On the Lattice of L-closure Operators

Raji George[1], T. P. Johnson[2]

[1]Department of Mathematics, St. Peter's College, Kolenchery - 682311
Ernakulum Dt, Kerala State, India.
bobyraji@gmail.com

[2]Applied Sciences and Humanities Division, School of Engineering
Cochin University of Science and Technology, Cochin-22, Kerala State, India.
jjthenayan@gmail.com

Abstract: ***In this paper we proved that the lattice of all L-closure operators on a fixed set X is not modular. We identified the infra L-closure operators and ultra L-closure operators. Also established the relation between ultra L-topologies and ultra L-closure operators.***

Keywords: L - closure operator, ultra L-topology, ultra L-closure operator, infra L-closure operator.

1. Introduction

In 1965 L. A Zadeh [11] introduced fuzzy sets as a generalization of ordinary sets. After that C. L. Chang [2] introduced fuzzy topology and that led to the discussion of various aspects of L-topology by many authors. The Čech closure spaces introduced by Čech E. [1] is a generalization of the topological spaces. The theory of fuzzy closure spaces has been established by Mashhour and Ghanim [4] and Srivastava et. al [6],[7]. The definition of Mashhour and Ghanim is an analogue of Čech closure spaces and Srivastava et. al. have introduced it as an analogue of Birkhoff closure spaces in [7]. Based on [7], Rekha Srivastava and Manjari Srivastava studied the subspace of a fuzzy closure space. The notion of T_0-fuzzy closure spaces and T_1 fuzzy closure spaces were also introduced in [6]. In [5] P. T. Ramachandran studied the properties of lattice of closure operators. In [3] T. P. Johnson studied some properties of the lattice $L(X)$ of all fuzzy closure operators on a fixed set X. In [9] Wu-Neng Zhou introduced the concept of L-closure spaces and the convergence in L-closure spaces. In this paper we study the lattice $LC(X)$ of L-closure operators and L-closure spaces which is a generalization of the concept of fuzzy closure spaces. Here we proved that the complete lattice $LC(X)$ is not modular. Also we identify the infra L-closure operator and ultra L-closure operator and establish the relation between ultra L-topology and ultra L-closure operator. We proved that an L-closure operator is an ultra L-closure operator if and only if it is the L-closure operator associated with an ultra L-topology. Also proved that infra Lclosure operators are less than or equal to any nonprincipal ultra L-closure operator and no nonprincipal ultra L-closure operator has a complement so that the lattice of L-closure operators is not complemented in general.

2. Preliminaries

A completely distributive lattice L is called a Fuzzy lattice, if there is an order reversing involution from L to L. Let X be any nonempty set and L is a Fuzzy lattice. The fundamental definition of L-fuzzy set theory and L-fuzzy topology are assumed to be familiar to the reader as in [10].Here we call L-fuzzy subsets as L subsets and L-fuzzy topology as L-topology.

2.1 Definition

A Čech fuzzy closure operator on a set X is a function $\chi : I^X \to I^X$, satisfying the following three axioms
1. $\chi(\underline{0}) = \underline{0}$
2. $f \leq \chi(f)$ for every f in I^X.
3. $\chi(f \vee g) = \chi(f) \vee \chi(g)$ where $I = [0, 1]$

For convenience it is called fuzzy closure operator on X and (X, χ) is called fuzzy closure space. In [9] Wu-Neng Zhou defined L-closure operator as follows.

2.2. Definition

A mapping $C: L^X \to L^X$ is called an L-closure operator or an L-closure, if it satisfies the following conditions for any $A, B \in L^X$:
1. $C(0_X) = 0_X$
2. $A \leq C(A)$
3. $A \leq B$ implies $C(A) \leq C(B)$
4. $C(C(A)) = C(A)$

But in this paper we take the definition of L-closure operator as a generalization of fuzzy closure operator in [4]

2.3. Definition

Let X be a nonempty set and L be a Fuzzy lattice. An L-closure operator on L^X is a mapping $\psi: L^X \to L^X$ satisfying the following conditions:
1. $\psi(\underline{0}) = \underline{0}$
2. $f \leq \psi(f)$
3. $\psi(f \vee g) = \psi(f) \vee \psi(g)$ for every $f, g \in L^X$.

The pair (X, ψ) is called an L-closure space. An L-subset f of X is said to be an L-closed set in (X, ψ) if $\psi(f) = f$. An L subset f of X is open if its complement is closed in (X, ψ). The set of all open L subsets of (X, ψ) form an L-topology on X called the L-topology associated with the L-closure operator ψ.

Let F be an L-topology on a set X. Then a function $\psi : L^X \to L^X$ defined by $\psi(f) = \bar{f}$ for all $f \in L^X$, where $\bar{f}$ denotes the closure of f with respect to F is called the L-closure operator associated with the L-topology F.

An L-closure operator on a set X is called L-topological if it is the L- closure operator associated with an L-topology on X. That is $\psi(\psi(f)) = \psi(f)$ for all $f \in L^X$

Note that different L-closure operators can have the same associated L-topology. But different L -topologies cannot have the same associated L-closure operator.

3. Lattice of L-closure operators

Let ψ_1 and ψ_2 be L-closure operators on X. Then $\psi_1 \leq \psi_2$ if and only if $\psi_2(f) \leq \psi_1(f)$ for every f in L^X. The relation $\leq$ defined above is a partial order on the set of all L-closure operators on L^X. We denote the poset by LC(X). Then LC(X) is a lattice. The L-closure operator D on X defined by $D(f) = f$ for every f in L^X is called the discrete L-closure operator. The L-closure operator I on X defined by $I(f) = \begin{cases} \underline{0} & \text{if } f=\underline{0} \\ \underline{1} & \text{otherwise} \end{cases}$

is called the indiscrete L-closure operator.

Remark 3.1

D and I are the L-closure operators associated with the discrete and indiscrete L-topologies on X respectively. Moreover D is the unique L-closure operator who's associated L-topology is discrete. Also I and D are the smallest and the largest elements of LC(X) respectively.

Theorem 3.1.

LC(X) is a complete lattice.

Proof. Can be easily proved.

Definition 3.1.

Lattice of L closure operators LC(X) is modular if and only if

$\chi \geq \eta \Rightarrow \chi \wedge (\psi \vee \eta) = (\chi \wedge \psi) \vee \eta, \forall\ \chi, \psi, \eta \in LC(X)$

Theorem 3.2.

LC(X) is not modular

Proof. Let X be any set and $x \in X$. Define ψ_x, χ_x, η_x from $L^X \to L^X$ by $\psi_x(\underline{0}) = \underline{0}$

$$\psi_x(f)(y) = \begin{cases} f(y) & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

$\chi_x(\underline{0}) = \underline{0}$

$$\chi_x(f)(y) = \begin{cases} 1 & \text{if } y \neq x \\ f(y) & \text{if } y = x \end{cases}$$

$\eta_x(\underline{0}) = \underline{0}$

$$\eta_x(f)(y) = \begin{cases} 1 & \text{if } y \neq x \\ \beta & \text{if } y = x \end{cases} \quad \text{and } \beta \geq f(y)$$

Then $\chi_x(f)(y) \leq \eta_x(f)(y), \forall y$. Hence $\chi_x \geq \eta_x$

$\chi_x \wedge \psi_x = \inf(\chi_x, \psi_x)$

$= \sup(\chi_x(f)(y), \psi_x(f)(y))$

$= 1$

$(\chi_x \wedge \psi_x) \vee \eta_x = \inf(1, \eta_x(f)(y))$

$= f(y)$

$\psi_x \vee \eta_x = \sup(\psi_x, \eta_x)$

$= \inf(\psi_x(f)(y), \eta_x(f)(y))$

$= f(y)$

$\chi_x \wedge (\psi_x \vee \eta_x) = \sup(\chi_x(f)(y), f(y))$

$= 1$

Therefore $\chi_x \wedge (\psi_x \vee \eta_x) \neq (\chi_x \wedge \psi_x) \vee \eta_x$

So LC(X) is not modular.

Definition 3.2.

An L-closure operator on X is called an infra L-closure operator if the only L-closure operator on X strictly smaller than it is I.

Let X be any set and $a, b \in X$ such that $a \neq b$. Define $\psi_{a,b} : L^X \to L^X$ by

$$\Psi_{a,b}(f) = \begin{cases} f & \text{if } f = \underline{0} \\ g_{\alpha,b} & \text{if } f = a_\alpha \\ 1 & \text{otherwise} \end{cases}$$

α is a dual atom in L and $g_{\alpha,b}$ is defined by

$$g_{\alpha,b}(a) = \begin{cases} 1 & \text{if } a \neq b \\ \alpha & \text{if } a = b \end{cases}$$

In the topological context Ramachandran [5] proved that a closure operator on X is an infra closure operator if and only if it is of the form $V_{a,b}$ for some a, b in X, $a \neq b$, where $V_{a,b}$ is defined by

$$V_{a,b}(A) = \begin{cases} \varphi & \text{if } A = \varphi \\ X \setminus \{b\} & \text{if } A = \{a\} \\ X & \text{otherwise} \end{cases}$$

Analogously in the L-topological context we prove the following theorem.

Theorem 3.3.

An L-closure operator is an infra L-closure operator if and only if it is of the form $\psi_{a,b}$ for some $a, b \in X, a \neq b$.

Proof. Let ψ is an L-closure operator on X strictly smaller than $\psi_{a,b}$, then $\psi(a_\alpha)$ will be strictly greater than $\psi_{a,b}(a_\alpha) = g_{\alpha,b}$ and hence equal to $\underline{1}$ so that $\psi(f) = \underline{1}, \forall f \in L^X$ other than $\underline{0}$. Hence $\psi = I$. Thus all L-closure operators of the form $\psi_{a,b}$ are infra L-closure operators.

Conversely let ψ be any L closure operator other than me. Then we can find a nonzero L subset f such that $\psi(f) \neq I$

(f) = $\underline{1}$ (i.e. $\psi(f) \neq \underline{1}$) and elements a_α, b_β, where $\alpha, \beta \in L$ such that $a_\alpha \leq f$ and b_β not in $\psi(f)$. Then b_β is not an element of $\psi(a_\alpha)$. That is $b_\beta \not\leq \psi(a_\alpha) \Rightarrow g_{a,b} \not\leq \psi(a_\alpha)$. That is $\psi_{a,b}(a_\alpha) \not\leq \psi(a_\alpha)$. Also $\psi_{a,b}(k) = \underline{1}$ for every nonzero L subset k other than a_α. So $\psi_{a,b}(f) \geq \psi(f)$, $\forall f$. That is $\psi_{a,b} \leq \psi$. Thus all infra L-closure operators are of the form $\psi_{a,b}$ for $a, b \in X$ such that $a \neq b$.

3.2 Remark

When L = **I** there is no infra L-closure operator.

Definition 3.3

An L-topology F on X is an ultra L-topology if the only L-topology on X strictly finer than F is the discrete L-topology.

Let X be a nonempty set and L is a finite pseudo complemented chain.
If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geq b_\lambda\}$, then a principal ultra L-topology = $\mathcal{G}(a, \mathcal{U}(b_\lambda), a_\beta) = \mathcal{G}(a_\beta)$, which is the simple extension of $\mathcal{G}$ by a_β i.e. $\mathcal{G}(a_\beta) = \{f \vee (g \wedge a_\beta), f, g \in \mathcal{G}, a_\beta \notin \mathcal{G}\}$, where $a, b \in X$, λ and β are the atom and dual atom in L respectively.

Let X be a nonempty set and L is a Boolean lattice. If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geq b_\lambda\}$, where $a, b \in X$, λ is an atom, then a principal ultra L-topology denoted by $\mathcal{G}(\beta_j)$ = L-topology generated by any (m-1), $\mathcal{G}(a_{\beta i})$ among m, $\mathcal{G}(a_{\beta i})$, i = 1, 2, , ..., m, j = 1, 2, ,..., m, $i \neq j$ if there are m dual atoms $\beta_1, \beta_2, ..., \beta_m$, where $\mathcal{G}(a_{\beta i})$ = simple extension of $\mathcal{G}$ by $a_{\beta i}$. Let X be an infinite set and L is a finite pseudo complemented chain.

If $\mathcal{G} = \mathcal{G}(a, \mathcal{U}) = \{f|f(a) = 0\} \cup \mathcal{U}$ where $\mathcal{U}$ is a non principal ultra L-filter not containing a_λ, $0 \neq \lambda \in L$. Then the non principal ultra L-topology = $\mathcal{G}(a, \mathcal{U}, a_\beta) = \mathcal{G}(a_\beta)$, is the simple extension of $\mathcal{G}$ by a_β, where $a \in X$, β is the dual atom in L.

Let X be an infinite set and L is a Boolean lattice. If $\mathcal{G} = \mathcal{G}(a, \mathcal{U})$, $a \in X$, then a non principal ultra L-topology $\mathcal{G}(\beta_j)$ = L-topology generated by any (m -1), $\mathcal{G}(a_{\beta i})$ among m, $\mathcal{G}(a_{\beta i})$, i = 1, 2, ,..., m, j = 1, 2, ..., m, $i \neq j$, if there are m dual atoms $\beta_1, \beta_2, ,..., \beta_m$ where $(a_{\beta i})$ = simple extension of $\mathcal{G}$ by $a_{\beta i}$. Here m can be assumed infinite value.
If X is a non empty set and L is a diamond lattice $\{0, \alpha, \beta, 1\}$ then the L-closure operator ψ associated with an ultra L-topology $\mathcal{G}(a, \mathcal{U}, a_\beta)$, $a \in X$, is given by

$$\psi(f) = \begin{cases} f & \text{if } f=\underline{0} \text{ or } a_\alpha \leq f \text{ or } cf \in \mathcal{U} \\ f \vee a_\alpha & \text{otherwise} \end{cases}$$

In topological context it is known that [5] a closure operator on X is an ultraclosure operator if and only if it is the closure operator associated with some ultra topology on X and in L-topological context we prove the following theorem.

Theorem 3.4

Let X is a non empty set and L is a diamond lattice $\{0, \alpha, \beta, 1\}$. Then an L-closure operator on X is an ultra L-closure operator if and only if it is the L-closure operator associated with some ultra L-topology on X .

Proof .Let $\mathcal{G}(a, \mathcal{U}, a_\beta)$ be an ultra L-topology on X and ψ be the associated L-closure operator . Let ψ' be an L-closure operator on X strictly larger than ψ. Then there exists an L subset f of X such that $\psi'(f) < \psi(f)$.

But $\psi'(f) \neq (f)$. Then $\psi(f) = f \vee a_\alpha$ and $\psi'(f) = f$, which means that complement of f is open in (X, ψ') and not open in (X, ψ). Also every open set in (X, ψ) is open in (X, ψ') . Thus the associated L- topology of ψ' is strictly larger than the ultra L-topology and hece is discrete. Thus ψ' = D.

Hence the L-closure operator associated with an ultra L-topology is an ultra L-closure operator.

Next to prove that every ultra L-closure operator is the L-closure operator associated with an ultra L-topology.

Let ψ be an L-closure operator on X other than D. It suffices to prove that there exists an L-closure operator associated with an ultra L-topology larger than ψ. Since $\psi \neq D$ there exists an element a of X such that a_α is not open in (X, ψ). Now consider $\mathcal{G} = \{f|f(a) = 0\} \cup \mathcal{U}$ where $\mathcal{U}$ is an ultra L-filter not containing a_λ, $0 \neq \lambda \in L$. Then a_α is not an element of $\mathcal{G}$. Now consider the ultra L-topology $\mathcal{G}(a, \mathcal{U}, a_\alpha)$ = simple extension of $\mathcal{G}$ by a_α. Let ψ' be the L-closure operator associated with it . Then $\psi \leq \psi'$. Otherwise if $\psi' \leq \psi$, then every open set in ψ' is open in ψ'. But a_α is open in ψ' . So it must be open in ψ, which is a contradiction.

Remark 3.3

In a similar way we can prove the above theorem when L is a finite pseudo complemented chain or other Boolean lattice.

Definition 3.4

Let $x \in X$, $\lambda \in L$. An L point x_λ is defined by $x_\lambda(y) = \begin{cases} \lambda \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$ where $0 < \lambda \leq 1$

Definition 3.5

An L-closure operator ψ on X is T_1 if every L point is closed. That is $\psi(x_\lambda) = x_\lambda$, $\forall x \in X, \lambda \in L$.

Definition 3.6 [8]

Let $\psi_1 = \{f|\psi(f) = f\}$. A fuzzy closure space (X, ψ) is called quasi-separated if and only if for any two fuzzy points x_λ and y_γ with $x_\lambda \in C(y_\gamma)$, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leq C(y_\gamma)$ and $y_\gamma \in g \leq C(x_\lambda)$.

Theorem 3.5 [8]

A fuzzy closure space is quasi-separated if and only if every fuzzy point in X is Čech-fuzzy closed.

Result

Let $\psi_1 = \{f \in L^X | \psi(f) = f\}$. An L-closure space (X, ψ) is said to be T_1 if for every pair of distinct L points x_λ and y_γ, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leq C(y_\gamma)$ and $y_\gamma \in g \leq C(x_\lambda)$.

Proof. Necessary part
Suppose that the L-closure operator ψ is T_1. Then by definition $\psi(x_\lambda) = x_\lambda$ Then by theorem 3.5 the L-closure space(X, ψ) is quasi separated. Hence for every pair of distinct L points x_λ and y_γ, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leq C(y_\gamma)$ and $y_\gamma \in g \leq C(x_\lambda)$. Sufficient part Suppose that for every pair of distinct L points x_λ and y_γ, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leq C(y_\gamma)$ and $y_\gamma \in g \leq C(x_\lambda)$. Then by definition (X, ψ) is quasi separated. Then by theorem 3.5, (X, ψ) is a T_1 L-closure space.
Proposition [8] An L-closure space (X, ψ) is T_1 if and only if the associated L topological space (X, F) is T_1

Theorem 3.6.

Infra L-closure operators are less than or equal to any non principal ultra L-closure operator.

Proof. Let $\psi_{a,b}$ be an infra L-closure operator and ψ be a non principal ultra L-closure operator. Since $\psi_{a,b}\, \psi_{a,b}(f) = \underline{1}$

Theorem 3.7

No non principal ultra L-closure operator has a complement.

Proof. Assume the contrary. Let ψ be a non principal ultra L-closure operator with a complement ψ' in the lattice LC(X). Since ψ' is not indiscrete there exists an infra L-closure operator $\psi_{a,b} \leq \psi'$ by the proof of the theorem 3.3. But $\psi_{a,b} \leq \psi$ by theorem 3.6. This contradicts the fact that ψ and ψ' are complements in the lattice LC(X) and hence the proof of the theorem.

Remark 3.4.

The lattice of L-closure operators is not complemented in general.

If L is a diamond lattice, the principal ultra L-closure operator associated with the principal ultra L-topology $\mathfrak{S}(a, (b_\beta), a_\beta)$ is given by $\varphi a,b(f) = f$ if $f=\underline{0}$ or $a_\alpha \leq f$ or $cf \in \mathscr{U}(b_\beta)$
$f \vee a_\alpha$ otherwise

Theorem 3.8.

An infra L-closure operator $\psi_{a,b}$ and $\varphi_{b,a}$ are incomparable if L is a diamond lattice.

Proof. We have $\psi_{a,b}(a_\alpha) = g_{\alpha,b}$, $\Phi_{b,a}(a_\alpha) = a_\alpha \vee b_\beta$. Since α and β are not comparable, $\psi_{a,b}$ and $\varphi_{b,a}$ are not comparable.

Remark 3.5.

In a similar way, we can discuss the above theorem if L is a finite pseudo complemented chain or other Boolean lattices.

4 Conclusion

In this paper we identified the infra L-closure operator and ultra L-closure in LC(X) and established the relation between ultra L-closure topology and ultra L-closure operator if there is a dual atom in the lattice L . Also it is proved that LC(X) is not modular and not complemented in general.

5 Future Scope

The problem of finding whether this lattice is atomic and dually atomic under any condition on the fuzzy lattice L, is not yet solved. Also, the problem of semi-modularity and semi-complementation is not yet analyzed.

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Author Profile

Raji George received the B.Sc. and M.Sc. degrees in Mathematics from St. Peter's College, Kolenchery in 1984 and 1986 respectively, B.Ed. from Govt. Trainings College, Trichur, and M.Phil degree from Madurai Kamaraj University. She is currently Associate Professor in St. Peter's College, Kolenchery.