Certain Subordination Results for a Class of Analytic Functions Defined by the Generalized Derivative Operator

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Abstract: In this paper, we discuss several interesting subordination results for a class of analytic functions defined by using a generalized derivative operator which was introduced and studied by Al-Abbadi and Darus[1]. A number of interesting consequences of some of these results are also discussed, 2000 Mathematics Subject Classification. 30C45, 30C80.

Keywords: Subordination, Salagean Differential Operator, Subordinating factor sequence, Hadamard product, Convolution

1. Introduction

Let A be a class of functions f(z) analytic in the unit disk U = {z : |z| < 1} and normalized by

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]

(1.1)

Let A(s) denote the class of functions of the form

\[ f(z) = z + \sum_{k=1}^{\infty} a_k s^k z^k \]

(1.2)

and s ∈ N = {1, 2, ...}, which are analytic in the open unit disk U on the complex plane ℂ. We further let c_α(s) be the class consisting of functions g which are convex of order α in U, i.e.

\[ c_\alpha(s) = \{ g(z) : \Re \left( 1 + \frac{z^2 g''(z)}{g'(z)} \right) > \alpha, \quad z \in U \} \]

for 0 ≤ α < 1

Al-Abbadi and Darus[1], introduce the class \( m \), \( n \), \( \lambda_1 \), \( \lambda_2 \) \( \alpha \) consisting of functions f(z) satisfying

\[ \Re \left( \frac{z \mu_{m,n}^{\lambda_1,\lambda_2} f(z)}{\mu_{m,n}^{\lambda_1,\lambda_2} f(z)} \right) > \alpha, \quad z \in U, \]

(1.3)

Where for f ∈ A = A(1), the generalized derivative operator \( \mu_{\lambda_1,\lambda_2}^{m,n} f(z) \) is defined by

\[ \mu_{\lambda_1,\lambda_2}^{m,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(1 + \lambda_1 (k-1))^{m-1}}{(1 + \lambda_2 (k-1))^m} C(n,k) a_k z^k, \]

\[ o \leq \alpha < 1, \quad \lambda_2 \geq \lambda_1 \geq 0, \quad (n, m) \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \]

\[ C(n,m) = \binom{n+k-1}{n} (n+1)_{k-1} / (1)_{k-1}, \]

Note that the series expansion (1.2) is equivalent to (1.1) with omitted coefficient as follows:

s = 1 : no coefficient is omitted
s = 2 : a_2 = 0 omitted
s = 3 : a_2 = a_3 = 0 omitted and so on.

Consequently, A(1) = A and A(s) ⊆ A(1).

2. Definitions and preliminaries

Theorem 2.1. [1] If f(z) ∈ A(s) given by (1.2), satisfies the coefficient inequality:

\[ \sum_{k=1}^{\infty} \frac{(k-\alpha)(1+\lambda_1 (k-1))^{m-1}}{(1+\lambda_2 (k-1))^m} C(n,k) a_k \leq 1 - \alpha \]

(2.1)

(s ∈ N = {1, 2, 3, ...}, n ∈ N, m ∈ N \[ \cup \{0\} \] and C(n,k) = \( \binom{n+k-1}{n} \)), then f(z) ∈ \( \mu_{\lambda_1,\lambda_2}^{m,n} (s, \alpha) \).

for 0 ≤ α < 1, \( \lambda_1 \geq \lambda_2 \geq 0 \).

Let us denote by \( \mu_{\lambda_1,\lambda_2}^{m,n} (s, \alpha) \) the class of functions f(z) defined by (1.2) whose coefficients satisfies the condition (2.1).

Definition 1. (Hadamard product or convolution)

If f, g ∈ A(s), where f(z) is as defined in (1.2) and g(z) is given by

\[ g(z) = z + \sum_{k=1}^{\infty} b_k z^k \]

the Hadamard product (or convolution) \( f \ast g \) of f(z) and g(z) is defined by

\[ (f \ast g)(z) = z + \sum_{k=1}^{\infty} a_k b_k z^k = (g \ast f)(z) \]

(2.2)
Definition 2. (Subordination Principle.)

Let \( f(z) \) and \( g(z) \) be analytic in the unit disk \( U \). Then \( f(z) \) is said to be subordinate to \( g(z) \) in \( U \) and we write
\[
\text{if there exist a Schwarz function } w(z), \text{ analytic in } U \text{ with } w(0) = 0, \ |w(z)| < 1 \text{ such that}
\]
\[
f(z) = g(w(z)), \ z \in U,
\]
In particular, if the function \( g(z) \) is univalent in \( U \), then \( f(z) \) is subordinate to \( g(z) \) if
\[
f(0) = g(0), \ f(u) \subset g(u)
\]
Definition 3. (Subordinating factor sequence)

A sequence \( \{C_k\}_{k=1}^{\infty} \) of complex number is said to be a subordinating factor sequence if whenever \( f(z) \) of the form
\[
f(z) = \sum_{k=1}^{\infty} a_k z^k \in A(s)
\]
we have the following theorem:

Theorem A: (Wilf[2])

The sequence \( \{C_k\}_{k=1}^{\infty} \) is a subordinating factor sequence if
\[
\sum_{k=1}^{\infty} a_k C_k z^k < f(z), \ z \in U, \ a_1 = 1.
\]

We have the following theorem:

Main Result

Subordination result for the class

\[
\text{Main Result}
\]

Subordination result for the class

**Theorem 3.1.** Let \( f(z) \in \varphi_{\beta, \gamma}^{n, m}(s, \alpha) \subset \varphi_{\beta, \gamma}^{n, m}(s, \alpha) \)

Where
\[
\varphi_{\beta, \gamma}^{n, m}(s, \alpha) = \{ f \in A(s): \sum_{k=1}^{\infty} \frac{(k-\alpha)(1+\lambda)(k-1)^{m-1}}{(1+\lambda)(k-1)^n} C(n,k) |a_k| \leq 1 - \alpha \}
\]
Then
\[
(\text{a})
\]
\[
(\text{b})
\]
\[
\text{Re}(f(z)) > \frac{[(1-\alpha)(1+\lambda)]^m + 2(1-\alpha)(n+1)(1+\lambda)^{m-1}}{(1-\alpha)(n+1)(1+\lambda)^{m-1}}, \ (z \in U).
\]

**PROOF OF THEOREM 3.1**

Let \( f(z) \) defined by (2.1) be any member of the class
\[
\varphi_{\beta, \gamma}^{n, m}(s, \alpha)
\]
and suppose that
\[
g(z) = z + \sum_{k=1}^{\infty} b_k z^k \in C_\alpha(s).
\]

Then
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we have that

\[
1 - 1 - r > 0; \ (|z| = r > 1).
\]

Thus, (3.5) holds true in \( U \) and consequently proves (3.1).

To show that the constant \( \frac{1}{2} \), we need to show that

\[
\left| \frac{z(2-\alpha)(n+1)(1+\lambda_1)^{m-1} - (1-\alpha)(1+\lambda_2)^m z^2}{2(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \right| \leq \frac{1}{2}, \quad (|z| = 1).
\]

This implies that,

\[
\frac{1}{2} \leq \left| \frac{z(2-\alpha)(n+1)(1+\lambda_1)^{m-1} - (1-\alpha)(1+\lambda_2)^m z^2}{2(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \right| \leq 1.
\]

Hence,

\[
\min \left\{ \frac{z(2-\alpha)(n+1)(1+\lambda_1)^{m-1} - (1-\alpha)(1+\lambda_2)^m z^2}{2(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}}, \ f_0(z) \right\} \leq 1, \quad (z \in U).
\]

Next we show that

\[
\text{Re}(f(z)) > \frac{[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]}{2(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \cdot z.
\]

Now taking

\[
g(z) = \frac{z}{1-z} \in C\sigma(s)
\]

in (3.1) we have the following:

\[
(2-\alpha)(n+1)(1+\lambda_1)^{m-1} \cdot \text{Re}(f(z)) < \frac{z}{1-z}
\]

Therefore,

\[
\text{Re}(f(z)) > \frac{[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]}{2(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \cdot z.
\]

which implies that

\[
(2-\alpha)(n+1)(1+\lambda_1)^{m-1} \cdot \text{Re}(f(z)) < \frac{1}{2}, \quad (z \in U).
\]

Hence, we have

\[
\text{Re}(f(z)) > \frac{[(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}]}{2(1-\alpha)(1+\lambda_2)^m + (2-\alpha)(n+1)(1+\lambda_1)^{m-1}} \cdot z.
\]

which is (3.2) require to complete the proof of theorem 3.1.

**SOME APPLICATION OF THE THEOREM 3.1**

Taking \( n = 1 \) in theorem 3.1, we obtain the following:

**Corollary 1.** If the function \( f(z) \) defined by (1.2) satisfies
\begin{align}
\tag{4.1} \sum_{k+r+1}^\infty \frac{(2k-2\alpha)(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_1)(k-1)^m} |a_k| \leq \rho
\end{align}

\((s \in \mathbb{N} = \{1, 2, 3, \ldots\}; m \in \mathbb{N} \cup \{0\}; 0 \leq \alpha < 1; \lambda_2 \geq \lambda_1 \geq 0; \rho > 0; \text{ then for every } g \in C_\alpha(s), \text{ one has}
\begin{align}
\tag{4.2} \frac{(4-2\alpha)(1+\lambda_1)^{m-1}}{2[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^m]} (f*g)(z) < g(z)
\end{align}

\((z \in U; 0 \leq \alpha < 1; g \in C_\alpha(s); \lambda_2 \geq \lambda_1 \geq 0; m \in \mathbb{N} \cup \{0\}; \rho \geq 0)\)

and,
\begin{align}
\tag{4.3} \text{Re}(f(z)) > \frac{[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^m]}{(4-2\alpha)(1+\lambda_1)^{m-1}},
\end{align}

\((z \in U)\).

The contact factor
\begin{align}
\frac{(4-2\alpha)(1+\lambda_1)^{m-1}}{2[(4-2\alpha)(1+\lambda_1)^{m-1} + \rho(1+\lambda_2)^m]}
\end{align}

cannot be replaced by any larger one.

Remark 1: When \(\alpha = 0; \lambda_1 = \lambda_2 = 0; \rho = 1; m = 0\) in corollary 1, we have the result obtained by Selvaraj and Karthikeyan[3].

Taking \(\lambda_1 = \lambda_2 = 0\); and \(m = 0\) in Theorem 3.1; we obtain the following:

**Corollary 2.** If the function \(f(z)\) defined by (1.2) satisfies
\begin{align}
\sum_{k+r+1}^\infty (k+\alpha)C(n,k)|a_k| \leq \rho,
\end{align}

\((\rho > 0, n \in \mathbb{N}; 0 \leq \alpha < 1); \text{ then for every function } g \in C_\alpha(s), \text{ one has}
\begin{align}
\tag{4.4} \frac{4-2\alpha}{2(4-2\alpha+\rho)} (f*g)(z) < g(z)
\end{align}

\((z \in U; 0 \leq \alpha < 1; \rho > 0)\)

and,
\begin{align}
\tag{4.5} \text{Re}(f(z)) > \frac{4-2\alpha + \rho}{4-2\alpha},
\end{align}

\((z \in U)\).

The constant factor
\begin{align}
\frac{4-2\alpha + \rho}{4-2\alpha}
\end{align}

cannot be replaced by any larger one.

Remark 2: When \(\alpha = \rho = \frac{1}{2}\) in corollary 2, we have the result obtained by Aouf et al [4].

Taking \(\lambda_1 = \lambda_2 = 1, \alpha = 0, m = 2, \text{ in Theorem 3.1}; \text{ we obtain the following}

**Corollary 3.** If the function \(f(z)\) defined by (1.2) satisfies
\begin{align}
\sum_{k+r+1}^\infty C(n,k)|a_k| \leq \rho,
\end{align}

\((\rho > 0, n \in \mathbb{N}); \text{ then for every function } g \in C_\alpha(s), \text{ one has}
\begin{align}
\tag{4.6} \frac{n+1}{2(n+1+\rho)} (f*g)(z) < g(z)
\end{align}

\((z \in U; 0 \leq \alpha < 1; \rho > 0)\)

and,
\begin{align}
\tag{4.7} \text{Re}(f(z)) > \frac{n+1}{1+(n+1)(4-2\alpha)}
\end{align}

\((z \in U)\).

The constant factor
\begin{align}
\frac{n+1}{1+(n+1)(4-2\alpha)}
\end{align}

cannot be replaced by any larger one.

Remark 3: When \(\rho = m(m > 0)\), in the corollary 3, we have the result obtained by Atiya et al [5].

Taking \(\alpha = 0\) in Theorem 3.1, we obtain the following:

**Corollary 4.** If the function \(f(z)\) defined by (1.2) satisfies
\begin{align}
\sum_{k+r+1}^\infty \frac{k(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} C(n,k)|a_k| \leq 1,
\end{align}

\((s, n \in \mathbb{N}; m \in \mathbb{N} \cup \{0\}; \lambda_2 \geq \lambda_1 \geq 0)\)

and,
\begin{align}
\tag{4.8} \text{Re}(f(z)) > \frac{[(1+\lambda_1)^m+2(n+1)(1+\lambda_2)^m]}{(n+1)(1+\lambda_1)^m},
\end{align}

\((z \in U)\).

The constant factor
\begin{align}
\frac{(n+1)(1+\lambda_1)^m}{(1+\lambda_2)^m+2(n+1)(1+\lambda_1)^m}
\end{align}

cannot be replaced by any larger one.

Remark 4: When \(m = n = 1, \lambda_1 = 0, \lambda_2 = 1\) in the corollary 4, we have the result obtained by Sukhjit[6] and Selvara et al [3].

**References**


