

Certain Subordination Results for a Class of Analytic Functions Defined by the Generalized Derivative Operator

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Abstract: In this paper, we discuss several interesting subordination results for a class of analytic functions defined by using a generalized derivative operator which was introduced and studied by Al-Abbadi and Darus[1]. A number of interesting consequences of some of these results are also discussed, 2000 Mathematics Subject Classification. 30C45, 30C80.

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1. Introduction

Let A be a class of functions $f(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized by

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

Let $A(s)$ denote the class of functions of the form

$$(1.2) \quad f(z) = z + \sum_{k=s+1}^{\infty} a_k z^k$$

and $s \in \mathbb{N} = \{1, 2, \dots\}$, which are analytic in the open unit disk U on the complex plane \mathbb{C} . We further let $c_\alpha(s)$ be the class consisting of functions g which are convex of order α in U . i.e. $c_\alpha(s) =$

$$\left\{ g \in A(s) : \operatorname{Re} \left(1 + \frac{z g''(z)}{g'(z)} \right) > \alpha, \quad z \in U \right\}$$

for $0 \leq \alpha < 1$

Al-Abbadi and Darus[1], introduce the class $\wp_{\lambda_1, \lambda_2}^{n,m}(s, \alpha)$ consisting of functions $f(z)$ satisfying

$$(1.3) \quad \operatorname{Re} \left\{ \frac{z(\mu_{\lambda_1, \lambda_2}^{n,m} f(z))}{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)} \right\} > \alpha, \quad (z \in U),$$

Where for $f \in A = A(1)$, the generalized derivative operator

$\mu_{\lambda_1, \lambda_2}^{n,m} : A \rightarrow A$ is defined by

$$(1.4) \quad \mu_{\lambda_1, \lambda_2}^{n,m} f(z) = z + \sum_{k=s+1}^{\infty} \frac{(1 + \lambda_1(k-1))^{m-1}}{(1 + \lambda_2(k-1))^m} C(n, k) a_k z^k,$$

$0 \leq \alpha < 1, \lambda_2 \geq \lambda_1 \geq 0, (n, m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}),$

$$C(n, m) = \binom{n+k-1}{n} = (n+1)_{k-1} / (1)_{k-1},$$

Note that the series expansion (1.2) is equivalent to (1.1) with omitted coefficient as follows:

$s = 1$: no coefficient is omitted

$s = 2$: $a_2 = 0$ omitted

$s = 3$: $a_2 = a_3 = 0$ omitted and so on.

Consequently, $A(1) = A$ and $A(s) \subseteq A(1)$.

2. Definitions and preliminaries

Theorem 2.1. [1] If $f(z) \in A(s)$ given by (1.2), satisfies the coefficient inequality:

$$(2.1) \quad \sum_{k=s+1}^{\infty} \frac{(k-\alpha)(1 + \lambda_1(k-1))^{m-1}}{(1 + \lambda_2(k-1))^m} C(n, k) |a_k| \leq 1 - \alpha$$

($s \in \mathbb{N} = \{1, 2, 3, \dots\}$), $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $C(n, k) = \binom{n+k-1}{n}$, then $f(z) \in \wp_{\lambda_1, \lambda_2}^{n,m}(s, \alpha)$.

for $0 \leq \alpha < 1, \lambda_1 \geq \lambda_2 \geq 0$.

Let us denote by $\wp_{\lambda_1, \lambda_2}^{n,m}(s, \alpha)$ the class of functions $f(z)$ defined by (1.2) whose coefficients satisfies the condition (2.1).

Definition 1. (Hadamard product or convolution)

If $f, g \in A(s)$, where $f(z)$ is as defined in (1.2) and $g(z)$ is given by

$$g(z) = z + \sum_{k=s+1}^{\infty} b_k z^k$$

the Hadamard product (or convolution) $f * g$ of $f(z)$ and $g(z)$ is defined by

$$(2.2) \quad (f * g)(z) = z + \sum_{k=s+1}^{\infty} a_k b_k z^k = (g * f)(z)$$

Definition 2. (Subordination Principle.)

Let $f(z)$ and $g(z)$ be analytic in the unit disk U . Then $f(z)$ is said to be subordinate to $g(z)$ in U and we write

$$f(z) \prec g(z), \quad z \in U,$$

if there exist a Schwarz function $w(z)$, analytic in U with $w(0) = 0, |w(z)| < 1$ such that

$$(2.3) \quad f(z) = g(w(z)), \quad z \in U$$

In particular, if the function $g(z)$ is univalent in U , then $f(z)$ is subordinate to $g(z)$ if

$$(2.4) \quad f(0) = g(0), \quad f(U) \subset g(U)$$

Definition 3. (Subordinating factor sequence)

A sequence $\{C_k\}_{k=1}^\infty$ of complex number is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U , the subordination is given by

$$\sum_{k=1}^\infty a_k c_k z^k \prec f(z), \quad z \in U, \quad a_1 = 1.$$

We have the following theorem:

Theorem A: (Wilf[2])

The sequence $\{C_k\}_{k=1}^\infty$ is a subordinating factor sequence if and only if

$$(2.5) \quad \operatorname{Re} \left\{ 1 + 2 \sum_{k=1}^\infty c_k z^k \right\} > 0, \quad (z \in U).$$

The object of this present work is to derive subordination results for functions in the classes $\mathcal{P}_{\lambda_1, \lambda_2}^{n, m^*}(s, \alpha)$, and to also consider some interesting consequences of our result.

3. Main Result

Subordination result for the class

Theorem 3.1. Let $f(z) \in \mathcal{P}_{\lambda_1, \lambda_2}^{n, m^*}(s, \alpha) \subset \mathcal{P}_{\lambda_1, \lambda_2}^{n, m}(s, \alpha)$

Where

$$\mathcal{P}_{\lambda_1, \lambda_2}^{n, m^*}(s, \alpha)$$

$$= \{f \in A(s):$$

$$\sum_{k=s+1}^\infty \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} C(n, k) \mid a_k \mid \leq 1-\alpha \}$$

Then

(a)

(3.1)

$$\frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} (f * g)(z) \prec g(z)$$

$$(z \in U, \quad 0 \leq \alpha < 1; \quad \lambda_2 \geq \lambda_1 \geq 0; \quad n \in \mathbb{N}; m \in \mathbb{N}_0),$$

for every $g \in C_\alpha(s)$.

And the constant factor

$$\frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]}$$

is best possible.

(b)

(3.2)

$$\operatorname{Re}(f(z)) > -$$

$$\frac{[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]}{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}, \quad (z \in U).$$

PROOF OF THEOREM 3.1

Let $f(z)$ defined by (2.1) be any member of the class

$$\mathcal{P}_{\lambda_1, \lambda_2}^{n, m^*}(s, \alpha)$$

$$g(z) = z + \sum_{k=s+1}^\infty b_k z^k \in C_\alpha(s).$$

Then

$$\frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} (f * g)(z) =$$

$$\frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} \left(z + \sum_{k=s+1}^\infty a_k b_k z^k \right)$$

Thus, by definition (2.3) the subordination (3.1) will hold if the sequence,

(3.4)

$$\left\{ \frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} a_k \right\}_{k=1}^\infty$$

is a subordinating factor sequence with $a_1 = 1$.

Therefore by Theorem A, it is sufficient to show that

(3.5)

$$\operatorname{Re} \left\{ 1 + \sum_{k=1}^\infty \frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{2[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} a_k z^k \right\} > 0; \quad (z \in U)$$

Now,

$$\operatorname{Re} \left\{ 1 + \sum_{k=1}^\infty \frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]} a_k z^k \right\} =$$

$$\operatorname{Re} \left\{ 1 + \frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} z + \right.$$

$$\left. \frac{1}{(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \times \right.$$

$$(3.6) \quad \left. \sum_{k=s+1}^\infty (2-\alpha)(n+1)(1+\lambda_1)^{m-1} a_k z^k \right\},$$

$$> \left\{ 1 - \frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} r \right.$$

$$\left. - \frac{1}{(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \right.$$

$$\left. \times \sum_{k=s+1}^\infty \frac{(k-\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} C(n, k) \mid a_k \mid r^k \right\}$$

(because $(k-\alpha)(1+\lambda_1(k-1))^{m-1} C(n, k)$ is an increasing function of k .)

$$> \left\{ 1 - \frac{(2-\alpha)(n+1)(1+\lambda_1)^{m-1}}{(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} r \right.$$

$$1 - \frac{(1 + \lambda_2)^m (1 - \alpha)}{(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}} r$$

$$1 - \left\{ 1 - \frac{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} + (1 + \lambda_2)^m (1 - \alpha)}{(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}} \right\} r$$

= 1 - r > 0; (|z| = r > 1).

Thus, (3.5) holds true in U and consequently proves (3.1).

To show that the constant

$$\frac{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

Is best possible, we need to show that

(3.8)

$$\text{Min} \left\{ \text{Re} \frac{z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} f_0 \right\}$$

$$= -\frac{1}{2} (z \in U).$$

To do this, we consider the function $f_0(z)$ defined by

$$f_0(z) = \frac{z(2 - \alpha)(1 + \lambda_1)^{m-1}(n + 1) - (1 - \alpha)(1 + \lambda_2)^m z^2}{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}}$$

which is a member of the class $\mathcal{P}_{\lambda_1, \lambda_2}^{n, m^*}(s, \alpha)$. Thus from relation (3.1), we obtain

(3.9)

$$\frac{z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} < \frac{z}{1 - z}$$

Now, by using the fact that

(3.10) $|\text{Re } z| \leq |z|$

we have that

$$\left| \text{Re} \frac{z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} \right|$$

$$\leq \frac{|z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2|}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

$$= \frac{|z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2|}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

$$= \frac{|z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z|}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

$$\leq \frac{|z| |(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z|}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

$$\leq \frac{|[(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z]|}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} \quad (3.11)$$

$$\leq \frac{|[(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} + (1 - \alpha)(1 + \lambda_2)^m z]|}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

$$\leq \frac{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} + (1 - \alpha)(1 + \lambda_2)^m}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}$$

$$= \frac{1}{2}, \quad (|z| = 1).$$

This implies that,

$$\left| \text{Re} \frac{z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} \right| \leq \frac{1}{2}$$

i.e.,

$$-\frac{1}{2} \leq \text{Re} \frac{z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} \leq \frac{1}{2} \quad (3.12)$$

Hence,

$$\text{Min} \left\{ \text{Re} \frac{z(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1} - (1 - \alpha)(1 + \lambda_2)^m z^2}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} f_0(z) \right\}$$

$$= -\frac{1}{2} (z \in U).$$

Next we show that

$$\text{Re}(f(z)) > \frac{[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}},$$

(z ∈ U)

Now taking

$$g(z) = \frac{z}{1 - z} \in C_\alpha(s)$$

in (3.1) we have the following:

(3.13)

$$\frac{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} f(z) < \frac{z}{1 - z}$$

Therefore,

(3.14)

$$\text{Re} \left\{ \frac{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} f(z) \right\} > -\frac{1}{2}$$

Since

(3.15) $\text{Re} \left(\frac{z}{1 - z} \right) > -\frac{1}{2}, \quad |z| < r$

which implies that

(3.16)

$$\frac{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}}{2[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]} \text{Re}(f(z)) > -\frac{1}{2}$$

Hence, we have

$$\text{Re}(f(z)) > -\frac{[(1 - \alpha)(1 + \lambda_2)^m + (2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}]}{(2 - \alpha)(n + 1)(1 + \lambda_1)^{m-1}},$$

(z ∈ U).

which is (3.2) require to complete the proof of theorem 3.1 .

SOME APPLICATION OF THE THEOREM 3.1

Taking n = 1 in theorem 3.1; we obtain the following:

Corollary 1. If the function $f(z)$ defined by (1.2) satisfies

$$(4.1) \quad \sum_{k=s+1}^{\infty} \frac{(2k-2\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} |a_k| \leq \rho$$

($s \in \mathbb{N} = \{1, 2, 3, \dots\}$); $m \in \mathbb{N} \cup \{0\}$; $0 \leq \alpha < 1$;
 $\lambda_2 \geq \lambda_1 \geq 0$; $\rho \geq 0$; then for every $g \in C_\alpha(s)$,
 one has

$$(4.2) \quad \frac{(4-2\alpha)(1+\lambda_1)^{m-1}}{2[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^m]} (f * g)(z) \prec g(z)$$

($z \in U$; $0 \leq \alpha < 1$; $g \in C_\alpha(s)$; $\lambda_2 \geq \lambda_1 \geq 0$;
 $m \in \mathbb{N} \cup \{0\}$; $\rho \geq 0$)

and,
 (4.3)

$$\operatorname{Re}(f(z)) > -\frac{[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^m]}{(4-2\alpha)(1+\lambda_1)^{m-1}},$$

($z \in U$).

The contact factor

$$\frac{(4-2\alpha)(1+\lambda_1)^{m-1}}{2[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^m]}$$

cannot be replaced by any larger one.

Remark 1: When $\alpha = 0$; $\lambda_1 = \lambda_2 = 0$; $\rho = 1$; $m = o$ in corollary 1, we have the result obtained by Selvaraj and Karthikeyan[3].

Taking $\lambda_1 = \lambda_2 = 0$; and $m = 0$ in Theorem 3.1; we obtain the following:

Corollary 2. If the function $f(z)$ defined by (1.2) satisfies

$$(4.4) \quad \sum_{k=s+1}^{\infty} (k-\alpha)C(n,k) |a_k| \leq \rho,$$

($\rho > 0, n \in \mathbb{N}; 0 \leq \alpha < 1$); then for every function,
 $g \in C_\alpha(s)$, one has

$$(4.5) \quad \frac{4-2\alpha}{2(4-2\alpha+\rho)} (f * g)(z) \prec g(z)$$

($z \in U$; $0 \leq \alpha < 1$; $\rho > 0$)

and,

$$(4.6) \quad \operatorname{Re}(f(z)) > -\left(\frac{(4-2\alpha)+\rho}{4-2\alpha}\right), \quad (z \in U).$$

The constant factor

$$\frac{(4-2\alpha)+\rho}{4-2\alpha}$$

cannot be replaced by any larger one.

Remark 2: When $\alpha = \rho = \frac{1}{2}$ in corollary 2, we have the result obtained by Aouf et al [4].

Taking $\lambda_1 = \lambda_2 = 1, \alpha = 0, m = 2$, in Theorem 3.1; we obtain the following:

Corollary 3. If the function $f(z)$ defined by (1.2) satisfies

$$(4.7) \quad \sum_{k=s+1}^{\infty} C(n,k) |a_k| \leq \rho,$$

($\rho > 0, n \in \mathbb{N}$;) then for every function, $g \in C_\alpha(s)$, one has

$$(4.8) \quad \frac{n+1}{2(n+1+\rho)} (f * g)(z) \prec g(z)$$

($z \in U$; $0 \leq \alpha < 1$; $\rho > 0$)

and,

$$(4.9) \quad \operatorname{Re}(f(z)) > -\left(1 + \frac{\rho}{n+1}\right), \quad (z \in U).$$

The constant factor

$$\frac{n+1}{2(n+1+\rho)}$$

cannot be replaced by any larger one.

Remark 3: When $\rho = m(m > 0)$, in the corollary 3, we have the result obtained by Attiya et al [5].

Taking $\alpha = 0$ in Theorem 3.1; we obtain the following:

Corollary 4. If the function $f(z)$ defined by (1.2) satisfies

$$(4.10) \quad \sum_{k=s+1}^{\infty} \frac{k(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} C(n,k) |a_k| \leq 1,$$

($s, n \in \mathbb{N}; m \in \mathbb{N} \cup \{0\}; \lambda_2 \geq \lambda_1 \geq 0$)

and,

$$(4.12) \quad \operatorname{Re}(f(z)) > -\frac{[(1+\lambda_2)^m + 2(n+1)(1+\lambda_1)^{m-1}]}{(n+1)(1+\lambda_1)^{m-1}},$$

($z \in U$).

The constant factor

$$\frac{(n+1)(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m + 2(n+1)(1+\lambda_1)^{m-1}}$$

cannot be replaced by any larger one.

Remark 4: When $m = n = 1, \lambda_1 = 0, \lambda_2 = 1$ in the corollary 4, we have the result obtained by Sukhjit[6] and Selvara et al [3].

References

- [1] M. H. Al-Abbad and Maslina Darus, "On Subclass of Analytic Univalent Functions Associated with Negative Coefficients," International Journal of Mathematics and Mathematical Sciences, Vol. 2010, Article ID 343580, pp. 1-11, (2010).
- [2] H.S. Wilt "subordinating factors sequences for some convex maps of unit circle", proceeding of the American Mathematical society, Vol.12, pp 689-693, (1961).
- [3] C. Selvaraj and K.R. Karthikeyan, Certain Subordination Result for A Class of Analytic Functions Defined By The Generalized Integral Operator, International Journal of Computational and Mathematical sciences, pp.166-169, 2008.
- [4] M.K. Aouf, A. Shamandy, A.O Mostafa and F. El-Emam, subordination result associated with β .

Uniformly Convex and Starlike Functions, Proc. Pakistan Acad. Sci. 46(2): pp. 97-101, 2009.

- [5] A.A Attiya, Nak Eun Cho and M.A Kutbi, Subordination Properties for certain analytic functions, International Journal of Mathematics and Mathematical Sciences, Vol.2008, article ID 63825, pp. 1-8, 2008.
- [6] Sukhjit Singh, "A Subordination Theorem For Spirallike Function", International Journal of Mathematics and Mathematical sci., Vol . 24, N07, pp. 433-435, 2000