

An Introduction to Laplace Transform

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Abstract: An introduction to Laplace Transform is the topic of this paper. It deals with what Laplace Transform is, and what is it actually used for. The definition of Laplace Transform and most of its important properties have been mentioned with detailed proofs. This paper also includes a brief overview of Inverse Laplace Transform. A number of methods used to find the time domain function from its frequency domain equivalent have been explained with detailed explanations. It also includes the formulation of Laplace Transform of certain special function like the Heaviside's Unit Step Function and the Dirac Delta Function. A few practical life applications of Laplace Transform have also been stated.

Keywords: Laplace Transform, Heaviside's, properties, Dirac Delta

1. Introduction

This paper deals with a brief overview of what Laplace Transform is and its application in the industry. The Laplace Transform is a specific type of integral transform. Considering a function $f(t)$, its corresponding Laplace Transform will be denoted as $L[f(t)]$, where L is the operator operated on the time domain function $f(t)$. The Laplace Transform of a function results in a new function of complex frequency s . Like the Fourier Transform, the Laplace Transform is also used in solving differential and integral equations. It is also predominantly used in the analysis of transient events in the electrical circuits where frequency domain analysis is used.

2. Definition of Laplace Transform

Consider a function of time $f(t)$. If this function satisfies certain conditions and the if the integral,

$$\phi(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Exists, then $\phi(s)$ represents the Laplace Transform of $f(t)$, i.e.

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt \quad \dots (1)$$

3. Properties and Theorems of Laplace Transform

3.1 Linearity Property

If k_1 and k_2 are constants, then,

$$L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)] \quad \dots (2)$$

3.2 Change of Scale Property

A linear multiplication or division of a constant with the variable is known as scaling. Thus, if $L[f(t)] = \phi(s)$, then by change of scale property,

$$L[f(at)] = \frac{1}{a} \phi\left(\frac{s}{a}\right) \quad \dots (3)$$

3.3 First Shifting Theorem

The First Shifting Theorem of Laplace Transform states that if $L[f(t)] = \phi(s)$, then

$$L[e^{-at} f(t)] = \phi(s + a) \quad \dots (4)$$

Proof By definition,

$$\begin{aligned} L[e^{-at} f(t)] &= \int_0^{\infty} e^{-st} \{ e^{-at} f(t) \} dt \\ &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \end{aligned}$$

$$\therefore L[e^{-at} f(t)] = \phi(s + a)$$

3.4 Second Shifting Theorem

The Second Shifting Theorem of Laplace Transform states that if $L[f(t)] = \phi(s)$, then the Laplace Transform of the following function,

$$g(t) = f(t - a) \quad \text{when } t > a$$

$$= 0 \quad \text{when } t < a$$

Is expressed as

$$L[g(t)] = e^{-as} \phi(s) \quad \dots (5)$$

Proof By definition, $L[g(t)] = \int_0^{\infty} e^{-st} g(t) dt$

$$= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt$$

$$= 0 + \int_0^{\infty} e^{-st} f(t - a) dt$$

Now put $t - a = u, \therefore dt = du$

$$\therefore L[g(t)] = \int_0^{\infty} e^{-s(a+u)} f(u) du$$

$$= e^{-as} \int_0^{\infty} e^{-su} f(u) du$$

$$= e^{-as} \int_0^{\infty} e^{-st} f(t) dt$$

$$\therefore L[g(t)] = e^{-as} \phi(s)$$

3.5 Multiplication of powers of the variable

The variable that has been used so far is 't'. Thus, if we multiply powers of t with the original function f (t), the Laplace transform can be expressed as

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \phi(s) \quad \dots (6)$$

Proof This result can be proved by the use of Mathematical Induction.

Step 1 To prove that the result is true when n=1.

$$\text{Let } L[f(t)] = \phi(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Differentiating with respect to x and applying the rule of differentiation under the integral sign,

$$\phi'(s) = \int_0^{\infty} \frac{\partial}{\partial s} [e^{-st} f(t) dt]$$

$$= - \int_0^{\infty} e^{-st} t f(t) dt$$

$$= -L[tf(t)]$$

$$\therefore L[tf(t)] = (-1) \frac{d}{ds} \phi(s)$$

Which proves the result for n=1.

Step 2 Since the result holds true for n=1, it can be assumed that the result is true when n is any natural number 'k'.

$$\therefore L[t^k f(t)] = (-1)^k \frac{d^k}{ds^k} \phi(s)$$

Step 3 To prove that the result holds true when n=k+1.

From Step 2,

$$(-1)^k \frac{d^k}{ds^k} \phi(s) = L[t^k f(t)] = \int_0^{\infty} e^{-st} t^k f(t) dt$$

Differentiating with respect to x and applying the rule of differentiation under the integral sign,

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}} \phi(s)$$

$$= \int_0^{\infty} \frac{\partial}{\partial s} [e^{-st} t^k f(t) dt]$$

$$= - \int_0^{\infty} e^{-st} t^{k+1} f(t) dt$$

$$= -L[t^{k+1} f(t)]$$

$$\therefore L[t^{k+1} f(t)] = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} \phi(s)$$

Which proves the result for n=k+1.

Thus, by the rule of Mathematical Induction, it can be said that the result is true for any value of n.

3.6 Division of variable

If $L[f(t)] = \phi(s)$, then the Laplace Transform when the function is divided by the variable can be expressed as,

$$L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} \phi(s) ds \quad \dots (7)$$

Proof By definition, $\phi(s) = \int_0^{\infty} e^{-st} f(t) dt$

Integrating both sides with respect to s between the limits s to ∞ and then changing the order of integration on the RHS,

$$\int_s^{\infty} \phi(s) ds = \int_0^{\infty} \left[\int_s^{\infty} e^{-st} f(t) ds \right] dt$$

$$= \int_0^{\infty} \left[\frac{e^{-st}}{-t} f(t) \right]_s^{\infty} dt$$

$$= \int_0^{\infty} e^{-st} \frac{f(t)}{t} dt$$

$$= L\left[\frac{1}{t} f(t)\right]$$

$$\therefore L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} \phi(s) ds$$

4. Laplace Transform of Derivatives

Let f (t) be the time domain function. The Laplace Transform of its derivative can be expressed as

$$L[f'(t)] = sL[f(t)] - f(0) \quad \dots (8)$$

Proof By definition, $L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$

Integrating by parts,

$$L[f'(t)] = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} (-s) e^{-st} f(t) dt$$

$$= f(0) + s \int_0^{\infty} e^{-st} f(t) dt$$

$$\therefore L[f'(t)] = sL[f(t)] - f(0)$$

Differentiating equation (8) again with respect to variable t,

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

Thus, in general, the nth derivative can be expressed as,

$$L[f^n(t)] = s^n L[f(t)] - \dots - s^2 f^{n-3}(0) - s f^{n-2}(0) - f^{n-1}(0) \quad \dots (9)$$

The above mentioned results are put to incredible use in

solving Differential Equations.

5. Laplace Transform of Integrals

When the time domain function is integrated, its Laplace Transform can be expressed as,

$$L \left[\int_0^t f(u)du \right] = \frac{1}{s} \phi(s) \quad \dots (10)$$

Proof By definition,

$L \left[\int_0^t f(u)du \right] = \int_0^\infty e^{-st} \left[\int_0^t f(u)du \right] dt$. Integrating by parts,

$$= \left[\int_0^t f(u)du \left\{ -\frac{e^{-st}}{s} \right\} \right]_0^\infty - \int_0^\infty \left[\left\{ -\frac{e^{-st}}{s} \right\} \frac{d}{dt} \int_0^t f(u)du \right] dt$$

But $\frac{d}{dt} \int_0^t f(u)du = f(t)$

$$\therefore L \left[\int_0^t f(u)du \right] = \int_0^\infty \frac{1}{s} e^{-st} f(t) dt = \frac{1}{s} L[f(t)]$$

$$\therefore L \left[\int_0^t f(u)du \right] = \frac{1}{s} \phi(s)$$

The above mentioned result can be generalized as,

$$L \left[\int_0^t \int_0^t \dots \int_0^t f(u) (du)^n \right] = \frac{1}{s^n} L[f(t)] \quad \dots (11)$$

6. Inverse Laplace Transform

6.1 Definition

If $L[f(t)] = \phi(s) = \int_0^\infty e^{-st} f(t)dt$, then $f(t)$ is called the Inverse Laplace Transform of $\phi(s)$. It can be denoted as,

$$L^{-1} \phi(s) = f(t) \quad \dots (12)$$

Thus, the frequency domain function $\phi(s)$ can be converted to its corresponding time domain equivalent $f(t)$ using the Laplace Inverse operator (L^{-1}).

7. Different methods of obtaining Inverse Laplace Transform

There are numerous ways to obtain the Inverse Laplace Transform of a given frequency domain function. The choice of the method employed in solving a problem on Inverse Laplace Transform depends on the nature and structure of the problem itself. Often it would be noted

that a single problem can be solved by multiple methods. A few methods have been explained below.

7.1 Using Standard Results

A few standard results which can be used to find the inverse Laplace Transform have been tabulated below. These results can be easily proven using the standard definitions as mentioned in equations (1) and (12).

Table 1

Frequency Domain Function	Inverse Laplace Transform
$\frac{1}{s}$	$L^{-1} \left[\frac{1}{s} \right] = 1$
$\frac{1}{s+a}$	$L^{-1} \left[\frac{1}{s+a} \right] = e^{-at}$
$\frac{1}{s-a}$	$L^{-1} \left[\frac{1}{s-a} \right] = e^{at}$
$\frac{1}{s^n}$	$L^{-1} \left[\frac{1}{s^n} \right] = \frac{t^{n-1}}{\Gamma n}$
$\frac{1}{s^2+a^2}$	$L^{-1} \left[\frac{1}{s^2+a^2} \right] = \frac{1}{a} \sin at$
$\frac{s}{s^2+a^2}$	$L^{-1} \left[\frac{s}{s^2+a^2} \right] = \cos at$
$\frac{1}{s^2-a^2}$	$L^{-1} \left[\frac{1}{s^2-a^2} \right] = \frac{1}{a} \sinh at$
$\frac{s}{s^2-a^2}$	$L^{-1} \left[\frac{s}{s^2-a^2} \right] = \cosh at$

7.2 Using First Shifting Theorem

As seen in equation (4), the First Shifting Theorem can be expressed as,

$$L[e^{-at} f(t)] = \phi(s+a)$$

This means that if $f(t) = L^{-1}[\phi(s)]$, then,

$$L^{-1}[\phi(s+a)] = e^{-at} f(t)$$

$$L^{-1}[\phi(s+a)] = e^{-at} L^{-1}[\phi(s)] \quad \dots (13)$$

7.3 Use of Partial Fractions

Whenever possible, it is always easier to solve a problem on Inverse Laplace Transform by expressing the given function $\phi(s)$ into a sum of linear or quadratic partial fraction as,

$\phi(s) = \frac{A}{(s+a)^r} + \frac{Bs+C}{(s^2+a^2)^r}$, and then use standard results given in table 1 to find corresponding Inverse Laplace Transform.

7.4 Using Change of Scale Property

From equation 3, the change of scale property can be expressed as,

$$L[f(at)] = \frac{1}{a} \phi\left(\frac{s}{a}\right)$$

Thus if $f(t) = L^{-1}[\phi(s)]$, taking Inverse Laplace Transform,

$$L^{-1}\left[\frac{1}{a} \phi\left(\frac{s}{a}\right)\right] = f(at) \quad \dots (14)$$

7.5 Convolution Theorem

7.5.1 Definition

If $f_1(t)$ and $f_2(t)$ are two functions, then the following integral

$$\int_0^t f_1(u)f_2(t-u)du$$

Is called the convolution of $f_1(t)$ and $f_2(t)$ and is denoted as $f_1(t) * f_2(t)$

$$\therefore f_1(t) * f_2(t) = \int_0^t f_1(u)f_2(t-u)du \quad \dots (15)$$

7.5.2 Theorem

Let $L[f_1(t)] = \phi_1(s)$ and $L[f_2(t)] = \phi_2(s)$, then,

$$L^{-1}[\phi_1(s) \phi_2(s)] = \int_0^t f_1(u)f_2(t-u)du \quad \dots (16)$$

Where $f_1(t) = L^{-1}[\phi_1(s)]$ and $f_2(t) = L^{-1}[\phi_2(s)]$

7.6 Using Differentiation of $\phi(s)$

If $L[f(t)] = \phi(s)$, then using $n=1$ in equation 6,

$$L[tf(t)] = -\phi'(s)$$

$$\therefore tf(t) = -L^{-1}[\phi'(s)]$$

$$\therefore tL^{-1}[\phi(s)] = -L^{-1}[\phi'(s)]$$

$$\therefore L^{-1}[\phi(s)] = -\frac{1}{t}L^{-1}[\phi'(s)] \quad \dots (17)$$

This method is particularly used to find the Inverse Laplace Transform of functions having $\tan^{-1} X$, $\cot^{-1} X$ and $\log X$ terms.

7.7 Using Integration of $f(t)$

Equation (10) gives us the result of the Laplace Transform when the function $f(t)$ is integrated as shown,

$$L\left[\int_0^t f(u)du\right] = \frac{1}{s} \phi(s)$$

$$\therefore \int_0^t f(u)du = L^{-1}\left[\frac{1}{s} \phi(s)\right]$$

But by definition, $f(u) = L^{-1}[\phi(s)]$

$$\therefore L^{-1}\left[\frac{1}{s} \phi(s)\right] = \int_0^t L^{-1}[\phi(s)]ds \quad \dots (18)$$

8. Laplace Transform of Periodic Functions

Considering $f(t)$ to be a periodic function with period a , it's Laplace Transform can be expressed as,

$$L[f(t)] = \frac{1}{1 - e^{-as}} \int_0^a e^{-st}f(t)dt \quad \dots (19)$$

Proof Since $f(t)$ is periodic with period a , $f(t) = f(t+a) = f(t+2a) = \dots$ and so on.

$$\begin{aligned} L[f(t)] &= \int_0^\infty e^{-st}f(t)dt \\ &= \int_0^a e^{-st}f(t)dt + \int_a^{2a} e^{-st}f(t)dt \\ &\quad + \dots \infty \end{aligned}$$

Now $\int_a^{2a} e^{-st}f(t)dt = \int_0^a e^{-s(u+a)}f(u+a)du$ where $t = u + a$

$$= e^{-as} \int_0^a e^{-su}f(u+a)du$$

$$= e^{-as} \int_0^a e^{-st}f(t+a)dt$$

$$= e^{-as} \int_0^a e^{-st}f(t)dt \quad \text{since } f(t+a) = f(t)$$

Similarly, the next integral can be proved as

$e^{-2as} \int_0^a e^{-st}f(t)dt$ and so on with all further integrals.

$$L[f(t)] = [1 + e^{-as} + e^{-2as} + \dots \infty] \int_0^a e^{-st}f(t)dt$$

$$\therefore L[f(t)] = \frac{1}{1 - e^{-as}} \int_0^a e^{-st}f(t)dt$$

9. Heaviside's Unit Step Function

Heaviside's Unit Step Function can have only two possible values either 0 or 1. It can be defined as,

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

The function takes a jump of unit magnitude at $x=0$.

Taking the Laplace transform of the above function,

$$L[H(t)] = \int_0^{\infty} e^{-st} H(t) dt$$

$$\therefore L[H(t)] = \int_0^{\infty} e^{-st} dt$$

$$= L[1]$$

$$\therefore L[H(t)] = \frac{1}{s} \quad \dots (20)$$

9.1 Displaced Unit Step Function

If the origin is shifted to t=a, i.e. if the function is zero before t=a, and takes a jump of unit magnitude at t=a, then the function is called the Displaced Unit Step Function.

$$H(t - a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Thus, instead of taking a jump at t=0, the function now takes a jump of unit magnitude at t=a, and its Laplace Transform can be expressed as,

$$L[H(t - a)] = \int_0^{\infty} e^{-st} H(t - a) dt$$

$$= \int_0^a 0 dt + \int_a^{\infty} e^{-st} dt$$

$$= \left[-\frac{e^{-st}}{s} \right]_a^{\infty}$$

$$\therefore L[H(t - a)] = \frac{e^{-as}}{s} \quad \dots (21)$$

9.2 Effect of Multiplication of H(t - a)

Often in practical applications it is required to find the Laplace Transform when the time domain function itself is multiplied with the Unit Step Function, i.e. the function will be defined as,

$$f(t)H(t - a) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases}$$

Taking the Laplace Transform of the above function,

$$L[f(t)H(t - a)] = \int_0^{\infty} e^{-st} f(t)H(t - a) dt$$

$$= \int_0^a 0 dt + \int_a^{\infty} e^{-st} f(t) dt$$

Now, let t - a = u. ∴ dt = du.
When t=a, u=0. When t=∞, u=∞.

$$\therefore L[f(t)H(t - a)] = \int_0^{\infty} e^{-s(a+u)} f(a + u) du$$

$$= e^{-as} \int_0^{\infty} e^{-su} f(u + a) du$$

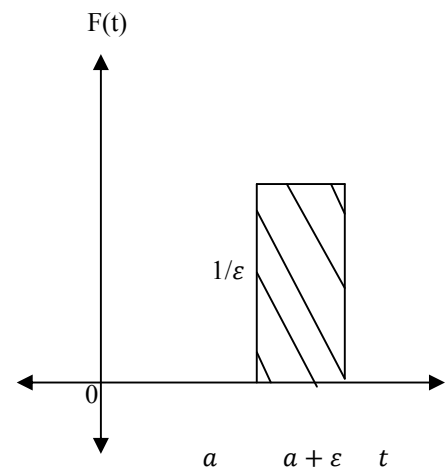
$$= e^{-as} \int_0^{\infty} e^{-st} f(t + a) dt$$

$$\therefore L[f(t)H(t - a)] = e^{-as} L[f(t + a)] \quad \dots (22)$$

In a specific case where a=0,

$$\therefore L[f(t)H(t)] = L[f(t)]$$

10. Dirac Delta Function (Unit Impulse Function)



The function represented by the figure can be defined as,

$$F(t) = \begin{cases} 0, & t < a \\ \frac{1}{\epsilon}, & a \leq t \leq a + \epsilon \\ 0, & t > a + \epsilon \end{cases}$$

As ε → 0, the function F(t) tends to infinity at a, and is zero everywhere else. But the integral of F(t) is unity. Thus, the integral $\lim_{\epsilon \rightarrow 0} \int_0^{\infty} F(t) dt (= 1)$, represents a unit impulse at t = a. Hence the limiting form of F(t) as ε → 0, is called as the Unit Impulse Function or the Dirac Delta Function and is denoted by δ(t - a).

$$\delta(t - a) = \lim_{\epsilon \rightarrow 0} F(t)$$

10.1 Laplace Transform of Dirac Delta Function

Taking the Laplace Transform of F (t) defined earlier,

$$L[F(t)] = \int_0^{\infty} e^{-st} F(t) dt$$

$$\begin{aligned}
 &= \frac{1}{\varepsilon} \int_a^{a+\varepsilon} e^{-st} dt \\
 &= \frac{1}{\varepsilon} \left[\frac{e^{-st}}{-s} \right]_a^{a+\varepsilon} \\
 &= \frac{1}{s} e^{-as} \left[\frac{1 - e^{-\varepsilon s}}{\varepsilon} \right]
 \end{aligned}$$

$$\therefore L[\delta(t - a)] = \lim_{\varepsilon \rightarrow 0} L[F(t)]$$

$$= \frac{1}{s} e^{-as} \lim_{\varepsilon \rightarrow 0} \left[\frac{1 - e^{-\varepsilon s}}{\varepsilon} \right]$$

$$\therefore L[\delta(t - a)] = e^{-as} \quad \dots (23)$$

If a=0, then $L[\delta(t)] = 1$.

11. Applications

Laplace Transforms are put to incredible amount of use in solving differential equations and in circuit analysis which involves the components like resistors, inductors and capacitors. Most often, during circuit analysis, the time domain equations are first written and then Laplace Transform of the time domain equation is taken to convert it to its frequency domain equivalent. However, it is also possible to convert the circuit impedance into its frequency domain equivalent and then proceed, both of which produce the same result.

12. Conclusion

This paper thus, consisted of a brief overview of what Laplace Transform is, and what is it used for. The primary use of Laplace Transform of converting a time domain function into its frequency domain equivalent was also discussed. Major properties of Laplace Transform and a few special functions like the Heaviside's Unit Step Function and Dirac Delta Functions were also discussed in detail. It also included a detailed explanation of Inverse Laplace Transform and the various methods that can be employed in finding the Inverse Laplace Transform. It goes without saying that Laplace Transform is put to tremendous use in many branches of Applied Sciences.

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