Review of g*G-axioms in Topological Spaces

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Abstract: ***In this paper we introduced g*G-axioms in topological spaces. It was Gupta who first used connected sets to define a new kind of separation axioms known as G-axioms. Some of the G-axioms are G_1, G_2 and G_2' axioms. Replacing open sets doing separation by g*-open sets in these axioms, new concepts of $g*G_1$, $g*G_2$ and $g*G_2'$ axioms are introduced and studied in this work. These new extensions are termed as some g*G-axioms. Non coincidence of these axioms is shown by various counter examples.***

Keywords: Connected set, g*-open, g*-neighborhood, g*-closure

1. Introduction

Replacing the sets being separated or doing separation in the separation axioms by different types of sets, several extensions of separation axioms have been introduced by mathematicians from time to time. Gupta [2] used the notion of connectedness to extend the study of separation axioms in a way like Aull [1]. Sivakamasundari [4] introduced the concept of gG-axioms. A space X is said to be G_1 (resp. G_2 [2]) if for any connected subset M of X and a point $x \notin M$, there exist open (resp. disjoint open) sets U and V such that $x \in U, M \subseteq V$ and $U \cap M = \phi, \{x\} \cap V = \phi$ (resp. $x \in U$, $M \subseteq V$). A space X is said to be G_2' [2] if any two disjoint connected subsets M and N of X are separated by disjoint open sets U and V. Generalizations of these axioms, by replacing the open sets doing separation by g*-open sets, are proposed in the work.

Throughout the sequel, the space X will mean topological space X with topology τ on which no separation axioms are assumed. If Y is a subspace of X, then τ_Y denotes relativized topology on Y. cl(A), gcl(A), g*cl(A) and X – A are used to indicate closure, g-closure, g*-closure and complement of a subset A of X. G*O(X, τ) denotes the class of g*-open sets in X.

2. Preliminaries

2.1 Definition

A space X is connected if and only if there do not exist disjoint non-empty open sets H and K such that $X = H \cup K$. in this definition 'open' can be replaced by 'closed' [7]. Apparently, X is connected if and only if there are no open-closed subsets of X other than ϕ and X [7]. Clearly, singletons (points) are connected sets in every topology. If two sets A and B have the property that $cl(A) \cap B = A \cap cl(B) = \phi$, they are called separated. A subset C in a topological space is connected if it cannot be written as the union of two separated sets [5]. If A is a connected subset of a space X and $A \subseteq B \subseteq cl(A)$, then B is connected [7].

2.2 Definition [3]

A set A is **g-closed** if and only if $cl(A) \subseteq A$, whenever $A \subseteq U$ and U is open. The complement of g-closed set is known as a g-open set.

2.3 Definition [6]

A set A is **g*-closed** if and only if $cl(A) \subseteq A$, whenever $A \subseteq U$ and U is g-open. The complement of g*-closed set is known as a g*-open set. The family of g*-open sets is denoted by G*O(X, τ).

2.4 Definition [6]

A map $f : X \longrightarrow Y$ is called g*-continuous, if the inverse image of every closed set in Y is g*-closed in X.

2.5 Definition [6]

A map f: $X \longrightarrow Y$ is called g*c-irresolute if the inverse image of every g*-closed set in Y is g*-closed in X.

2.6 Proposition [6]

If a map $f : X \longrightarrow Y$ is continuous then it is g*-continuous

2.7Proposition [6]

If a map f: $X \longrightarrow Y$ is bijective, open and g*-continuous then f is g*c-irresolute

2.8 Definition [4]

A space X is said to be gG_1 if for any point $x \in X$ and any connected subset M of X not containing x, there exist g-open sets U and V such that $x \in U, M \subseteq V$ and $U \cap M = \phi, \{x\} \cap V = \phi.$

2.9 Definition [4]

A space X is said to be gG_2 if for every connected set F and a point $x \in F$ there exist g-open sets U and V such that $x \in U, F \subseteq V$ and $U \cap V = \phi.$

2.10 Definition [4]

A space X is said to $gG_2^{'}$ if for any two disjoint connected sets M and N of X, there exist disjoint g-open sets U and V such that $M \subseteq U$ and $N \subseteq V.$

3. g*G-AXIOMS

3.1 g^*G_1-Spaces

3.1.1 Definition

A space X is said to be g^*G_1 if for any point $x \in X$ and any connected subset M of X not containing x, there exist g*-open sets U and V such that $x \in U, M \subseteq V$ and $U \cap M = \phi, \{x\} \cap V = \phi.$

3.1.2 Proposition

Every G_1 space is g^*G_1.

Proof: The proof follows from the fact that every open set is g*-open. The converse is not true as seen from the following example.

3.1.3 Example

Let X = {a, b, c}, τ = {ϕ, X, {a}, {b, c}}. Then the space X is g^*G_1 but not G_1.

3.1.4 Theorem

Every g^*G_1 space is gG_1-space.

Proof: The proof follows from every g*-open set is g-open. The converse is not true as seen from the following example.

3.1.5 Example

Let X = {a, b, c}, τ = {ϕ, X, {a}, {b, c}}. Then the space X is g^*G_1 but not gG_1.

3.1.6 Definition

A subset M of X is called a g*-neighborhood of X if there is a g*-open set U such that $x \in X \subseteq M.$

3.1.7 Theorem

In a space X, the following are equivalent

(a) X is g^*G_1
(b) Every connected subset of X is g*-closed.
(c) For any two disjoint connected subsets M and N of X, there exist g*-open sets U and V such that $M \subseteq U, N \subseteq V$ and $U \cap N = \phi, V \cap M = \phi.$

Proof

(a) $\rightarrow$ (b) Let M be a connected subset of X and $x \notin M.$ Then $x \in X - M.$ By hypothesis, there exists a g*-open set U such that $x \in U$ and $U \cap M = \phi.$ Hence $U \subseteq X - M.$ That is $x \in U \subseteq X - M.$ This implies X – M is a g*-neighborhood of x. Since x is arbitrary, X – M is a g*-neighborhood of each point of X – M. This implies X – M is g*-open. Hence M is g*-closed.

(b) $\rightarrow$ (c) Let M and N be two disjoint connected subsets of X. Then by (b) M and N are g*-closed and hence X – M and X – N are g*-open sets containing N and M as M and N are disjoint. Put U = X – N and V = X – M. Then $M \subseteq X - N = U,\ N \subseteq X - M = V \,\&\, U \cap M = \phi$ & $V \cap N = \phi.$ So (c) holds.

(c) $\rightarrow$ (a) Since singleton set is a connected subset (a) follows form (c) by taking M to be a singleton set.

3.1.8 Theorem

A space X is g^*G_1 if and only if for any connected set M and $x \notin M,$ there exists a g*-open set U such that $x \in U$ and $U \cap M = \phi.$

Proof

Let X be a g^*G_1. From the definition 3.1.1., the criterion follows. Conversely, since there exists a g*-open set U containing x such that $U \cap M = \phi.$. So X – M is a g*-neighborhood of x and hence X – M is g*-open. So that M is g*-closed. Then by Theorem 3.1.7 (b), X is g^*G_1.

3.1.9 Theorem

Every open subspace Y of a g^*G_1-space X, is g^*G_1.

Proof

Let A be a connected set in $Y \subseteq X.$ Then A is connected in X. If $y \in Y$ does not belong to A, then by g^*G_1 axiom on X, there exist g*-open sets U and V in X such that $y \in U, A \subseteq V \,\&\, U \cap A = \phi, \{y\} \cap V = \phi.$ Consequently $(Y - U)$ and $(Y - V)$ are g*-open sets in Y satisfying $y \in Y \cap U, A \subseteq Y \cap V$ and $A \cap (Y \cap U) = (U \cap A) \cap Y = \phi \cap Y = \phi$ and similarly $\{y\} \cap (Y \cap V) = \phi.$ Hence Y is g^*G_1.

3.1.10 Theorem

If f is a continuous map from X to Y. Then f(A) is connected in Y if A is connected in X.

Proof

Suppose f(A) is disconnected, then $f(A) = H \cup K$, where H and K are disjoint open sets in Y. Hence $A = f^{-1}(H \cup K) = f^{-1}(H) \cup f^{-1}(K)$. Also since f is continuous, so $f^{-1}(H)$ and $f^{-1}(K)$ are disjoint open sets in X which obviously shows that A is disconnected, a contradiction. Hence f(A) is connected.

3.1.11 Theorem

If f is one-one, onto continuous and open mapping from a space X to another space Y. Then X is $g*G_1$ if Y is $g*G_1$-space.

Proof

Let X→Y be a continuous map and M be a connected subset in X such that $x \notin M$. Then due to continuity, f(M) is connected in Y by Theorem 3.1.10. and $f(x) \notin f(M)$, as f is one-one. Since Y is $g*G_1$, there exist a g*-open sets U and V in Y such that $f(x) \in U, f(M) \subseteq V$ and $U \cap f(M) = \phi$, $V \cap f(x) = \phi$. This implies that $f^{-1}(U) \& f^{-1}(V)$ are g*-open sets in X [Proposition 2.6. and Proposition 2.7, f is g*c-irresolute] and $x \in f^{-1}(U), M \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap M = \phi$, $f^{-1}(V) \cap \{x\} = \phi$. Hence X is $g*G_1$.

3.2 $g*G_2$-space

3.2.1 Definition

A space X is said to be $g*G_2$ if for every connected set F and a point $x \notin F$ there exist g*-open sets U and V such that $x \in U, F \subseteq V$ and $U \cap V = \phi$.

Evidently, every G_2-space is $g*G_2$-space. However, the converse is not true as shown by the following example.

3.2.2 Example

Let X = {a, b, c}, τ ={ϕ, X, {a}, {b, c}}. Then the space X is $g*G_2$ but not G_2.

3.2.3 Theorem

Every $g*G_2$-space is gG_2-space.

Proof: The proof follows from every g*-open set is g-open.

3.2.4 Definition

A space X is said to be $g*T_2$ if for each pair of distinct points in X there exist distinct g*-open sets U and V in X such that $x \in U \& y \in V$.

3.2.5 Theorem

Every $g*G_2$-space is $g*T_2$.

Proof

Let X be $g*G_2$-space and $x \neq y \in X$. Then $x \notin \{y\}$ a connected set. So by hypothesis there exist disjoint g*-open sets U and V such that $x \in U \& \{y\} \subseteq V$. (i.e.)$y \in V$. Hence X is $g*T_2$.

3.2.7 Theorem

Every open subspace Y of a $g*G_2$-space X, is $g*G_2$.

Proof

Similar to the proof of Theorem 3.1.9.

3.2.8 Definition

A Point $x \in X$ is said to be a g*-limit of a subset A of X if every g*-open set containing x, contains atleast one point of A other than x.

3.2.9 Theorem

Every connected subset M of a $g*G_2$-space X is g*-closed.

Proof

Let M be a connected subset of a $g*G_2$-space and Let $x \in X$ be such that $x \notin M$. Then by hypothesis, there exist a disjoint g*-open sets G and H such that $x \in G, M \subseteq H \& G \cap H = \phi$, which implies that $G \cap M = \phi$. Since $M \subseteq H$, so x is not a g*-limit point of M. Hence $x \notin g*cl(M)$. Thus $x \notin M$ implies $x \notin g*cl(M)$ showing $g*cl(M) \subseteq M$. Hence M is g*-closed.

In View of Theorem 3.2.9 and Theorem 3.1.7.(b) every $g*G_2$ space is $g*G_1$. However the converse is not true as seen from the following example.

3.2.10 Example

The modified fort space (P55, [5]) is an example of space which is $g*G_1$ but not $g*G_2$. Let $X = N \cup \{x_1, x_2\}$, where N is an infinite set and x_1, x_2 distinct points. Let τ = {all subsets of N}∪{all sets containing x_1 or x_2 if and only if they contains all but a finite number of points in N}. Then the only connected subsets of X are the one point subsets X. (X,τ) is not a G_2-space as x_1 and x_2 do not have disjoint neighborhoods. As GO(X, τ) = P(X) = G*O(X, τ). X is also not a gG_2 and $g*G_2$ space but (X,τ) is a gG_1 and $g*G_1$ space.

3.2.11 Theorem

A space X is $g*G_2$ if and only if for any point x in X and any connected set M not containing x, $g*cl(U)\cap M=\phi$, where U is a g*-neighborhood of x.

Proof

Let $x\notin M$, where M is any connected set in X. Then by $g*G_2$ axiom there exist disjoint g*-open sets U and V containing x and M respectively. Clearly then $U\subseteq X-V$ and $g*cl(U)\subseteq g*cl(X-V)=X-V$. This implies $g*cl(U)\cap M=\phi$ as $M\subseteq V$.

Conversely, let for any connected set M and $x\notin M.$, $g*cl(U)\cap M=\phi$, where U is a g*-neighborhood of x. Hence $M\subseteq X-g*cl(U)$, a g*-open set. Since U is g*-neighborhood of x there exists a g*-open set, say V such that $x\in V\subseteq U$. Hence there exist g*-open sets V and X – g*cl(U) such that $x\in V$ and $M\subseteq X-g*cl(U)$. Also $V\cap(X-g*cl(U))=\phi$. Hence X is $g*G_2$

3.2.12 Theorem

A space X is $g*G_2$ if and only if every connected set is g*-closed and for every g*-closed connected set F and a point $x\notin F$, there exist disjoint g*-open sets U and V such that $x\in U, F\subseteq V$.

Proof

Let X be a $g*G_2$-space. By Theorem 3.2.9 every connected subset is g*-closed and the rest of the hypothesis follows by $g*G_2$-axiom.

Conversely in X, for a g*-closed connected set F and a point $x\notin F$, there exist disjoint g*-open sets U and V such that $x\in U, F\subseteq V$ and that every connected set is g*-closed. So the proof follows. Since only g*-closed connected sets are all connected sets in X.

3.2.13 Theorem

If f is continuous and open bijection from X to a $g*G_2$-spaces Y. Then X is $g*G_2$-also.

Proof

Similar to the proof of Theorem 3.1.11

3.3. $g*G_2'$-spaces

3.3.1 Definition

A space X is said to $g*G_2'$ if for any two disjoint connected sets M and N of X, there exist disjoint g*-open set U and V such that $M\subseteq U, N\subseteq V$.

3.3.2 Remark

1. Every G_2'-space is $g*G_2'$
2. Every $g*G_2'$-space is gG_2'

3.3.3 Definition

A space X is said to a g*-irresolutely normal if for any pair of disjoint g*-closed sets A and B there exist g*-open sets U and V such that $A\subseteq U, B\subseteq V$and $U\cap V=\phi$.

3.3.4 Theorem

Every g*-irresolutely normal $g*G_1$-space X is $g*G_2'$

Proof

Let A and B be disjoint connected sets in X. Since X is $g*G_1$, so by Theorem 3.1.7(b) A and B are disjoint g*-closed sets. By application of g*-irresolute normality, there exist disjoint g*-open sets U and V such that $A\subseteq U, B\subseteq V$. Hence X is $g*G_2'$

3.3.5 Theorem

Every open subset Y of a $g*G_2'$ space X is $g*G_2'$

Proof

Similar to the proof of Theorem 3.1.9

3.3.6 Theorem

If f: X →Y is a continuous and open bijection. If Y is $g*G_2'$-space, then X is $g*G_2'$.

Proof

Let M and N be disjoint connected sets in X. Then f(M) and f(N) are connected sets in Y and $f(M)\cap f(N)=\phi$. Since Y is $g*G_2'$-space, there exist disjoint g*-open sets U and V such that $f(M)\subseteq U$ and $f(N)\subseteq V$. Hence $M\subseteq f^{-1}(U)$ and $N\subseteq f^{-1}(V)$. Also By Proposition 2.6. and Proposition 2.7. $f^{-1}(U)$ and $f^{-1}(V)$ are g*-open sets. Since f is bijective and $U\cap V=\phi$. We get $\phi=f^{-1}(U\cap V)=f^{-1}(U)\cap f^{-1}(V)$. Hence X is $g*G_2'$-space.

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