On Fixed Point Theorem in Fuzzy2- Metric Spaces for Integral type Inequality

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Abstract: ***In the present paper we are proving a common fixed point theorem for fuzzy 2 - metric spaces for rational expressions. 2010 Mathematics Subject Classification: Primary 47H10, 54H25.***

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1. Introduction and Preliminaries

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [37] in 1922. In the general setting of complete metric space, this theorem runs as the follows, Theorem 1.1(Banach's contraction principle) Let (X, d) be a complete metric space, c∈ (0, 1) and f: X→X be a mapping such that for each x, y∈X, d $(fx, fy) \leq$ c d(x, y) Then f has a unique fixed point a∈X, such that for each x∈X,$\lim_{n\to\infty} f^n x = a$. After the classical result, R.Kannan [31] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions. In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

Theorem 1.2(Branciari) Let (X, d) be a complete metric space, c∈ (0, 1) and let f: X→X be a mapping such that for each x, y ∈ X, $\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt$

Whereφ: [0,+∞) →[0,+∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+∞) , non negative, and such that for each ε >o, $\int_0^{\varepsilon} \varphi(t)dt$, then f has a unique fixed point a∈X such that for each x∈X, $\lim_{n\to\infty} f^n x = a$ After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive condition of integral type for a different contractive mapping satisfying various known properties. A fine work has been done by B.E. Rhoades [2] extending the result of BriancARI by replacing the condition [1.2] by the following;

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq$$
$$\int_0^{max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}} \varphi(t)dt (1.3)$$

The aim of this paper is to generalize some mixed type of contractive conditions to the mapping and then a pair of mappings, satisfying a general contractive mapping such as R. Kannan type [31], S.K. Chatrterjee type [36],T. Zamfirescu type [38],Turkoglu[39] etc. In 1965, the concept of fuzzy sets was introduced by Zadeh [42].After that many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [10], Erceg [12], Kaleva and Seikkala [25, 26], Kramosil and Michalek [27], have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors [3-5, 13, 19, 22, 23, 24, 29, 30, 34, and 35] have also studied the fixed point theory in the fuzzy metric spaces and [6-9, 21, 28,] have studied for fuzzy mappings which opened an avenue for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. Gähler in a series of papers [15, 16, and 17] investigated 2-metric spaces. Sharma, Sharma and Iseki [33] studied for the first time contraction type mappings in 2-metic space. We [40, 41] have also worked on 2-Metric spaces and 2- Banach spaces for rational expressions.

Definition 1.1: A binary operation *: $[0, 1]^3$→ [0, 1] is called a continuous t-norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever
$a_1 \geq a_2, \ b_1 \geq b_2, \ c_1 \geq c_2$ for all $a_1, b_1, c_1, a_2, b_2, c_2$ are in [0, 1].

Definition1. 2: the 3-tuple (X, M,*) is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions: for all x, y, z, u ∈ X and $t_1, t_2, t_3 >$ 0,

(1) M(x, y, z, 0) = 0,
(2) M(x, y, z, t) = 1 for all t > 0 (only when the three simplex x, y, z degenerate)
(3) M(x, y, z, t) = M(x, z, y, t) = M (y, z, x, t) =……………..
(4) M(x, y, z, w,t_1, t_2, t_3) ≥*(M(x, y, u, t_1)*M(x, u, z,t_2)*M (u, y, z,t_3)
(5) M(x, y, z,): [0, 1) × [0, 1] is left continuous.

Definition 1.3: Let (X, M,*) be a fuzzy- 2 metric space.

(1) A sequence $\{x_n\}$ in fuzzy -2 metric space X is said to be convergent to a point x ∈ X (denoted by

$$\lim_{n\to\infty} x_n = x \quad or \quad x_n \to x$$

if for any $\lambda \in (0,1)$ and t > 0, there exists $n_0 \in$ N such that for all n ≥ n_0 and $a \in$ X, M $(x_n, x, a, t) > 1 - \lambda$.

That is

$$\lim_{n\to\infty} M(x_n, x, a, t) = 1 \text{ for all } a \in X \text{ and } t > 0.$$

(2) A sequence $\{x_n\}$ in fuzzy- 2 metric space X is called a Cauchy sequence, if for any $\lambda \in (0,1)$ and t > 0, there exists

$n_0 \in$ N such that for all m, n ≥ n_0 and a ∈ X, M $(x_n, x_m, a, t) > 1 - \lambda$.

(3) A fuzzy- 2 metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.4: Self A function M is continuous in fuzzy 2-metric space if $x_n \to x$, $y_n \to y$, then

$$\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t) \text{ for all } a \in X \text{ and } t > 0.$$

Definition 1.5: Two mappings A and S on fuzzy 2-metric space X are weakly commuting if

M (ASu, SAu,a, t) ≥ M (Au, Su, a, t) $for\ all\ a, u \in X\ and\ t > 0$.

2. Some Basic Results

Lemma 2.1: [19] for all $x, y \in X$, M(x, y) is non-decreasing.

Lemma 2.2: [9] Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M,*) with condition (FM-6) If there exists a number $q \in (0,1)$ such that M $(y_{n+2}, y_{n+1}, qt) \geq$ M (y_{n+1}, y_n, t) for $t > 0$ and n =1,2,3….then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.3: [30] for all $x, y \in X$ and for a number $q \in (0,1)$ such that M $(x, y, qt) \geq$ M (x, y, t) for $t > 0$ then $x = y$.

Lemma (2.1, 2.2, and 2.3) are also true for fuzzy2-metric spaces.

3. Main Result

Theorem 3.1: Let (X, M, *) be a complete fuzzy 2-metric space and let S and T be continuous mappings of X in X. then S and T have a common fixed point in X if there exists continuous mapping A of X into S(X) ∩ T (X) which commute weakly with S and T,

(3.1a)

$$\int_0^{M(Ax,Ay,a,qt)} \zeta(t)dt \geq \int_0^{\left\{\min\left\{\begin{array}{l} M(Ty, Ay, a, t), M(Sx, Ax, a, t), \\ M(Sx, Ty, a, t), \dfrac{M(Sx, Ty, a, t)}{M(Ax, Ty, a, t)}, \\ \dfrac{M(Ax, Sx, a, t)}{M(Ay, Sx, a, t)}, M(Ax, Ty, a, t), \end{array}\right\}\right\}} \zeta(t)dt$$

For all x, y, a ∈ X, t > 0 and $q \in (0,1)$

(3.1b)

$\lim_{n\to\infty} M(x, y, z, t) = 1$ for all $x, y, z \in X$.

Then F, T and A have a unique common fixed point in X.

Proof:

We define a sequence $\{x_n\}$ in X such that $Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = Tx_{2n}$ for n = 1, 2. . .

We shall prove that $\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1a), we write

$$\int_0^{M(Ax_{2n},Ax_{2n+1},a,qt)} \zeta(t)dt \geq \int_0^{\left\{\min\left\{\begin{array}{l} M(Tx_{2n+1},Ax_{2n+1},a,t), M(Sx_{2n},Ax_{2n},a,t), M(Sx_{2n},Tx_{2n+1},a,t), \\ \frac{M(Sx_{2n},Tx_{2n+1},a,t)}{M(Ax_{2n},Tx_{2n+1},a,t)}, \frac{M(Ax_{2n},Sx_{2n},a,t)}{M(Ax_{2n+1},Sx_{2n},a,t)}, M(Ax_{2n},Tx_{2n+1},a,t), \end{array}\right\}\right\}} \zeta(t)dt$$

$$\int_0^{M(Ax_{2n},Ax_{2n+1},a,qt)} \zeta(t)dt \geq \int_0^{\left\{\min\left\{\begin{array}{l} M(Ax_{2n},Ax_{2n+1},a,t), M(Ax_{2n+1},Ax_{2n},a,t), M(Ax_{2n+1},Ax_{2n},a,t), \\ \frac{M(Ax_{2n+1},Ax_{2n},a,t)}{M(Ax_{2n},Ax_{2n},a,t)}, \frac{M(Ax_{2n},Ax_{2n+1},a,t)}{M(Ax_{2n+1},Ax_{2n+1},a,t)}, M(Ax_{2n},Ax_{2n},a,t) \end{array}\right\}\right\}} \zeta(t)dt$$

$$= \int_0^{\left\{\min\left\{\begin{array}{l} M(Ax_{2n},Ax_{2n+1},a,t), M(Ax_{2n+1},Ax_{2n},a,t), M(Ax_{2n+1},Ax_{2n},a,t), \\ M(Ax_{2n+1},Ax_{2n},a,t), M(Ax_{2n},Ax_{2n+1},a,t), 1 \end{array}\right\}\right\}} \zeta(t)dt$$

$$\geq \int_0^{\left\{\min\left\{M(Ax_{2n-1},Ax_{2n},a,\frac{t}{q}), M(Ax_{2n},Ax_{2n-1},a,\frac{t}{q})\right\}\right\}} \zeta(t)dt$$

Therefore

$$\int_0^{M(Ax_{2n},Ax_{2n+1},a,qt)} \zeta(t)dt \geq \int_0^{\left\{\min\left\{M(Ax_{2n-1},Ax_{2n},a,\frac{t}{q})\right\}\right\}} \zeta(t)dt$$

By induction

$$\int_0^{M(Ax_{2k},Ax_{2m+1},a,qt)} \zeta(t)dt \geq \int_0^{\left\{\min\left\{M(Ax_{2m},Ax_{2k-1},a,\frac{t}{q})\right\}\right\}} \zeta(t)dt$$

for every k and m in N, further if 2m+1>2k, then

$$\int_0^{M(Ax_{2k},Ax_{2m+1},a,qt)} \zeta(t)dt \geq \int_0^{\left\{M(Ax_{2k},Ax_{2m},a\frac{t}{q})\right\}} \zeta(t)dt$$

$$\cdots\cdots\cdots$$

$$\geq \int_0^{\left\{M(Ax_0,Ax_{2m+1},a\frac{t}{q^{2k}})\right\}} \zeta(t)dt \ldots\ldots(3.1c)$$

If 2k>2m+1, then

$$\int_0^{M(Ax_{2k},Ax_{2m+1},\,a,\,qt)} \zeta(t)dt \ge \int_0^{\left\{M(Ax_{2k-1},Ax_{2m},\,a,\,\frac{t}{q})\right\}} \zeta(t)dt$$

....

$$\ge \int_0^{\left\{M(Ax_{2k-(2m+1)},Ax_0,\,a,\,\frac{t}{q^{2m+1}})\right\}} \zeta(t)dt.......(3.1d)$$

By simple induction with (3.1c) and (3.1d) we have

$$\int_0^{M(Ax_n,Ax_{n+p},\,a,\,qt)} \zeta(t)dt \ge \int_0^{\left\{M(Ax_0,\,Ax_p,\,a,\,\frac{t}{q^n})\right\}} \zeta(t)dt$$

For n = 2k, p = 2m+1 or n = 2k+1, p = 2m+1

$$\int_0^{M(Ax_n,Ax_{n+p},\,a,\,qt)} \zeta(t)dt \ge \int_0^{\left\{M(Ax_0,\,Ax_1,\,a,\,\frac{t}{2q^n})\right\}*\left\{M(Ax_1,\,Ax_p,\,a,\,\frac{t}{q^n})\right\}} \zeta(t)dt$$

........(3.1*e*)

If n = 2k, p =2m or n =2k+1, p = 2m ,

For every positive integer p and n in N, by nothing that

$$\int_0^{\left\{M(Ax_0,Ax_p,\,a,\,\frac{t}{q^n})\right\}} \zeta(t)dt \to 1 \text{ as } n \to \infty$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete there exists $z \in X$, such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z.$

It follows that $Az = Sz = Tz$ and Therefore

$$\int_0^{M(Az,AAz,a,qt)} \zeta(t)dt \ge \int_0^{\left\{\min\left\{\begin{matrix} M(TAz,\,AAz,\,a,\,t), M(Sz,Az,a,t), \\ M\,(Sz,\,TAz,a,t),\,\dfrac{M(Sz,\,TAz,a,t)}{M(Az,\,TAz,a,t)}, \\ \dfrac{M(Az,\,Sz,a,t)}{M(AAz,\,Sz,a,t)}, M(Az,\,TAz,a,t), \end{matrix}\right\}\right\}} \zeta(t)dt$$

$$\int_0^{M(Az,A^2z,a,qt)} \zeta(t)dt \geq \int_0^{\{M(Sz,TAz,a,t)\}} \zeta(t)dt$$

$$\geq \int_0^{\{M(Sz,ATz,a,t)\}} \zeta(t)dt$$

$$\geq \int_0^{\{M(Az,A^2z,a,t)\}} \zeta(t)dt$$

$$\geq \int_0^{\left\{M(Az,A^2z,a,\frac{t}{q^n})\right\}} \zeta(t)dt.$$

Since,

$$\lim_{n\to\infty} M(Az, AA^2z, a, \frac{t}{q^n}) = 1 \to Az = A^2z$$

Thus z is common fixed point of A, S and T.

For uniqueness, let v $(v \neq z)$ be another common fixed point of S, T and A, By (3.1a) we write

$$\int_0^{M(Az,Av,a,qt)} \zeta(t)dt \geq \int_0^{\left\{\min\left\{\begin{matrix} M(Tv,Av,a,t), M(Sz,Az,a,t), \\ M(Sz,Tv,a,t), \dfrac{M(Sz,Tv,a,t)}{M(Az,Tv,a,t)}, \\ \dfrac{M(Az,Sz,a,t)}{M(Av,Sz,a,t)}, M(Az,Tv,a,t), \end{matrix}\right\}\right\}} \zeta(t)dt$$

$$\int_0^{M(Az,Av,a,qt)} \zeta(t)dt \geq \int_0^{\{M(z,v,a,t)\}} \zeta(t)dt$$

This implies that

$$\int_0^{M(z,v,a,qt)} \zeta(t)dt \geq \int_0^{\{M(z,v,a,t)\}} \zeta(t)dt$$

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