Applications of Meromorphic Multivalent Functions Associated with Differential Subordination

S. M. Khairnar¹, R. A. Sukne²

¹Professor, Head and Dean (R & D), Maharashtra Academy of Engineering, Alandi, Pune-412105, India
smkhairnar2007@gmail.com

²Assistant Professor in Mathematics, Dilkap Research Institute of Engineering & Management Studies, Neral, Tal. Karjat, Dist. Raigad, India
rasukne@gmail.com

Abstract: In this paper authors introduced subclasses \(D^*_p(\alpha, \beta)\), \(F^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)\), \(G^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)\), and \(H^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)\) as well as \(D^*_p(\alpha, \beta)\) and \(F^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)\), \(G^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)\), and \(H^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)\) of meromorphic multivalent functions in the punctured unit disk \(D^* = \{z: 0 < |z| < 1\} = D(0)\). By using the method of differential subordinations, we derive some certain properties of meromorphically multivalent functions.

Keywords: Meromorphically multivalent function, Analytic function, Subordination, Differentiation.

1 Introduction

Let \(S\) denote the class of functions of the form:
\[
f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_{p+m-1}z^{p+m-1}, \quad (p \in N)
\]
(1)

Which are analytic and \(p\)-valent in the punctured unit disk \(D^* = \{z: 0 < |z| < 1\} = D(0)\). Let \(f(z)\) and \(g(z)\) be analytic in \(D^*\), then we say that \(f(z)\) is subordinate to \(g(z)\) in \(D^*\). Where \(f(z) < g(z)\), if there exists an analytic function \(h(z)\) in \(D^*\), such that \(|h(z)| \leq |z|\) and \(f(z) = g(h(z))\) \((z \in D^*)\). If \(g(z)\) is univalent in \(D^*\) then the subordination
\[
f(z) < g(z) \quad (D^*) \iff f(0) = g(0) \quad \text{and} \quad f(D^*) \subset g(D^*)
\]

Let \(q(z) = 1 + q_1z + q_2z^2 + \cdots\) be analytic in \(D^*\) Such that
\[
|q(z)| < \frac{1+2z}{1-2z} \quad (z \in D^*)
\]
(2)

\[
|q(z) - 1 - 2\alpha| < 2\alpha \left(\frac{1}{2} \leq \beta < \alpha \leq \frac{1}{2}\right)
\]
(3)

Re \(q(z) > \frac{1-2\alpha}{2\alpha}\) \((2\beta = -1, \quad z \in D^*)\) (4)

Recently, several authors proved some interesting properties of meromorphically multivalent functions. In the present topic, we are going to prove some subordination properties for the class \(S\).

When \(g(z) = f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} b_{p+m-1}z^{p+m-1}, \quad (p \in N)\). We define the Hadamard product (convolution) of \(f(z)\) and \(g(z)\) by
\[
f \ast g(z) = f(z) + \sum_{n=0}^{\infty} a_n b_{p+n-1}z^{p+n-1}
\]
(5)

Where \((p \in N, m \in N_0 = N \cup \{0\}, z \in D)\). We define a linear operator \(L^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)(z)\) by
\[
L^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{\sigma(\xi + n)(p + n)}{(\delta - \nu)} + 1^m a_{p+n-1}z^{p+n-1}
\]
(6)

\[
\psi^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)(z)
\]
(7)

Where \(\psi^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; f(z)) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{\sigma(\xi + n)(p + n)}{(\delta - \nu)} + 1^m x^{p+n-1}\).

Throughout this paper we are taking \(\sigma \in \delta \cup \nu\) and \(L^m_p(\sigma, \eta, \xi, \epsilon, \delta, \nu; \varphi)(z) = \psi^m_p(f(z))\)

\[
\sum_{m=0}^{\infty} \frac{\sigma(\xi + n)(p + n)}{(\delta - \nu)} + 1^m x^{p+n-1}
\]

\[
f(z) \prec \varphi(z) \quad (\varphi \in \mathbb{C})
\]

By using the method of differential subordinations, we derive some certain properties of meromorphically multivalent functions.

Keywords: Meromorphically multivalent function, Analytic function, Subordination, Differentiation.

1.1 Definitions

Definition 1.1.1 Function \(f \in S\) is said to be in the class \(F^m_p(\sigma; \xi, \epsilon, \delta, \nu; \varphi)\) if it satisfies the following subordination condition
\[
f(z) \prec \varphi(z) \quad (\sigma \geq 0, z \in D^*, \varphi \in \mathbb{R}, \varphi(z) \neq 0)\)

For simplicity we can write

\[
f(z) \prec \varphi(z) \quad (\sigma \geq 0, z \in D^*, \varphi \in \mathbb{R}, \varphi(z) \neq 0)\)

where \(\varphi \in \mathbb{R}\) and \(\varphi(z) \neq 0\) for all \(z \in D^*\).

**Definition 1.1.2** Function \( f \in S \) is said to be in the class \( G_p^m(\alpha; \xi, \eta, \epsilon, \delta, \sigma, \varphi) \) if it satisfies the following subordination condition:

\[
-\frac{1}{p(1+z)^m} \left( \frac{p - z \varphi'(z)}{\varphi(z)} \right) > 0,
\]

where \( \alpha, \delta > 0, \epsilon > 0, \eta \geq 0, \xi \geq 0, \sigma, \varphi \geq 0, z \in D \).

**Definition 1.1.3** Function \( f \in S \) is said to be in the class \( H_p^m(\alpha; \xi, \eta, \epsilon, \delta, \sigma, \varphi) \). If it satisfies the following subordination condition:

\[
-\frac{1}{p(1+z)^m} \left( \frac{p - z \varphi'(z)}{\varphi(z)} \right) > 0,
\]

where \( \alpha, \delta > 0, \epsilon > 0, \eta \geq 0, \xi \geq 0, \sigma, \varphi \geq 0, z \in D \).

**Definition 1.1.4** Class \( \mathcal{H}^m_p(\alpha; \xi, \eta, \epsilon, \delta, \sigma, \varphi) \) consists of functions \( f(z) \in S \) satisfying the following subordination condition:

\[
-\frac{1}{p(1+z)^m} \left( \frac{p - z \varphi'(z)}{\varphi(z)} \right) > 0,
\]

where \( \alpha, \delta > 0, \epsilon > 0, \eta \geq 0, \xi \geq 0, \sigma, \varphi \geq 0, z \in D \).

**Remark 1.2.3** For \( \alpha = 0, \xi = \eta = 1, \sigma = \frac{1}{2} \), we have the class \( F_p^m(0; \frac{1}{2}, 1, 1, 1, 1, 1; \varphi) = F_p^m(\epsilon, \delta, \sigma, \varphi) \). Where the class \( F_p^m(\epsilon, \delta, \sigma, \varphi) \) consists of functions \( f(z) \in S \) satisfying the following subordination condition:

\[
-\frac{1}{p(1+z)^m} \left( \frac{p - z \varphi'(z)}{\varphi(z)} \right) > 0,
\]

where \( \alpha, \delta > 0, \epsilon > 0, \eta \geq 0, \xi \geq 0, \sigma, \varphi \geq 0, z \in D \).

**Remark 1.2.4** Putting \( p = 1, m = 0, k = 2, \xi = \eta = \delta = 1, \epsilon = \sigma = \frac{1}{2} \), and \( \varphi(z) = 1 + \frac{z}{1-z} \), in definition 1.1.1, we have the class \( \mathcal{B}^m_1(\alpha; \xi, \eta, \epsilon, \delta, \sigma, \varphi) \) consisting of functions \( f(z) \in S \) satisfying the following subordination condition:

\[
-\frac{1}{p(1+z)^m} \left( \frac{p - z \varphi'(z)}{\varphi(z)} \right) > 0,
\]

where \( \alpha, \delta > 0, \epsilon > 0, \eta \geq 0, \xi \geq 0, \sigma, \varphi \geq 0, z \in D \).

**Lemmas**

**Lemma 3.1.1** Let \( f(z) \) be analytic and starlike univalent in \( D \) with \( h(0) = 0 \). If \( g(z) \) is analytic in \( D \) and \( zg(z) < \xi(\varphi(z)) \), then \( g(z) < g(0) + \int_0^{|z_0|} \frac{h(\xi)\,d|z|}{1-2\xi} \).

**Lemma 3.1.2** Let \( q(z) \) be analytic and other than constant in \( D \) with \( q(0) = 1 \). If \( |q(z_0)| \leq 1 \) and

\[
\text{Re}(q(z_0)) = \min_{|z| \leq |z_0|} |\text{Re}(q(z))|,
\]

then \( \frac{1}{2} |q(z_0)|^2 \leq 1 + 2\text{Re}(q(z_0)) \).

**Lemma 3.1.3** Let \( \varphi(z) \) be analytic in \( D \) with \( \varphi(0) = 1 \). If \( q(z) \) is analytic in \( D \) with \( q(0) = 1 \), then the following subordination:

\[
q(z) + \frac{q'(z)}{q(z)+r} < \varphi(z) \Rightarrow q(z) < \varphi(z).
\]
Lemma 1.3.4 Let \(d, r \in \mathbb{C}\); and \(\emptyset(z)\) is convex and univalent in \(D\) with \(\emptyset(0) = 1\) and \(\text{Re}[d \emptyset(z) + r] > 0\).
Also let \(q(z) < \emptyset(z)\). If \(q(z)\) is analytic in \(D\) with \(q(0) = 1\), then the following subordination
\[
q(z) + \frac{z q'(z)}{d + r} < \emptyset(z) \Rightarrow q(z) < \emptyset(z) \quad (z \in D).
\]

Lemma 1.3.5 Let \(f \in F^m_{p,k}(\alpha; \sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi)\) then
\[
\left[\frac{1}{2} \left(1 + z^2\right) + p(1 - \emptyset(z))\right] > 0.
\]
Where
\[
0 < \sigma < \frac{1}{2}, \xi > 0, \eta > 0, 0 < \epsilon \leq \frac{1}{2}, \delta, \sigma, \varphi > 0, z \in D.
\]

Proof: For \((j \in \{0, 1, 2, \ldots, k - 1\})\) we have obtained
\[
f_{p,k}^m[\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z] = \sum_{k=0}^{n} \epsilon_k f^m_{p,k}(\epsilon_k z).
\]
Hence
\[
f_{p,k}^m[\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z] = \sum_{k=0}^{n} \epsilon_k f^m_{p,k}(\epsilon_k z).
\]
And
\[
f_{p,k}^m[\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z] = \sum_{k=0}^{n} \epsilon_k f^m_{p,k}(\epsilon_k z).
\]
Replacing \(m\) by \(m + 1\) in (A) we get
\[
f_{p,k}^{m+1}[\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z] = \sum_{k=0}^{n} \epsilon_k f^m_{p,k}(\epsilon_k z).
\]
And
\[
f_{p,k}^{m+1}[\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z] = \sum_{k=0}^{n} \epsilon_k f^m_{p,k}(\epsilon_k z).
\]
(10)

(From 7) and (10) we obtained
\[
\left[\frac{1}{2} \left(1 + z^2\right) + p(1 - \emptyset(z))\right] > 0.
\]
Noting that \(\emptyset(z)\) is convex and univalent in \(D\). It can be concluded that Lemma 1.3.5 holds true.

Using equations (5) and (6) we get
\[
z f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z) + \left(\frac{1}{2} + \frac{1}{3}\right) f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z)
\]
\[
= \frac{1}{3} \sum_{k=0}^{n} \epsilon_k f^m_{p,k}(\epsilon_k z) = \frac{1}{3} f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z).
\]
And
\[
\emptyset(z) = \frac{\left(\frac{1}{2} + \frac{1}{3}\right) f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z)}{f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z)}
\]
Then
\[
\emptyset(z) = \frac{\left(\frac{1}{2} + \frac{1}{3}\right) f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z)}{f_{p,k}^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma, \varphi; z)}
\]
(11)

2 Main Results

Theorem 2.1 Let \(0 < a \leq 1\) and \(0 < b < 1\). If \(f(z) \in S\) satisfies \(f(z) \neq 0\) \(z \in D^*\) and
\[
\left|\frac{1}{p} f(z) + \frac{1}{b} \right| < \delta < \left|\frac{1}{p} f(z) + 1\right| < \delta < \left|\frac{1}{p} f(z) + 1\right| < \delta < \left|\frac{1}{p} f(z) + 1\right| < \delta
\]

Where \(\delta\) is the minimum positive root of the equation
\[
\alpha \sin \left(\sin \left(\frac{\pi}{2}\right)\right) - x^2 + x \left(1 - \frac{a}{2}\right) \sin \left(\sin \left(\frac{\pi}{2}\right)\right) = 0
\]

Therefore,
\[
\left|\frac{1}{p} f(z) + \frac{1}{b} \right| < \delta < \left|\frac{1}{p} f(z) + 1\right| < \delta
\]

The bound \(b\) is the best possible for each \(0 < a \leq 1\).

Proof: Let
\[
g(x) = \frac{1}{2} \sin \left(\sin \left(\frac{\pi}{2}\right)\right) x^2 - x + \left(1 - \frac{a}{2}\right) \sin \left(\sin \left(\frac{\pi}{2}\right)\right) = 0
\]

There exist two roots for the Equation
\[
\frac{1}{2} \sin \left(\sin \left(\frac{\pi}{2}\right)\right) x^2 - x + \left(1 - \frac{a}{2}\right) \sin \left(\sin \left(\frac{\pi}{2}\right)\right) = 0
\]

Since \(g(0) = \frac{1}{2} \sin \left(\sin \left(\frac{\pi}{2}\right)\right) (0)^2 - 0 + \left(1 - \frac{a}{2}\right) \sin \left(\sin \left(\frac{\pi}{2}\right)\right) = 0\)

Therefore, \(g(x) = 0\) for \(x = 0\) and \(x = 1\).
\[
= \left(1 - \frac{a}{2}\right) \sin \left(\frac{\pi b}{2}\right) > 0
\]
\[
g(1) = \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) \cdot (1)^2 - 1 + \left(1 - \frac{a}{2}\right) \sin \left(\frac{\pi b}{2}\right) = \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) + \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) = 2 \frac{a}{2} \sin \left(\frac{\pi b}{2}\right) < 0
\]
Hence we get \(0 < \frac{a}{2 - a} \delta < \delta < 1\) \hspace{1cm} (15)

Put \(z^p f(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z)\) \hspace{1cm} (16)

Then from the assumption of the theorem, we see that \(q(z)\) is analytic in \(D\) with \(q(0) = 1\) and \(\frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z) \neq 0\) for all \((z \in D)\). Taking the logarithmic differentiations on both sides of \(z^p f(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z)\), we obtained
\[
z^p f(z) + p = \frac{(2-a)q(z)}{a+(2-a)q(z)} \hspace{1cm} (17)
\]
\[
z^p f(z) \left[\frac{z^p f(z)}{f(z)} + p\right] = \frac{(2-a)q(z)^2}{a+(2-a)q(z)} \hspace{1cm} \forall (z \in D) \hspace{1cm} (18)
\]

Thus the inequality \(|\frac{z^p f(z)}{f(z)} + p| < \delta \hspace{1cm} (z \in D)\).

It is equivalent to \(\frac{a}{2+(2-a)q(z)} < \delta z\).

By using Lemma 1, above inequality leads to
\[
f_0 \left[\frac{a}{2+(2-a)q(z)}\right] dt < \delta z. \text{Or to} \hspace{1cm} 1 - \frac{2}{a+(2-a)q(z)} < \delta z \hspace{1cm} (20)
\]

In view of above results it can be written as
\[
q(z) < \frac{1 + \frac{a}{1 - \delta z}}{1 - \delta z} \hspace{1cm} (21)
\]

Now by taking \(\alpha = \frac{a}{2 - a}, \beta = -\frac{\delta}{2}\) in \((1,2)\), we have
\[
0 < \frac{\alpha + \beta}{2} < 1\hspace{1cm} (\text{in } D)
\]
\[
\frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z) < 1\hspace{1cm} (22)
\]

It is easy to see that
\[
\left|z^p f(z) \left[\frac{z^p f(z)}{f(z)} + p\right]\right| < \delta z \hspace{1cm} (z \in D).\hspace{1cm} (23)
\]

Since \(z^p f(z) - \frac{a}{2} = \frac{1 + \frac{a}{1 - \delta z}}{1 - \delta z}\).

It follows from \((3)\) that
\[
\sup_{x \in D} \left|\arg \left(z^p f(z) - \frac{a}{2}\right)\right| = \arg \sin \left(\frac{\theta}{2 - a + a \delta^2}\right) = \frac{b}{2}\hspace{1cm} (24)
\]

Hence, we conclude that the bound \(b\) is the best possible for all \(a \in [0,1]\). Next, we derive the following.

**Theorem 2.2** If \(f(z) \in S\) satisfies \(f(z) \neq 0, (z \in D)\) and
\[
\text{Re}\left[1 - 2e \cdot \log(1 - z)\right] > 1 - 2e \cdot \log 2 \hspace{1cm} (z \in D).\hspace{1cm} (25)
\]

Then \(q(z)\) is analytic in \(D\) with \(q(0) = 1\) and \(q(z) \neq 0\) for \(z \in D\). In accordance with \((17)\) and \((20)\), we obtained
\[
1 - \frac{2\alpha}{a} \leq \frac{1 + \frac{a}{1 - \delta z}}{1 - \delta z} \hspace{1cm} (26)
\]

Let \(\frac{1}{q(z)} < \frac{1}{2} + \frac{2\alpha}{a} \hspace{1cm} (27)
\]

Now by Lemma 1, we obtain
\[
\frac{1}{q(z)} < 1 - 2e \cdot \log(1 - z).
\]
Since the function \(1 - 2e \cdot \log(1 - z)\) is convex univalent in \(D\) and
Differentiating both sides we get
\[ q(z) \frac{dq(z)}{dz} + \frac{1}{3} p + \frac{1}{3p} \left( \frac{f_m(s, \xi, \eta, \delta, \sigma, z)}{f_m(s, \xi, \eta, \delta, \sigma, z)} \right) q(z) \]

\[ = \frac{-1}{3p} \frac{f_m(s, \xi, \eta, \delta, \sigma, z)}{f_m(s, \xi, \eta, \delta, \sigma, z)} \left( \frac{\phi_m(s, \xi, \eta, \delta, \sigma, z)}{\phi_m(s, \xi, \eta, \delta, \sigma, z)} \right) + \frac{1}{3p} \left( \frac{f_m(s, \xi, \eta, \delta, \sigma, z)}{f_m(s, \xi, \eta, \delta, \sigma, z)} \right) q(z) \]

\[ \Rightarrow q(z) \frac{dq(z)}{dz} + \frac{1}{3} p + \frac{1}{3p} \left( \frac{f_m(s, \xi, \eta, \delta, \sigma, z)}{f_m(s, \xi, \eta, \delta, \sigma, z)} \right) q(z) = \frac{-1}{3p} \frac{f_m(s, \xi, \eta, \delta, \sigma, z)}{f_m(s, \xi, \eta, \delta, \sigma, z)} \left( \frac{\phi_m(s, \xi, \eta, \delta, \sigma, z)}{\phi_m(s, \xi, \eta, \delta, \sigma, z)} \right) \]

\[ \Rightarrow q(z) = -\frac{1}{3p} \frac{f_m(s, \xi, \eta, \delta, \sigma, z)}{f_m(s, \xi, \eta, \delta, \sigma, z)} \left( \frac{\phi_m(s, \xi, \eta, \delta, \sigma, z)}{\phi_m(s, \xi, \eta, \delta, \sigma, z)} \right) \]

This implies that \( f \in \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).

Corollary 2.1.3 Let \( \varphi \in P \) with
\[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + p(1 - \varphi(z)) > 0. \]

Where
\[ 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}. \]

Then \( G_m^p(\sigma, \xi, \eta, \delta, \sigma, \varphi) \subset \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).

Corollary 2.1.4 Let \( \varphi \in P \) with
\[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + p(1 - \varphi(z)) > 0. \]

Where
\[ 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}. \]

Then \( X_m^p(\sigma, \xi, \eta, \delta, \sigma, \varphi) \subset \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).

Corollary 2.1.5 Let \( \varphi \in P \) with
\[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + p(1 - \varphi(z)) > 0. \]

Where
\[ 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}. \]

Then \( G_{m+1}^p(\sigma, \xi, \eta, \delta, \sigma, \varphi) \subset \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).

Corollary 2.1.6 Let \( \varphi \in P \) with
\[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + p(1 - \varphi(z)) > 0. \]

Where
\[ 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}. \]

Then \( X_{m+1}^p(\sigma, \xi, \eta, \delta, \sigma, \varphi) \subset \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).

Corollary 2.1.7 Let \( \varphi \in P \) with
\[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + p(1 - \varphi(z)) > 0. \]

Where
\[ 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}. \]

Then \( H_{m+1}^p(\sigma, \xi, \eta, \delta, \sigma, \varphi) \subset \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).

Corollary 2.1.8 Let \( \varphi \in P \) with
\[ \frac{1}{3} \left( 2 + \frac{1}{a} \right) + p(1 - \varphi(z)) > 0. \]

Where
\[ 0 < \sigma \leq \frac{1}{2}, \xi > 0, \eta \geq 0, 0 < \varepsilon \leq \frac{1}{2}. \]

Then \( \exists_{m+1}^p(\sigma, \xi, \eta, \delta, \sigma, \varphi) \subset \mathbb{P}(s, \sigma, \xi, \eta, \delta, \sigma, \varphi) \).
2.2 Integral Representation

In this section we are going to prove integral representations associated with the function classes \(F_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\), \(G_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) and \(H_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\).

**Theorem 2.2.1** Let \(f \in F_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) then
\[
F_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi) = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

Where \(w(z)\) is analytic in \(D\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in D)\).

**Proof** Suppose that \(f \in F_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\).

\[
\frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

Upon integration which yields,

\[
\log\left(z^p \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) \right) = -\frac{1}{p} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

From above equations we get
\[
\left(\psi^m f\right)(z) = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

Thus by applying method of the proof of theorem 2.2.3 we find that
\[
f_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi) = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

And \(|w(z)| < 1\) \((z \in D)\).

**Corollary 2.2.1** Let \(f \in G_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) then
\[
g_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi) = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

Where \(g_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) is analytic in \(D\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in D)\).

**Corollary 2.2.2** Let \(f \in H_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) then
\[
\psi^m f(z) = -\frac{1}{p} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

And \(|w(z)| < 1\) \((z \in D)\).

**Corollary 2.2.3** Let \(f \in H_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) then
\[
h_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi) = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

Where \(h_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) is analytic in \(D\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in D)\).

2.3 Convolution Properties

In this part we are going to derive several convolution properties for the function classes \(F_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\), \(G_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) and \(H_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\).

**Theorem 2.3.1** Let \(f \in F_p^m(\sigma, \xi, \eta, \varepsilon, \delta, \sigma; \varphi)\) then
\[
f(z) = \frac{-\pi \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j))}{\kappa k} \sum_{j=0}^{k-1} \varphi(w(\varepsilon z^j)) ^{-1} \, dz.
\]

Where \(w(z)\) is analytic in \(D\) with \(w(0) = 0\) and \(|w(z)| < 1\) \((z \in D, j = 1, 2)\).
\[
-\sum_{\alpha=0}^{n-1} \varphi(w(\zeta)) \exp \left( \frac{-p}{k} \sum_{j=0}^{n-1} f_{0}^{j} \frac{\varphi(w(\zeta))^{j}}{\zeta} \frac{d\zeta}{\zeta} \right)
\]

\[
* = \sum_{\alpha=0}^{n-1} \frac{1}{m(n+1)}
\]

Where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in D) \).

**Proof:** Since we know that linear operator given by

\[
\psi_p^m f(z) = \frac{1}{z^p} \sum_{n=0}^{\infty} [3(p + n) + 1]^m a_{p,n-1} z^{n-1}
\]

\[
= (\psi_{p,\xi,\eta,\epsilon,\delta,\sigma} \ast f)(z).
\]

Where

\[
\psi_p^m f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} [3(p + n) + 1]^m z^{n-1}
\]

and

\[
\psi_p^m f(z) = -p \sum_{j=0}^{n-1} f_{0}^{j} \frac{\varphi(w(\zeta))^{j}}{\zeta} \frac{d\zeta}{\zeta}.
\]

Thus from above equation, we can easily get the theorem.

\[
\psi_p^m f(z) = \frac{1}{z^p} \sum_{n=0}^{\infty} \frac{1}{m(n+1)}
\]

where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in D) \).

**Theorem 2.3.2** Let \( f \in L_p^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \) then

\[
f(z) = \left[ -p \sum_{j=0}^{n-1} f_{0}^{j} \varphi(w(\zeta)) \exp \left( \frac{-p}{k} \sum_{j=0}^{n-1} f_{0}^{j} \frac{\varphi(w(\zeta))^{j}}{\zeta} \frac{d\zeta}{\zeta} \right) \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{m(n+1)}
\]

where \( w(z) \) is analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in D, j = 1,2) \).

**Proof:** Since we know that linear operator given by,

\[
\psi_p^m f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} [3(p + n) + 1]^m a_{p,n-1} z^{n-1}
\]

\[
= (\psi_{p,\xi,\eta,\epsilon,\delta,\sigma} \ast f)(z).
\]

Where

\[
\psi_p^m f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} [3(p + n) + 1]^m z^{n-1}
\]

and

\[
\psi_p^m f(z) = -p \sum_{j=0}^{n-1} f_{0}^{j} \frac{\varphi(w(\zeta))^{j}}{\zeta} \frac{d\zeta}{\zeta}.
\]

We obtained

\[
-\sum_{\alpha=0}^{n-1} \varphi(w(\zeta)) \exp \left( \frac{-p}{k} \sum_{j=0}^{n-1} f_{0}^{j} \frac{\varphi(w(\zeta))^{j}}{\zeta} \frac{d\zeta}{\zeta} \right)
\]

\[
= \sum_{\alpha=0}^{n-1} \frac{1}{m(n+1)}
\]

Thus from above result, we can easily prove the Theorem.

**Theorem 2.3.3** Let \( f \in S \) and \( \varphi \in P \). Then \( f \in L_p^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \) if and only if

\[
f(z) = \left[ (-p z^{-p} + \sum_{n=0}^{\infty} [3(n + 1)^m (n - p)] z^{n-1} + pp(e^{i\theta}) (z^{n-1} + \sum_{n=0}^{\infty} [3(n + 1)^m z^{n-1}) \right.
\]

\[
\left. \neq 0 \right) \quad (z \in D; 0 \leq \theta < 2\pi).
\]

**Proof:** Suppose that \( f \in L_p^m(\sigma, \xi, \eta, \epsilon, \delta, \sigma; \varphi) \). Since the following subordination condition

\[
\left[ \frac{-p(\psi_p^m(f))^{j}}{p^j(\psi_p^m(\sigma, \xi, \epsilon, \delta, \sigma; \varphi))} \right] < \varphi(z) \quad (z \in D)
\]

is equivalent to

\[
\left[ \frac{-p(\psi_p^m(f))^{j}}{p^j(\psi_p^m(\sigma, \xi, \epsilon, \delta, \sigma; \varphi))} \right] \neq \varphi(e^{i\theta}) \quad (z \in D; 0 \leq \theta < 2\pi).
\]


Author Profile
Dr. S. M. Khairnar is Professor and Head of Department in Maharashtra Academy of Engineering. He is also working as Dean (R & D)

R. A. Sukne is Assistant Professor in Mathematics at Dilkap Research Institute of Engineering & Management Studies, Dist. Raigad.