A Newton's Method for Nonlinear Unconstrained Optimization Problems with Two Variables

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Abstract: ***In this paper a modification of the classical Newton Method for solving nonlinear, univariate and unconstrained optimization problems based on the development of the new variants is presented. Moreover it focuses on the applications of the method for various means such as harmonic mean, Arithmetic mean, Geometric mean etc. Furthermore, numerical examples are discussed and the graphs of the results provided using MATLAB.***

Keywords: Unconstrained Optimization, Newton's Method, Non-linear Programming, Power mean, initial guess.

1. Introduction

The celebrate Newton's method is

$$x_{n+1} = x_n - \frac{f_x'(x_n)}{f_x''(x_n)} \tag{1.1}$$

used to approximate the optimum of a function is one of the most fundamental tools in computational mathematics, Operation Research, Optimization and Control Theory.
Here, the equation (1.1) can be extended for two variables and is given by

$$\left.\begin{aligned} x_{n+1} &= x_n - \frac{f_x(x_n,y_n)}{f_x'(x_n,y_n)} \\ y_{n+1} &= y_n - \frac{f_y(x_n,y_n)}{f_y'(x_n,y_n)} \end{aligned}\right\} \tag{1.2}$$

The idea behind the Newton's method is to approximate the objective function locally by a quadratic function which agrees with the function at a point. The process can be repeated at the point that optimizes the approximate function.

In this paper, an attempt is made to extend the results discussed for one variable as in [4] to two variables. The paper is organized as follows: Section 2 deals with variants of Newton's method for nonlinear equations with two variables. The extension of this method for unconstrained optimization problems with two variables is provided in section 3. In section 4, some numerical examples are discussed to verify the feasibility of the proposed method. Finally, some concluding remarks are given at the last.

2. Variants of Newton's Method for Nonlinear Equations with two variables

The modified Newton's method provided for single variable [4] is extended for two variables and is given by

$$\left.\begin{aligned} x_{n+1} &= x_n - \frac{2f(x_n,y_n)}{f_x(x_n,y_n)+f_x(x_{n+1}^*,x_{n+1}^*)} \\ y_{n+1} &= y_n - \frac{2f(x_n,y_n)}{f_y(x_n,y_n)+f_y(x_{n+1}^*,x_{n+1}^*)} \end{aligned}\right\} \tag{2.1}$$

Where

$$\left.\begin{aligned} x_{n+1}^* &= x_n - \frac{f(x_n,y_n)}{f_x(x_n,y_n)} \\ y_{n+1}^* &= y_n - \frac{f(y_n,x_n)}{f_y(x_n,y_n)} \end{aligned}\right\} \tag{2.2}$$

is the Newton's iterate for two variables.

3. Extension of the above Method for Unconstrained Optimization Problems with two variables

The above method obtained for two variables shall be extended for the case of Unconstrained Optimization Problems with two variables. The procedure is provided below as in [4]:

Suppose that the function f(x,y) is a sufficiently differentiable function. Let $\alpha \in R, \beta \in R$ is an extremum point of f(x,y) then $x = \alpha, y = \beta$ is a root of

$$f_x'(x,y)=0 \;\&\; f_y'(x,y)=0 \tag{3.1}$$

Extending Newton's theorem, we have

$$\left.\begin{aligned} f_x'(x,y) &= f_x'(x_n,y_n) + \int_{x_n}^{x} f''(t)\,dt \\ f_y'(x,y) &= f_y'(x_n,y_n) + \int_{y_n}^{y} f''(t)\,dt \end{aligned}\right\} \tag{3.2}$$

By the rectangular rule according to which

$$\left.\begin{aligned} \int_{x}^{x_{n+1}} f''(t)\,dt &\cong (x_{n+1}-x_n)\, f_x''(x_n,y_n) \\ \int_{y}^{y_{n+1}} f''(t)\,dt &\cong (y_{n+1}-y_n)\, f_y''(x_n,y_n) \end{aligned}\right\} \tag{3.3}$$

and using $f_x'(x_n,y_n) = 0 \;\&\; f_y'(x_n,y_n) = 0$ we get

$$\left.\begin{aligned} x_{n+1} &\equiv x_{n+1}^N = x_n - \frac{f_x'(x_n,y_n)}{f_x''(x_n,y_n)} \\ y_{n+1} &\equiv y_{n+1}^N = y_n - \frac{f_y'(x_n,y_n)}{f_y''(x_n,y_n)} \end{aligned}\right\} \tag{3.4}$$

$f_x''(x_n, y_n) \neq 0 \;\&\; f_y''(x_n, y_n) \neq 0$

This is a well-known quadratically convergent Newton's method for unconstrained optimization problems. By the trapezoidal approximation

$$\left.\begin{aligned} \int_{x_n}^{x_{n+1}} f''(t)dt \cong \frac{1}{2}(x_{n+1}-x_n)\{f_x''(x_n,y_n)+f_x''(x_{n+1},y_{n+1})\} \\ \int_{y_n}^{y_{n+1}} f''(t)dt \cong \frac{1}{2}(y_{n+1}-y_n)\{f_y''(x_n,y_n)+f_y''(x_{n+1},y_{n+1})\} \end{aligned}\right\} \quad (3.5)$$

in combination with the approximation

$$\left.\begin{aligned} f_x''(x_{n+1},y_{n+1}) = f_x''\left(x_n - \frac{f_x'(x_n,y_n)}{f_x''(x_n,y_n)}\right) = f_x''(x_{n+1}^N, y_{n+1}^N) \\ f_y''(x_{n+1},y_{n+1}) = f_x''\left(y_n - \frac{f_y'(x_n,y_n)}{f_y''(x_n,y_n)}\right) = f_y''(x_{n+1}^N, y_{n+1}^N) \end{aligned}\right\} \quad (3.6)$$

and $f_x'(x_{n+1}, y_{n+1}) = 0$, $f_y'(x_{n+1}, y_{n+1}) = 0$

we get the following arithmetic mean Newton's method given by

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{2f_x'(x_n,y_n)}{f_x''(x_n,y_n)+f_x''(x_{n+1}^N,y_{n+1}^N)} \\ y_{n+1} = y_n - \frac{2f_y'(x_n,y_n)}{f_y''(x_n,y_n)+f_y''(x_{n+1}^N,y_{n+1}^N)} \end{aligned}\right\} \quad (3.7)$$

for unconstrained optimization problems. This formula is also derived independently.

If we use the midpoint rule of integration then we obtain a new formula given by

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{f_x'(x_n,y_n)}{\mathrm{sign}(f_x''(x_0,y_0))\left[\frac{f_x''^{\alpha}(x_n,y_n)+f_x''^{\alpha}(x_{n+1}^N,y_{n+1}^N)}{2}\right]^{\frac{1}{\alpha}}} \\ y_{n+1} = y_n - \frac{f_y'(x_n,y_n)}{\mathrm{sign}(f_y''(x_0,y_0))\left[\frac{f_y''^{\alpha}(x_n,y_n)+f_y''^{\alpha}(x_{n+1}^N,y_{n+1}^N)}{2}\right]^{\frac{1}{\alpha}}} \end{aligned}\right\} \quad (3.8)$$

This family may be called the α^{th}-power mean iterative family of Newton's method for unconstrained optimization problems.

Various Cases:

It is interesting to note that for different specific values of α, various new methods can be deduced from Formula (3.8) as follows:

i) For α=1(arithmetic mean), Formula (3.8) corresponds to a cubically convergent arithmetic mean Newton's method

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{2f_x'(x_n,y_n)}{f_x''(x_n,y_n)+f_x''(x_{n+1}^N,y_{n+1}^N)} \\ y_{n+1} = y_n - \frac{2f_y'(x_n,y_n)}{f_y''(x_n,y_n)+f_y''(x_{n+1}^N,y_{n+1}^N)} \end{aligned}\right\} \quad (3.9)$$

ii) For α= -1 (harmonic mean), Formula (3.8) corresponds to a cubically convergent harmonic mean Newton's method

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{f_x'(x_n,y_n)}{2}\left[\frac{1}{f_x''(x_n,y_n)}+\frac{1}{f_x''(x_{n+1}^N,y_{n+1}^N)}\right] \\ y_{n+1} = y_n - \frac{f_y'(x_n,y_n)}{2}\left[\frac{1}{f_y''(x_n,y_n)}+\frac{1}{f_y''(x_{n+1}^N,y_{n+1}^N)}\right] \end{aligned}\right\} \quad (3.10)$$

iii) For $\alpha \to 0$ (geometric mean), Formula (3.8) corresponds to a new cubically convergent geometric mean Newton's method

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{f_x'(x_n,y_n)}{\mathrm{sign}(f_x''(x_0,y_0))\sqrt{f_x''(x_n,y_n)f_x''(x_{n+1}^N,y_{n+1}^N)}} \\ y_{n+1} = y_n - \frac{f_y'(x_n,y_n)}{\mathrm{sign}(f_y''(x_0,y_0))\sqrt{f_y''(x_n,y_n)f_y''(x_{n+1}^N,y_{n+1}^N)}} \end{aligned}\right\} \quad (3.11)$$

iv) For α =2, (root mean square) Formula (3.8) corresponds to a new cubically convergent root mean square Newton's method

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{f_x'(x_n,y_n)}{\mathrm{sign}(f_x''(x_0,y_0))\sqrt{\frac{f_x''^2(x_n,y_n)+f_x''^2(x_{n+1}^N,y_{n+1}^N)}{2}}} \\ y_{n+1} = y_n - \frac{f_y'(x_n,y_n)}{\mathrm{sign}(f_y''(x_0,y_0))\sqrt{\frac{f_y''^2(x_n,y_n)+f_y''^2(x_{n+1}^N,y_{n+1}^N)}{2}}} \end{aligned}\right\} \quad (3.12)$$

Some other new third-order iterative methods based on centrodial mean, logarithmic mean etc, can also be obtained from Formula (3.6) respectively.

v) New cubically convergent iteration method based on centroidal mean is

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{3f_x'(x_n,y_n)\{f_x''(x_n,y_n)+f_x''(x_{n+1}^N,y_{n+1}^N)\}}{2\{f_x''^2(x_n,y_n)+f_x''^2(x_{n+1}^N,y_{n+1}^N)+f_x''(x_n,y_n)f_x''(x_{n+1}^N,y_{n+1}^N)\}} \\ y_{n+1} = y_n - \frac{3f_y'(x_n,y_n)\{f_y''(x_n,y_n)+f_y''(x_{n+1}^N,y_{n+1}^N)\}}{2\{f_y''^2(x_n,y_n)+f_y''^2(x_{n+1}^N,y_{n+1}^N)+f_y''(x_n,y_n)f_y''(x_{n+1}^N,y_{n+1}^N)\}} \end{aligned}\right\} \quad (3.13)$$

vi) New cubically convergent iteration method based on logarithmic mean is

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{f_x'(x_n,y_n)\{\log|f_x''(x_n,y_n)|-\log|f_x''(x_{n+1}^N,y_{n+1}^N)|\}}{f_x''(x_n,y_n)-f_x''(x_{n+1}^N,y_{n+1}^N)} \\ y_{n+1} = y_n - \frac{f_y'(x_n,y_n)\{\log|f_y''(x_n,y_n)|-\log|f_y''(x_{n+1}^N,y_{n+1}^N)|\}}{f_y''(x_n,y_n)-f_y''(x_{n+1}^N,y_{n+1}^N)} \end{aligned}\right\} \quad (3.14)$$

As given by [2], this method can also be modified and is given by

$$\left.\begin{aligned} x_{n+1} = x_n - \frac{f_x'(x_n,y_n)}{f_x''(x_n,y_n)+pf_x'(x_n,y_n)} \\ y_{n+1} = y_n - \frac{f_y'(x_n,y_n)}{f_y''(x_n,y_n)+pf_y'(x_n,y_n)} \end{aligned}\right\} \quad (3.15)$$

This overcomes the two main practical deficiencies of Newton's method, namely, the need for analytic derivatives and the possible failure to converge to the solutions from poor starting points.

4. Numerical Examples

We shall present here the following numerical examples

Table 1: Test Problems

No.	*Examples*	*Initial Guess*	*Optimum points*
1	$x-y+2x^2+2xy+y^2$	(0,0)	(-1,1.5)
2	$x^3-3xy+y^3$	(1,2)	(1,1)

1. Output using the Formula (3.9) to (3.15)
Enter the Initial Values for the x and y
0 0
Optimum values are:

x= -1.000000
y= 1.500000
No of Iterations = 14

2. Output using the Formula (3.9) to (3.15)
Enter the Initial Values for the x and y
1 2
Optimum values are:
x= 1.000000
y= 1.000000
No of Iterations = 8

The graphs of the result are obtained using MATLAB and are given below:

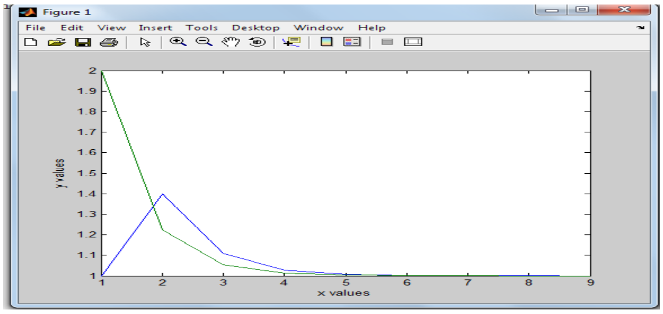

Figure 1. Testing data-optimum point

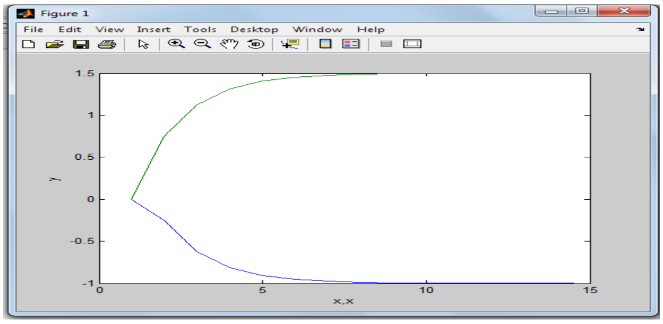

Figure 2. Testing data-optimum point

5. Conclusion

A nonlinear Newton's method which takes into account information about the objective functions for solving unconstrained optimization problems is presented here. Numerical examples showed that all the methods studied in various cases are efficient. In future, we will consider ways to accelerate these methods and perform new convergence analyses on them.

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