}{z}=\frac{-p}{k}\sum_{j=0}^{k-1}\frac{\varphi[w(\epsilon_k^j z)]-1}{z}\qquad (z\in D^*).$$

Upon integration which yields,

$$log\left(z^p f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)=-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}d\zeta.$$

Taking exponential theorem holds true.

Theorem 2.2.2 *Let* $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\psi_p^m f(z)=-p\int_0^z\left[\varsigma^{-p-1}\varphi[w(\varsigma)]exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^\varsigma\frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}d\zeta\right)\right]d\varsigma,$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1$ $(z\in D)$.

Proof Suppose that $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$(\psi_p^m f)'(z)=-\frac{p\,F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)}{z}\cdot\varphi[w(z)]$$

$$=-pz^{-p-1}\varphi[w(z)]\cdot exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}d\zeta\right),$$

Integrating above equation, theorem holds true.

Theorem 2.2.3 *Let* $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\psi_p^m f(z)=-p\int_0^z\left[\varsigma^{-p-1}\varphi[w_2(\varsigma)]exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\varphi[w_1(\zeta)]-1}{\zeta}d\zeta\right)\right]d\varsigma,$$

Where $w_j(z)$ is analytic in D with $w_j(0)=0$ and $|w_j(z)|<1$ $(z\in D,j=1,2)$.

Proof Suppose that $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\frac{-z\left(f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}=\varphi(w_1(z)).\qquad (z\in D)$$

Where $w_1(z)$ is analytic in D with $w_1(0)=0$ and $|w_1(z)|<1$. $(z\in D)$.

Thus by applying method of the proof of theorem 2.2.3 we find that

$$f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)=z^{-p}exp\left(-p\int_0^z\frac{\varphi[w_1(\zeta)]-1}{\zeta}d\zeta\right).$$

From above equations we get

$$(\psi_p^m f)'(z)=\frac{-pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}{z}\cdot\varphi[w_2(z)]$$

$$=-pz^{-p-1}\cdot\varphi[w_2(z)]\cdot exp\left(-p\int_0^z\frac{\varphi[w_1(\zeta)]-1}{\zeta}\mathrm{d}\zeta\right)$$

Where $w_j(z)$ is analytic in D with $w_j(0)=0$ and $|w_j(z)|<1$. $(z\in D,j=1,2)$. Integrating both sides of above integral we readily approach to the assertion theorem.

Corollary 2.2.1 *Let* $f\in G_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$g_p^m(\sigma,\eta,\xi\varepsilon,\delta,\upsilon;z)=z^{-p}exp\left(\frac{-p}{z}\int_0^z\frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right)$$

Where $g_p^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$$=\frac{1}{2}\left[\psi_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)\right]+\overline{\psi_p^m f(\bar{z})}=\frac{1}{z^p}+\cdots$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1$ $(z\in D)$.

Corollary 2.2.2 *Let* $f\in G_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\psi_p^m f(z)=-p\int_0^z\varsigma^{-p-1}\varphi[w(\varsigma)]exp\left(\frac{-p}{z}\int_0^z\frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right)dz.$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1$ $(z\in D)$.

Corollary 2.2.3 *Let* $f\in H_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$h_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\mathrm{z})=z^{-p}.exp\left(\frac{-p}{z}\int_0^z\frac{\varphi[w(\zeta)]-\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right).$$

Where

$$h_p^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]=\frac{1}{2}\left[\psi_p^m f(z)\right]+\overline{\psi_p^m f(-\bar{z})}$$

$$=\frac{1}{z^p}+\cdots\qquad (h\in S).$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1$. $(z\in D)$.

Corollary 2.2.4 *Let* $f\in H_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\psi_p^m f(z)=-p\int_0^z\varsigma^{-p-1}\varphi(w(\varsigma))exp\left(\frac{-p}{z}\int_0^z\frac{\varphi[w(\zeta)]-\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right)d\varsigma.$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1$. $(z\in D)$.

2.3 Convolution Properties

In this part we are going to derive several convolution properties for the function classes $F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$, $G_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ and $H_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Theorem 2.3.1 *Let* $f\in F_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$f(z)=$

$$\left[-p\int_0^z \varsigma^{-p-1}\,\varphi[w(\varsigma)]exp\left(\frac{-p}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\left[w\left(\epsilon_k^j\zeta\right)\right]-1}{\zeta}d\zeta\right)d\varsigma\right]$$
$$*\sum_{n=0}^{\infty}\left[\frac{1}{\Im n+1}\right]^m z^{n-p}$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1.$ $(z\in D)$

Proof: Since we know that linear operator given by

$$\psi_p^m f(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}[\Im(p+n)+1]^m a_{p+n-1}z^{p+n-1}$$
$$=\left(\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{p,m}*f\right)(z).$$

Where $\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{p,m}(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}[\Im(p+n)+1]^m z^{p+n-1}$ and $\psi_p^m f(z)=$

$$-p\int_0^z\left[\varsigma^{-p-1}\varphi[w(\varsigma)]exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^{\varsigma}\frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}d\zeta\right)\right]d\varsigma,$$
$$=\left[\sum_{n=0}^{\infty}\left[\frac{1}{\Im n+1}\right]^m z^{n-p}\right]*f(z).$$

Thus from above equation, we can easily get the theorem.

$$f(z)=\left[-p\int_0^z \varsigma^{-p-1}\,\varphi[w(\varsigma)]exp\left(\frac{-p}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\left[w\left(\epsilon_k^j\zeta\right)\right]-1}{\zeta}d\zeta\right)d\varsigma\right]$$
$$*\sum_{n=0}^{\infty}\left[\frac{1}{\Im n+1}\right]^m z^{n-p}.$$

Theorem 2.3.2 *Let* $f\in F_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$f(z)=\left[-p\int_0^z \varsigma^{-p-1}\,\varphi[w_2(\varsigma)]exp\left(\frac{-p}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi[w_1(\zeta)]-1}{\zeta}d\zeta\right)d\varsigma\right]$$

$*\sum_{n=0}^{\infty}\left[\frac{1}{\Im n+1}\right]^m z^{n-p}$. Where $w_j(z)$ is analytic in D with $w_j(0)=0$ and $|w_j(z)|<1$ $(z\in D, j=1,2)$.

Proof Since we know that linear operator given by,

$$\psi_p^m f(z)$$
$$=\frac{1}{z^p}+\sum_{n=0}^{\infty}[\Im(p+n)+1]^m a_{p+n-1}z^{p+n-1}$$
$$=\left(\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{p,m}*f\right)(z).$$

Where

$$\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{p,m}(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}[\Im(p+n)+1]^m z^{p+n-1}$$
$$\psi_p^m f(z)=$$
$$-p\int_0^z\left[\varsigma^{-p-1}\varphi[w_2(\varsigma)]exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^{z}\frac{\varphi[w_1(\zeta)]-1}{\zeta}d\zeta\right)\right]d\varsigma$$

We obtained

$$-p\int_0^z \varsigma^{-p-1}\,\varphi[w_2(\varsigma)]exp\left(-p\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\left[w_1\left(\epsilon_k^j\zeta\right)\right]-1}{\zeta}d\zeta\right)d\varsigma$$
$$=\left[\sum_{n=0}^{\infty}[\Im+1]^m z^{n-p}\right]*f(z).$$

Thus from above result, we can easily prove the Theorem.

Theorem 2.3.3 Let $f\in S$ and $\varphi\in P$. Then $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ if and only if

$$f(z)*\{(-pz^{-p}+\sum_{n=1}^{\infty}[\Im n+1]^m(n-p)z^{n-p})+$$
$$p\varphi(e^{i\theta})(z^{-p}+\sum_{n=1}^{\infty}[\Im n+1]^m z^{n-p})*\left(\frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{z^p(1-e^{\nu}z)}\right)$$
$$\neq 0.\ (z\in D^*; 0\le\theta<2\pi).$$

Proof Suppose that $f\in F_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$. since the following subordination condition

$$\frac{-z\left(\psi_p^m f\right)'(z)}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\prec\varphi(z).\quad(z\in D)$$

is equivalent to

$$\frac{-z\left(\psi_p^m f\right)'(z)}{pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\neq\varphi(e^{i\theta})\qquad(z\in D; 0\le\theta<2\pi).$$

It is easy to verify that the above condition can be written as

$$z\left(\psi_p^m f\right)'(z)+pf_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\varphi(e^{i\theta})\neq 0.\qquad(35)$$

On the other hand we find from (1.4) that

$$z\left(\psi_p^m f\right)'(z)=(-pz^{-p}+\sum_{n=1}^{\infty}[\Im n+1]^m(n-p)z^{n-p})$$
$$*f(z)\qquad(36)$$

Moreover, from the definition, we obtained

$$f_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)=\psi_p^m f(z)*\left(\frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{z^p(1-e^{\nu}z)}\right)$$
$$=(z^{-p}+\sum_{n=1}^{\infty}[\Im n+1]^m z^{n-p})*\left(\frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{z^p(1-e^{\nu}z)}\right)*f(z)$$
$$(37)$$

Substituting (36) and (37) in (35) we can easily arrive at the convolution property asserted by given theorem. In view of Corollaries 2.2.2 and 2.2.4 and by applying the method similar to method of Theorem 2.2.1 we can easily obtain the following results for the function classes $G_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)$ and $H_p^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)$.

Corollary 2.3.1 *Let* $f\in G_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon,\varphi)$, then

$$f(z)=\left[-p\int_0^z \varsigma^{-p-1}\,\varphi[w_2(\varsigma)]exp\left(\frac{-p}{2}\int_0^{\varsigma}\frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right)d\varsigma\right]$$
$$*\left(\sum_{n=0}^{\infty}\left[\frac{1}{\Im n+1}\right]^m z^{n-p}\right).$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1.$ $(z\in D)$.

Corollary 2.3.2 *Let* $f\in H_{p,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon,\varphi)$, Then

$$f(z)=\left[-p\int_0^z \varsigma^{-p-1}\,\varphi[w_2(\varsigma)]exp\left(\frac{-p}{2}\int_0^{\varsigma}\frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right)d\varsigma\right]$$
$$*\left(\sum_{n=0}^{\infty}\left[\frac{1}{\Im n+1}\right]^m z^{n-p}\right).$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1.$ $(z\in D)$.

Corollary 2.3.3 Let $f\in S$ and $\varphi\in P$. Then $f\in G_{p,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ if and only if

$$f*\left[(-pz^{-p}+\sum_{n=1}^{\infty}[\Im n+1]^m(n-p)z^{n-p})+\frac{p\varphi(e^{i\theta})}{2}I\right]$$
$$+\frac{p\varphi(e^{i\theta})}{2}\overline{(I*f)(\bar{z})}\neq 0\qquad(z\in D^*; 0\le\theta<2\pi).$$

Where I(z) is given by

$$I(z)=z^{-p}+\sum_{n=1}^{\infty}[\Im n+1]^m z^{n-p}.$$

Corollary 2.3.4 Let $f\in S$ and $\varphi\in P$. Then $f\in H_p^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$, If and only if

$$f*\left\{[-pz^{-p}+\sum_{n=1}^{\infty}(\Im n+1)^m(n-p)z^{n-p}]+\frac{p\varphi(e^{i\theta})}{2}I\right\}$$
$$-\frac{p\varphi(e^{i\theta})}{2}\overline{(I*f)(\overline{-z})}\qquad\neq 0.\qquad(z\in D^*; 0\le\theta<2\pi).$$

where $f(z)$ is given by above equation.

Remark By specializing the parameters $\sigma,\xi,\eta,\varepsilon,\delta,\upsilon,m,A,B$ and p in our results, we obtain corresponding results due to various researchers.

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