Applications of Meromorphic Multivalent Functions Associated with Differential Subordination

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Abstract: In this paper authors introduced subclasses 𝐃^∗ₜ抰(α, β), 𝐹^∗ₚ(α, η, ε, δ; ϕ; φ), 𝐆^∗ₚ(α, η, ε, δ; ϕ; φ), and 𝐃^∗ₚ(α, η, δ; ϕ; φ), as well as 𝐹^∗ₚ(α, η, ε, δ; ϕ; φ), 𝐆^∗ₚ(α, η, ε, δ; ϕ; φ), and 𝐃^∗ₚ(α, η, ε, δ; ϕ; φ) of meromorphic multivalent functions in the punctured unit disk 𝐃^∗ = {z: 0 < |z| < 1} = 𝐃(0). By using the method of differential subordinations, we derive some certain properties of meromorphically multivalent functions.

Keywords: Meromorphically multivalent function, Analytic function, Subordination, Differentiation.

1 Introduction

Let S denote the class of functions of the form:

\[ f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n-1} z^{p+n-1}, \quad (p \in N) \]  

Which are analytic and p-valent in the punctured unit disc \( \mathbb{D}^* = \{ z : 0 < |z| < 1 \} = \mathbb{D}(0) \). Let \( f(z) \) and \( g(z) \) be analytic in \( \mathbb{D} \), then we say that \( f(z) \) is subordinate to \( g(z) \) in \( \mathbb{D} \).

If \( f(z) < g(z) \), then there exists an analytic function \( h(z) \) in \( \mathbb{D} \) such that \( |h(z)| \leq |z| \) and \( f(z) = g(h(z)) \) \((z \in \mathbb{D})\). If \( g(z) \) is univalent in \( \mathbb{D} \) then the subordination

\[ f(z) \prec g(z) \quad (D) \iff f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D). \]

Let \( q(z) = 1 + q_1 z + q_2 z^2 + \cdots \) be analytic in \( \mathbb{D} \), such that

\[ \text{Re} \, q(z) > \frac{1 - 2\beta}{\alpha} \quad (z \in \mathbb{D}). \]

Recently, several authors proved some interesting properties of meromorphically multivalent functions. In the present topic, we are going to prove some subordination properties for the class S.

When \( g(z) = f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_{p+n-1} z^{p+n-1}, \quad (p \in N) \), we define the Hadamard product (convolution) of \( f(z) \) and \( g(z) \) by

\[ (f \ast g)(z) = f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n-1} b_{p+n-1} z^{p+n-1}. \]

We define the linear operator by

\[ L^p_m(\sigma, \eta, \xi, \epsilon, \delta, \sigma) f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{[\sigma \xi (n+1) + \eta \epsilon (n+1)]}{(\delta - \epsilon)} a_{p+n-1} z^{p+n-1}. \]

We define a linear operator by

\[ \psi^m_p(\eta, \xi, \epsilon, \delta, \sigma) f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{[\sigma \xi (n+1) + \eta \epsilon (n+1)]}{(\delta - \epsilon)} a_{p+n-1} z^{p+n-1}. \]

This paper authors introduced subclasses \( D^*_t(\alpha, \beta) \), \( F^*_p(\alpha, \eta, \epsilon, \delta; \phi; \phi) \), \( G^*_p(\alpha, \eta, \epsilon, \delta; \phi; \phi) \), and \( D^*_p(\alpha, \eta, \delta; \phi; \phi) \) of meromorphic multivalent functions associated with differential subordination. By using the method of differential subordinations, we derive some certain properties of meromorphically multivalent functions.

1.1 Definitions

Definition 1.1.1 Function \( f \in S \) is said to be in the class \( F^*_p(\alpha, \eta, \epsilon, \delta; \phi; \phi) \) if it satisfies the following subordination condition

\[ z^{\lceil \alpha (1+\sigma) \rceil} \psi^m_p(\sigma, \eta, \xi, \epsilon, \delta, \sigma) f(z) \prec \phi(z). \]

\[ f \in F^*_p(\alpha, \eta, \epsilon, \delta; \phi; \phi) \quad (\sigma \geq 0, z \in D^*, \phi \in P, \epsilon \in N). \]

For simplicity we can write

\[ f \in F^*_p(\sigma, \eta, \epsilon, \delta; \phi; \phi) \quad (\sigma \geq 0, z \in D^*, \phi \in P, \epsilon \in N). \]

\[ f \in F^*_p(\sigma, \eta, \epsilon, \delta; \phi; \phi) \quad (\sigma \geq 0, z \in D^*, \phi \in P, \epsilon \in N). \]
\[
F_{\nu, k}^m(\sigma; \xi, \eta, \epsilon, \delta, \nu, \varphi) = F_{\nu, k}^m(\sigma, \xi, \eta, \epsilon, \delta, \nu, \varphi).
\]

Where \(0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}, a > 0, \delta > 0, \nu > 0, \delta > \sigma, z \in D\).

**1.2 Remarks**

**Remark 1.2.1** Put \(p = 0, \xi = \eta = \sigma = 1, \epsilon = \nu = \sigma = \frac{1}{2}\) and \(\varphi(z) = \frac{1 + z}{1 - z}\) in definition 1.1.1, we have the class \(\mathcal{R}_p\), where

\[
T_p(z) = \frac{1}{2} f(z) - f(-z).
\]

**Remark 1.2.2** For \(\alpha = 0, \xi = \eta = \sigma = 1, \epsilon = \nu = \sigma = \frac{1}{2}\) we have the class \(F_{\nu, k}^m(0, \frac{1}{2}, 1, 1, \epsilon, \delta, \nu, \varphi) = F_{\nu, k}^m(\epsilon, \delta, \nu, \varphi)\).

Where the class \(F_{\nu, k}^m(\epsilon, \delta, \nu, \varphi)\) consisting of functions \(f(z) \in S\), this satisfies the following subordination condition

\[
-z (\varphi(z))' < \nu(z). \quad \varphi \in p \quad \text{and} \quad f_{\nu, k}^m(\epsilon, \delta, \nu, \varphi) = \frac{1 + \epsilon}{1 - \epsilon} \frac{1}{2} f(z) - f(-z).
\]

Such that \(0 < \alpha \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}\).

**Remark 1.2.3** For \(\alpha = 0, \xi = \eta = 1, \sigma = \frac{1}{2}\) we have the class \(F_{\nu, k}^m(0; \frac{1}{2}, 1, 1, 1, 1; \varphi) = F_{\nu, k}^m(\epsilon, \delta, \nu, \varphi)\), where the class \(F_{\nu, k}^m(\epsilon, \delta, \nu, \varphi)\) consisting of functions \(f(z) \in S\) which satisfies the following subordination condition

\[
-z (\varphi(z))' < \nu(z). \quad \varphi \in p \quad \text{and} \quad f_{\nu, k}^m(\epsilon, \delta, \nu, \varphi) = \frac{1 + \epsilon}{1 - \epsilon} \frac{1}{2} f(z) - f(-z).
\]

**Remark 1.2.4** Putting \(p = 0, \xi = \eta = 1, \sigma = 1, \epsilon = \nu = \sigma = \frac{1}{2}\) we have the class \(F_{\nu, k}^m(\epsilon, \delta, \nu, \varphi)\) consisting of functions \(f(z) \in S\) which satisfies the following subordination condition

\[
-z (\varphi(z))' < \nu(z). \quad \varphi \in p \quad \text{and} \quad f_{\nu, k}^m(\epsilon, \delta, \nu, \varphi) = \frac{1 + \epsilon}{1 - \epsilon} \frac{1}{2} f(z) - f(-z).
\]

**1.3 Preliminary Lemmas**

**Lemma 1.3.1** Let \(f(z)\) be analytic and starlike univalent in \(D\) with \(h(0) = 0\). If \(g(z)\) is analytic in \(D\) and \(zg(z) < f(z)\), then \(g(z) < g(0) + \int_0^z \frac{h(t)}{1 - t} dt\).

**Lemma 1.3.2** Let \(q(z)\) be analytic and other than constant in \(D\) with \(q(0) = 1\). If \(0 < |z_0| < 1\) and

\[
\text{Re}(q(z_0)) = \min_{|z| \leq 1} |\text{Re}(q(z))|.
\]

**Lemma 1.3.3** Let \(d, r \in C\); and \(\varphi(z)\) is convex and univalent in \(D\) with \(\phi(0) = 1\) and \(\text{Re}(d \phi(z) + r) > 0\). If \(q(z)\) is analytic in \(D\) with \(q(0) = 1\), then the following subordination

\[
qu(z) + \frac{q(z)}{d(z) + r} < 0 \Rightarrow (z) \Rightarrow (z) \leq \Phi(z).
\]
Lemma 1.3.4 Let \( d, r \in \mathbb{C} \); and \( \Theta(z) \) is convex and univalent in \( D \) with \( \Theta(0) = 1 \) and \( Re[d\Theta(z) + r] > 0 \).

Also let \( q(z) < \Theta(z) \). If \( q(z) \) is analytic in \( D \) with \( q(0) = 1 \), then the following subordination

\[
q(z) + \frac{q''(z)}{q'(z)} < 0 \Rightarrow \Theta(z) < q(z) < \Theta(z) \quad (z \in D).
\]

Lemma 1.3.5 Let \( f \in F^m_{p,k} (\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma) \) then

\[
-x(1 + \alpha)(f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma))' \left( \alpha + r f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma) \right) \left( \alpha + r f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma) \right)
\]

\[
p^1(1 + \alpha) f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma) \Rightarrow 0.
\]

And \( \Theta(z) \) is analytic in \( D \) and \( \Theta(0) = 1 \).

\[
\begin{align*}
\Omega(z) &= -p \left( \frac{1}{f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma, \xi, \eta, \sigma)} \right) \\
&= -p \left( \frac{1}{f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma, \xi, \eta, \sigma)} \right)
\end{align*}
\]

From above relations we obtained

\[
\begin{align*}
-x(1 + \alpha)(f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma))' \left( \alpha + r f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma) \right) &< 0 \quad (z \in D).
\end{align*}
\]

And \( \Theta(z) \) is analytic in \( D \) and \( \Theta(0) = 1 \).

\[
\begin{align*}
\Omega(z) &= -p \left( \frac{1}{f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma, \xi, \eta, \sigma)} \right)
\end{align*}
\]

\[
\Omega(z) = -p \left( \frac{1}{f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma, \xi, \eta, \sigma)} \right)
\]

\[
\begin{align*}
\Omega(z) &= -p \left( \frac{1}{f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma, \xi, \eta, \sigma)} \right)
\end{align*}
\]

\[
\begin{align*}
\Omega(z) &= -p \left( \frac{1}{f^m_{p,k}(\alpha; \sigma, \xi, \eta, \xi, \eta, \sigma, \xi, \eta, \sigma, \xi, \eta, \sigma)} \right)
\end{align*}
\]
\[= \left(1 - \frac{a}{2}\right) \sin \left(\pi \frac{b}{2}\right) > 0\]
\[g(1) = \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) \cdot 1^2 - 1 + \left(1 - \frac{a}{2}\right) \sin \left(\pi \frac{b}{2}\right)\]
\[= \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) + \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) = \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) < 0\]

Hence we get \(0 < \frac{a}{2} \delta \leq \delta < 1\) \( \text{(15)} \)

Put \(z^p f(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z)\) \( \text{(16)} \)

Then from the assumption of the theorem, we see that \(q(z)\) is analytic in \(D\) with \(q(0) = 1\) and \(\frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z) \neq 0\) for all \((z \in D)\). Taking the logarithmic differentiations on both sides of \(z^p f(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z)\), we obtained

\[z^p f(z) = \frac{(2-a)zq'(z)}{a+(2-a)q(z)}\] \( \text{(17)} \)

Thus the inequality

\[\left| \frac{1}{z^p f(z)} \left(z^p f'(z) + p\right) \right| < \delta \] \( \text{(18)} \)

It is equivalent to

\[\left| \frac{1}{z^p f(z)} \left(z^p f'(z) + p\right) \right| < \delta \] \( \text{(19)} \)

By using Lemma 1, above inequality leads to

\[f_0 \left[\frac{(2-a)zq'(z)}{a+(2-a)q(z)}\right] dt < \delta z. \text{ Or to} \]

\[1 - \frac{2}{a+(2-a)q(z)} < \delta z. \] \( \text{(20)} \)

In view of above results it can be written as

\[q(z) < \frac{1+a}{1-a} \frac{\delta z}{\delta} \] \( \text{(21)} \)

Now by taking \(\alpha = \frac{a}{2-a} \delta\) and \(\beta = -\frac{\delta}{2}\) in \((1,2)\), we have

\[g(\delta) = 0. \text{ This proves the statement.}\]

Next, we consider the function \(f(z)\) defined by

\[f(z) = z^p - \frac{\delta z}{\delta z} \quad (z \in D^*). \]

It is easy to see that

\[z^p f(z) \left[z^p f'(z) + p\right] = |\delta z| < \delta \quad (z \in D). \]

Since \(z^p f(z) - \frac{a}{2} = \frac{z^p + \frac{\delta z}{\delta z}}{\frac{\delta z}{\delta z}}\).

It follows from \((3)\) that

\[\sup_{z \in D} |z^p f(z) - \frac{a}{2}| = \sup_{z \in D} |\delta z| < \delta \quad (z \in D). \]

Hence, we conclude that the bound \(b\) is the best possible for all \(a \in (0,1]\). Next, we derive the following.

Theorem 2.2 If \(f(z) \in S\) satisfies \(f(z) \neq 0, (z \in D^*)\) and

where

\[\Re\left\{1 - 2e \cdot \log(1 - z)\right\} > 1 - 2e \cdot \log 2 \quad (z \in D). \]

From \(z = \frac{1}{q(z)}\) we obtained the inequality

\[\Re\left\{z^p f(z)\right\} > 1 - 2e \cdot \log 2. \]

To show that the bound \(\Re\left\{z^p f(z)\right\} > 1 - 2e \cdot \log 2\) \((z \in D)\), cannot be increased, we consider

\[f(z) = z^p \left(1 - 2e \cdot \log(1 - z)\right) \quad (z \in D^*). \]

We can verify that the function \(f(z)\) satisfies the inequality

\[\Re\left\{z^p f(z)\right\} < \epsilon \quad (z \in D). \]

On the other hand we have

\[Re z^p f(z) \rightarrow 1 - 2e \cdot \log 2 \text{ as } z \rightarrow -1. \]

Hence the Theorem holds good.

Theorem 2.3 Let \(f(z) \in S\) satisfies \(f(z) \neq 0, (z \in D^*). \)

If \(\left|\Re\left\{z^p f(z)\right\} - \tau\right| < \sqrt{\tau(\tau + 2p)}\)

\((z \in D)\) and \(\tau > 0, \text{ Then } Re z^p f(z) > 0 \text{ (28)}\)

Proof Let \(q(z) \in D\) be defined as

\[Re z^p f(z) = q(z) \quad \text{then } q(0) = 1, q(z) \neq 0, \text{ and}

\[\frac{z^p f'(z)}{z^p f(z)} + p \] \( \text{(29)} \)

\[Re q(z) > 0, |z| < |z_0| \text{ and } q(z_0) = ib \quad (z \in D), \text{ b is real and } b \neq 0. \]

Then by Lemma 1.2.2 we have

\[z_0 \cdot q'(z_0) \leq \frac{(1-b^2)}{2} \quad (30)\]

Thus it follows from above obtained results that

\[\left|\Re\left\{z^p f(z) - \tau\right\}\right| < \sqrt{\tau(\tau + 2p)}\]

\((z \in D)\) and \(\tau > 0. \text{ Therefore we have,}

\[Re q(z) > 0 \text{ for all } (2\epsilon D). \text{ This shows that}

\[Re z^p f(z) > 0 \text{ (31) } (z \in D). \text{ Theorem holds true.}

2.1 Inclusion relationships

Theorem 2.1.1 Let \(q \in P\) with \(s\)

\[\Re\left\{\frac{1}{2} z + p + (1 - q(z))\right\} > 0 \quad (32)\]

where

\[0 < s \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \epsilon \leq \frac{1}{2}\]

Then \(F_{p}^{m}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) < F_{p}^{m}(\alpha; \xi, \eta, \epsilon, \delta, \sigma; \varphi)\).

Proof Let \(f \in F_{p}^{m}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi)\) and

\[q(z) = \frac{x(z) \left[p f(z)\right]}{y(z) \left[p f(z)\right]} \quad (z \in D). \]

Then \(q(z)\) is analytic in \(D\) and \(q(0) = 1\) hence

\[q(z)f_{p,k}(\sigma, \xi, \eta, \epsilon, \delta, \sigma; z) = \frac{1}{3p} \left[p f(z)\right] + \left(1 + (1+3p) \psi_{p}^{m} f(z)\right)\]

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Differentiating both sides we get
\[ q'(z) + \frac{1}{3} + p + \left( \frac{\beta m_p(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)}{\rho_p \rho_k(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)} \right) q(z) \]
\[ = -\frac{1}{3p} \frac{\beta m_p(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)}{\rho_p \rho_k(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)} \left( 1 + \gamma(\phi(\theta))+\phi(\theta) \right) \]
\[ = \frac{p(1+\gamma(\phi(\theta))+\phi(\theta))}{p(1+\gamma(\phi(\theta))+\phi(\theta))} \left( \gamma(\phi(\theta)) + \phi(\theta) \right) \]
\[ = \frac{p(1+\gamma(\phi(\theta))+\phi(\theta))}{p(1+\gamma(\phi(\theta))+\phi(\theta))} \gamma(\phi(\theta)) + \phi(\theta) \]
\[ = \gamma(\phi(\theta)) + \phi(\theta) \]
\[ \therefore \quad q(z) < \gamma(\phi(\theta)) + \phi(\theta) \quad (z \in D). \]

Corollary 2.1.1 Let \( \phi \in P \) with
\[ \frac{2}{3} \left( 1 - \frac{1}{a} \right) + p \left( 1 - \phi(z) \right) > 0. \]
Where
\[ 0 < \rho \leq \frac{1}{2}, \xi \leq \frac{3}{4}, \gamma(\phi(\theta)) + \phi(\theta) \leq \frac{3}{4}, \rho(x, \omega, \sigma, \nu, y, \phi) \]
Then
\[ G_p^m(\gamma(\phi(\theta)) + \phi(\theta)) < \phi(\theta) \quad (z \in D). \]

Corollary 2.1.2 Let \( \phi \in P \) with
\[ \frac{2}{3} \left( 1 + \frac{1}{a} \right) + p \left( 1 - \phi(z) \right) > 0. \]
Where
\[ 0 < \rho \leq \frac{1}{2}, \xi \leq \frac{3}{4}, \gamma(\phi(\theta)) + \phi(\theta) \leq \frac{3}{4}, \rho(x, \omega, \sigma, \nu, y, \phi) \]
Then
\[ H_p^m(\gamma(\phi(\theta)) + \phi(\theta)) < \phi(\theta) \quad (z \in D). \]

Theorem 2.1.2 Let \( \phi \in P \) with
\[ \frac{2}{3} \left( 1 - \frac{1}{a} \right) + p \left( 1 - \phi(z) \right) > 0. \]
Where
\[ 0 < \rho \leq \frac{1}{2}, \xi \leq \frac{3}{4}, \gamma(\phi(\theta)) + \phi(\theta) \leq \frac{3}{4}, \rho(x, \omega, \sigma, \nu, y, \phi) \]
Then
\[ \gamma_p^m(\gamma(\phi(\theta)) + \phi(\theta)) < \phi(\theta) \quad (z \in D). \]

Proof Let \( f \in \gamma_p^m(\gamma(\phi(\theta)) + \phi(\theta)) \) and suppose that
\[ q(z) = \frac{-1}{3p} \frac{\beta m_p(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)}{\rho_p \rho_k(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)} f(z) + \frac{1 + \gamma(\phi(\theta))}{3p} \gamma_p^m(\gamma(\phi(\theta)) + \phi(\theta)) f(z) \]
Differentiating both sides we get
\[ q'(z) + \frac{1}{3} + p + \left( \frac{\beta m_p(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)}{\rho_p \rho_k(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)} \right) q(z) \]
\[ = -\frac{1}{3p} \frac{\beta m_p(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)}{\rho_p \rho_k(\sigma, \xi, \eta, \delta, \sigma, \nu, \omega)} \left( 1 + \gamma(\phi(\theta))+\phi(\theta) \right) \]
\[ = \frac{p(1+\gamma(\phi(\theta))+\phi(\theta))}{p(1+\gamma(\phi(\theta))+\phi(\theta))} \left( \gamma(\phi(\theta)) + \phi(\theta) \right) \]
\[ = \gamma(\phi(\theta)) + \phi(\theta) \]
\[ \therefore \quad q(z) < \gamma(\phi(\theta)) + \phi(\theta) \quad (z \in D). \]
Where \( 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2} \).

Then \( X_{n+1}(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi) \in X_p(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \).

### 2.2 Integral Representation

In this section we are going to prove integral representations associated with the function classes \( F_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \), \( G_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \) and \( H_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \).

#### Theorem 2.2.1

Let \( f \in F_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \) then

\[
F_{p,k}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) = \sum_{\xi_{j,k} = 0}^{\xi \sum_{j=0}^{1} \varphi(w(\xi)^{2})} \frac{\xi_{j,k}}{\xi}.
\]

Where \( \xi_{j,k} = 1 \sum_{j=0}^{1} \varphi(w(\xi)^{2}) \).

Where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \). And \( |w(z)| < 1 \).

Proof Suppose that \( f \in F_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \) then

\[
-\frac{z_{i}^{p_{m}}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi)}{\Gamma_{p,k}^{\sigma}} \sum_{\xi_{j,k} = 0}^{\xi \sum_{j=0}^{1} \varphi(w(\xi)^{2})} \frac{\xi_{j,k}}{\xi}.
\]

Where \( \xi_{j,k} = 1 \sum_{j=0}^{1} \varphi(w(\xi)^{2}) \).

Where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \).

Thus by applying method of the proof of theorem 2.2.3 we find that

\[
f_{p,k}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) = \sum_{\xi_{j,k} = 0}^{\xi \sum_{j=0}^{1} \varphi(w(\xi)^{2})} \frac{\xi_{j,k}}{\xi}.
\]

From above equations we get

\[
\psi_{m}^{f}(z) = \frac{-z_{i}^{p_{m}}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi)}{\Gamma_{p,k}^{\sigma}} \sum_{\xi_{j,k} = 0}^{\xi \sum_{j=0}^{1} \varphi(w(\xi)^{2})} \frac{\xi_{j,k}}{\xi}.
\]

Where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \).

#### Corollary 2.2.1

Let \( f \in G_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \) then

\[
g_{p,k}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) = \sum_{\xi_{j,k} = 0}^{\xi \sum_{j=0}^{1} \varphi(w(\xi)^{2})} \frac{\xi_{j,k}}{\xi}.
\]

Where \( \xi_{j,k} = 1 \sum_{j=0}^{1} \varphi(w(\xi)^{2}) \).

Where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \).

### 2.3 Convolution Properties

In this part we are going to derive several convolution properties for the function classes \( F_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \), \( G_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \) and \( H_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \).

#### Theorem 2.3.1

Let \( f \in F_{p,k}^{\sigma}(\sigma, \xi, \eta, \varepsilon, \delta; \sigma; \varphi) \) then

\[
f_{p,k}(z) = \sum_{\xi_{j,k} = 0}^{\xi \sum_{j=0}^{1} \varphi(w(\xi)^{2})} \frac{\xi_{j,k}}{\xi}.
\]
It is easy to verify that the above condition can be written as
\[
z(z) = \left(\frac{z}{u} + \frac{m}{n}\right)^n \neq 0.
\] (35)
On the other hand we find from (1.4) that
\[
z(z) = \left(-p + \sum_{n=1}^{\infty} m(n-1)p z^{n-p}\right) \neq 0.
\] (36)
Moreover, from the definition, we obtained
\[
f(z) = \left(-p + \sum_{n=1}^{\infty} m(n-1)p z^{n-p}\right) \neq 0.
\] (37)
Substituting (36) and (37) in (35) we can easily arrive at the convolution property asserted by given theorem. In view of Corollaries 2.2.2 and 2.2.4 and by applying the method similar to method of Theorem 2.2.1 we can easily obtain the following results for the function classes
\[
G_p^m(\sigma, \eta, \xi, \epsilon, \delta, \alpha, \varphi)
\] and
\[
H_p^m(\sigma, \eta, \xi, \epsilon, \delta, \alpha, \varphi).
\]

**Corollary 2.3.1** Let \( f \in L_p^m(\sigma, \eta, \xi, \epsilon, \delta, \alpha, \varphi) \) then
\[
f(z) = \left(-p + \sum_{n=1}^{\infty} m(n-1)p z^{n-p}\right) \neq 0.
\] (38)

**Corollary 2.3.2** Let \( f \in L_p^m(\sigma, \eta, \xi, \epsilon, \delta, \alpha, \varphi) \) then
\[
f(z) = \left(-p + \sum_{n=1}^{\infty} m(n-1)p z^{n-p}\right) \neq 0.
\] (39)

**Corollary 2.3.3** Let \( f \in S \) and \( \varphi \in P \). Then
\[
f(z) = \left(-p + \sum_{n=1}^{\infty} m(n-1)p z^{n-p}\right) \neq 0.
\] (40)

**Corollary 2.3.4** Let \( f \in S \) and \( \varphi \in P \). Then
\[
f(z) = \left(-p + \sum_{n=1}^{\infty} m(n-1)p z^{n-p}\right) \neq 0.
\] (41)

Remark By specializing the parameters \( \sigma, \xi, \eta, \epsilon, \delta, \alpha, \varphi, m, A, B \) and \( p \) in our results, we obtain corresponding results due to various researchers.

**References**


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