Certain Subordination Results for a Class of Analytic Functions Defined by the Generalized Derivative Operator

Oyekan, E.A1, Opoola, T.O2

1Department of Mathematics and Statistics, Bowen University, Iwo, Osun State, Nigeria shalonfa@yahoo.com
2Department of Mathematics, University of Ilorin, Ilorin, Nigeria opoolato@unilorin.edu.ng

Abstract: In this paper, we discuss several interesting subordination results for a class of analytic functions defined by using a generalized derivative operator which was introduced and studied by Al-Abbadi and Darus[1]. A number of interesting consequences of some of these results are also discussed, 2000 Mathematics Subject Classification. 30C45, 30C80.

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1. Introduction
Let A be a class of functions $f(z)$ analytic in the unit disk $U = \{ z : |z| < 1 \}$ and normalized by

\begin{equation}
(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\end{equation}

Let $A(s)$ denote the class of functions of the form

\begin{equation}
(1.2) \quad f(z) = z + \sum_{k=1}^{\infty} a_k^s z^k
\end{equation}

and $s \in \mathbb{N} = \{1, 2, \ldots\}$, which are analytic in the open unit disk $U$ on the complex plane $\mathbb{C}$. We further let $c_\alpha(s)$ be the class consisting of functions $g$ which are convex of order $\alpha$ in $U$. i.e. $c_\alpha(s) = \left\{ g \in A(s) : \text{Re} \left( 1 + \frac{zg''(z)}{g'(z)} \right) > \alpha, \quad z \in U \right\}$

for $0 \leq \alpha < 1$

Al-Abbadi and Darus[1], introduce the class $\mathcal{A}(m, n, \lambda, \alpha)$ consisting of functions $f(z)$ satisfying

\begin{equation}
(1.3) \quad \text{Re} \left( \frac{z(\mu_{m, n, \lambda, \alpha} f(z))}{\mu_{m, n, \lambda, \alpha} f(z)} \right) > \alpha, \quad (z \in U),
\end{equation}

where for $f \in A = A(1)$, the generalized derivative operator

\begin{equation}
(1.4) \quad \mu_{m, n, \lambda, \alpha} f(z) = z + \sum_{k=1}^{\infty} \frac{(1 + \lambda_k (k-1))^{m-1}}{(1 + \lambda_k (k-1))^m} C(n, k) a_k z^k,
\end{equation}

\begin{equation}
(2.1) \quad \sum_{k=1}^{\infty} \frac{(k - \alpha)(1 + \lambda_k (k-1))^{m-1}}{(1 + \lambda_k (k-1))^m} C(n, k) a_k \leq 1 - \alpha
\end{equation}

(s \in \mathbb{N} = \{1, 2, \ldots\}, n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$ and $C(n, k) = \binom{n+k-1}{n}$), then $f(z) \in \mathcal{A}(m, n, \lambda, \alpha)$.

for $0 \leq \alpha < 1, \quad \lambda_1 \geq \lambda_2 \geq 0$.

Let us denote by $\mathcal{A}(m, n, \lambda, \alpha)$ the class of functions $f(z)$ defined by (1.2) whose coefficients satisfies the condition (2.1).

Definition 1. (Hadamard product or convolution)
If $f, g \in A(s)$, where $f(z)$ is as defined in (1.2) and $g(z)$ is given by

\begin{equation}
(2.2) \quad g(z) = z + \sum_{k=1}^{\infty} b_k z^k
\end{equation}

the Hadamard product (or convolution) $f \ast g$ of $f(z)$ and $g(z)$ is defined by

Note that the series expansion (1.2) is equivalent to (1.1) with omitted coefficient as follows:

\begin{align*}
s = 1 : & \text{no coefficient is omitted} \\
s = 2 : & a_2 = 0 \text{ omitted} \\
s = 3 : & a_2 = a_3 = 0 \text{ omitted and so on.} \\
\text{Consequently, } A(1) = A \text{ and } A(s) \subseteq A(1).
\end{align*}
Definition 2. (Subordination Principle.)

Let \( f(z) \) and \( g(z) \) be analytic in the unit disk \( U \). Then \( f(z) \) is said to be subordinate to \( g(z) \) in \( U \) and we write

\[
f(z) \prec g(z), \quad z \in U,
\]

if there exist a Schwarz function \( w(z) \), analytic in \( U \) with \( w(0) = 0, \quad |w(z)| < 1 \) such that

\[
(2.3) \quad f(z) = w(g(z)), \quad z \in U
\]

In particular, if the function \( g(z) \) is univalent in \( U \), then \( f(z) \) is subordinate to \( g(z) \) if

\[
(2.4) \quad f(0) = g(0), \quad f(u) \subset g(u)
\]

Definition 3. (Subordinating factor sequence)

A sequence \( \{C_k\}_{k=1}^{\infty} \) of complex number is said to be a subordinating factor sequence if whenever \( f(z) \) of the form

\[
(3.1) \quad f(z) = \sum_{k=1}^{\infty} \alpha_k z^k
\]

is analytic, univalent and convex in \( U \), the subordination is given by

\[
\sum_{k=1}^{\infty} \alpha_k c_k z^k \prec f(z), \quad z \in U, \quad \alpha_1 = 1.
\]

We have the following theorem:

Theorem A: (Wilf[2])

The sequence \( \{C_k\}_{k=1}^{\infty} \) is a subordinating factor sequence if and only if

\[
(2.5) \quad \Re \left\{1 + 2 \sum_{k=1}^{\infty} a_k z^k \right\} > 0, \quad (z \in U).
\]

The object of this present work is to derive subordination results for functions in the classes \( \varphi_{a_1, a_2}^{n, m}(s, \alpha) \), and to also consider some interesting consequences of our result.

### 3. Main Result

Subordination result for the class

**Theorem 3.1.** Let \( f(z) \in \varphi_{a_1, a_2}^{n, m}(s, \alpha) \subset \varphi_{a_1, a_2}^{n, m}(s, \alpha) \)

Where

\[
\varphi_{a_1, a_2}^{n, m}(s, \alpha) = \{ f \in A(s): \sum_{k=1}^{\infty} \frac{(k - \alpha)(1 + \lambda_1(k - 1))^{m-1}}{(1 + \lambda_2(k - 1))^n} C(n, k) \mid a_k \leq 1 - \alpha \}
\]

Then

(a) \[
(3.1) \quad \Re \left\{1 + \frac{(2 - \alpha)(n+1)(1 + \lambda_1)}{(1 - \alpha)(1 + \lambda_2)^n + (2 - \alpha)(n+1)(1 + \lambda_1)^{m-1}} \right\} \geq 1,
\]

(b) \[
(3.2) \quad \Re \left\{1 + \frac{(2 - \alpha)(n+1)(1 + \lambda_1)}{(1 - \alpha)(1 + \lambda_2)^n + (2 - \alpha)(n+1)(1 + \lambda_1)^{m-1}} \right\} \geq 1 - \frac{1}{\lambda_2(k - 1)^m} \times \sum_{k=1}^{\infty} \frac{(k - \alpha)(1 + \lambda_1(k - 1))^{m-1}}{C(n, k) \mid a_k \mid r^k}
\]

(proof)

(b) \[
(3.2) \quad \Re \left\{1 + \frac{(2 - \alpha)(n+1)(1 + \lambda_1)}{(1 - \alpha)(1 + \lambda_2)^n + (2 - \alpha)(n+1)(1 + \lambda_1)^{m-1}} \right\} \geq 1 - \frac{1}{\lambda_2(k - 1)^m} \times \sum_{k=1}^{\infty} \frac{(k - \alpha)(1 + \lambda_1(k - 1))^{m-1}}{C(n, k) \mid a_k \mid r^k}
\]

(because \( (k - \alpha)(1 + \lambda_1(k - 1))^{m-1} C(n, k) \) is an increasing function of \( k \).)

\[
> \left\{1 - \frac{(2 - \alpha)(n+1)(1 + \lambda_1)}{(1 - \alpha)(1 + \lambda_2)^n + (2 - \alpha)(n+1)(1 + \lambda_1)^{m-1}} \right\} \geq 1 - \frac{1}{\lambda_2(k - 1)^m} \times \sum_{k=1}^{\infty} \frac{(k - \alpha)(1 + \lambda_1(k - 1))^{m-1}}{C(n, k) \mid a_k \mid r^k}
\]
Now, by using the fact that $s, \alpha > 0$, we need to show that $|zf(z)| < \frac{1}{2}$, where $|z| = 1$.

This implies that,
\[
\frac{z(2-\alpha)(n+1)(1+\lambda_2)^{m-1}-(1-\alpha)(1+\lambda_2)^m z^2}{2[(1-\alpha)(1+\lambda_2)^m+(2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} \leq \frac{1}{2}
\]

Hence,
\[
\frac{z(2-\alpha)(n+1)(1+\lambda_2)^{m-1}-(1-\alpha)(1+\lambda_2)^m z^2}{2[(1-\alpha)(1+\lambda_2)^m+(2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} f_0(z)
\]
\[
= -\frac{1}{2} (z \in U).
\]

Next we show that
\[
\text{Re}(f(z)) > -\frac{[1-\alpha](1+\lambda_2)^m+(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2(\alpha)(n+1)(1+\lambda_2)^{m-1}},
\]
\[
(z \in U).
\]

Now taking
\[
g(z) = \frac{z}{1-z} \in C_\alpha(s)
\]
in (3.1) we have the following:
\[
(2-\alpha)(n+1)(1+\lambda_1)^{m-1} \frac{f(z)}{2[(1-\alpha)(1+\lambda_2)^m+(2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} < \frac{z}{1-z}
\]

Therefore,
\[
(3.14)
\]

Since
\[
(3.15)
\]

which implies that
\[
(3.16)
\]

Hence, we have
\[
\text{Re}(f(z)) > -\frac{[1-\alpha](1+\lambda_2)^m+(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2(\alpha)(n+1)(1+\lambda_2)^{m-1}},
\]
\[
(z \in U).
\]

which is (3.2) require to complete the proof of theorem 3.1.

**SOME APPLICATION OF THE THEOREM 3.1**

Taking $n = 1$ in theorem 3.1, we obtain the following:

**Corollary 1.** If the function $f(z)$ defined by (1.2) satisfies
\[ (4.1) \sum_{k=0}^{\infty} \frac{(2k-2\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m}} |a_k| \leq \rho \]

\( s \in \mathbb{N} = \{1,2,3,\ldots\}; \ m \in \mathbb{N} \cup \{0\}; \ 0 \leq \alpha < 1; \ \lambda_2 \geq \lambda_1 \geq 0; \ \rho \geq 0; \ \text{then for every} \ g \in C_{\alpha}(s), \ \text{one has} \]

\[ (4.2) \]
\[ \frac{(4-2\alpha)(1+\lambda_1)^{m-1}}{2[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^{m}]} (f*g)(z) < g(z) \]

\( z \in \mathbb{U}; \ 0 \leq \alpha < 1; \ g \in C_{\alpha}(s); \ \lambda_2 \geq \lambda_1 \geq 0; \ m \in \mathbb{N} \cup \{0\}; \ \rho \geq 0 \)

\[ (4.3) \]
\[ \text{Re}(f(z)) > -\frac{[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^{m}]}{(4-2\alpha)(1+\lambda_1)^{m-1}}, \]

\( z \in \mathbb{U} \).

The contact factor

\[ (4-2\alpha)(1+\lambda_1)^{m-1} \]

\[ 2[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^{m}] \]

cannot be replaced by any larger one.

Remark 1: When \( \alpha = 0; \ \lambda_1 = \lambda_2 = 0; \ \rho = 1; \ m = \in \) in corollary 1, we have the result obtained by Selvaraj and Karthikeyan[3].

Taking \( \lambda_1 = \lambda_2 = 0; \) and \( m = 0 \) in Theorem 3.1; we obtain the following:

**Corollary 2.** If the function \( f(z) \) defined by (1.2) satisfies

\[ (4.4) \]
\[ \sum_{k=0}^{\infty} (k-\alpha)C(n,k) |a_k| \leq \rho, \]

\( (\rho > 0, n \in \mathbb{N}; 0 \leq \alpha < 1); \ \text{then for every function} \ g \in C_{\alpha}(s), \ \text{one has} \]

\[ (4.5) \]
\[ \frac{4-2\alpha}{2(4-2\alpha+\rho)} (f*g)(z) < g(z) \]

\( z \in \mathbb{U}; \ 0 \leq \alpha < 1; \ \rho > 0 \)

and,

\[ (4.6) \]
\[ \text{Re}(f(z)) > \frac{(4-2\alpha+\rho)}{4-2\alpha} \]

\( z \in \mathbb{U} \).

The constant factor

\[ (4-2\alpha) + \rho \]

\[ 4-2\alpha \]

cannot be replaced by any larger one.

Remark 2: When \( \alpha = \rho = \frac{1}{2} \) in corollary 2, we have the result obtained by Aouf et al [4].

Taking \( \lambda_1 = \lambda_2 = 1; \ \alpha = 0; \ m = 2, \) in Theorem 3.1; we obtain the following:

**Corollary 3.** If the function \( f(z) \) defined by (1.2) satisfies

\[ (4.7) \]
\[ \sum_{k=0}^{\infty} C(n,k) |a_k| \leq \rho, \]

\( (\rho > 0, n \in \mathbb{N}); \ \text{then for every function} \ g \in C_{\alpha}(s), \ \text{one has} \]

\[ (4.8) \]
\[ \frac{n+1}{2(n+1+\rho)} (f*g)(z) < g(z) \]

\( z \in \mathbb{U}; \ 0 \leq \alpha < 1; \ \rho > 0 \)

and,

\[ (4.9) \]
\[ \text{Re}(f(z)) > \frac{1+\rho}{n+1} \]

\( z \in \mathbb{U} \).

The constant factor

\[ \frac{n+1}{2(n+1+\rho)} \]

cannot be replaced by any larger one.

 Remark 3: When \( \rho = m(m > 0), \) in the corollary 3, we have the result obtained by Attiya et al [5].

Taking \( \alpha = 0 \) in Theorem 3.1; we obtain the following:

**Corollary 4.** If the function \( f(z) \) defined by (1.2) satisfies

\[ (4.10) \]
\[ \sum_{k=0}^{\infty} \frac{k(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m}} C(n,k) |a_k| \leq 1, \]

\( (s,n \in \mathbb{N}; m \in \mathbb{N} \cup \{0\}; \lambda_2 \geq \lambda_1 \geq 0) \)

and,

\[ (4.11) \]
\[ \frac{(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^{m}}{(4-2\alpha)(1+\lambda_1)^{m-1}}, \]

\( z \in \mathbb{U} \).

The constant factor

\[ \frac{(n+1)(1+\lambda_1)^{m-1}}{(1+\lambda_2)^{m} + 2(n+1)(1+\lambda_1)^{m-1}} \]

cannot be replaced by any larger one.

Remark 4: When \( m = n = 1, \lambda_1 = 0, \lambda_2 = 1 \) in the corollary 4, we have the result obtained by Sukhjit[6] and Selvaraj et al [3].

**References**


