

Martingale Approach to the Pricing of European Style Contingent Claims

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Abstract: This paper reviews the mathematical foundation of martingale theory to the pricing of contingent claims in financial markets. The martingale theory of the pricing of contingent claims in an ideal financial market is described within the context of the Europeans and Asian options markets.

Keywords: Martingale theory; Contingent claims; Ideal financial market; Europeans option; Asian option.

1. Introduction

Financial mathematics is evolving as a subject that utilizes diverse results from such sophisticated area of mathematics as stochastic analysis; numerical analysis; theory of differential equation; game theory and theory of dynamical system to mention just a few areas[3]. The foundation of financial mathematics as it is known today has its origin in the seminal papers by Fisher Black and Myron Scholes (1973) and by Robert Merton (1973), where the Ito's formula has been used for deriving the Black Scholes equation[3; 8]. Other notable contributions were made by Harrison and Kreps (1979) and Harrison and Pliska (1981) further showed that a natural mathematical framework for the analysis of financial markets is stochastic analysis and martingale theory. Since then this framework has played a dominating role in financial mathematics [12; 13]. This theory has today become a powerful and effective tool for quantitative analysis in many economical problems [8]. This paper employed the martingale theory to model the pricing and hedging of certain contingent claims under a number of assumptions. A European contingent claim was considered in a frictionless market. The paper addresses some of the key ideas that underlying the modern approach to the mathematical modeling of contingent claims in security markets.

2. Some basic notation from Stochastic Analysis

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, $(\mathcal{R}^d, \mathcal{R}^d)$ the d -dimensional Euclidean measurable space. A collection \mathcal{F} of Ω is called a σ -algebra, if it satisfies the following conditions:

- (i) $\Omega \in \mathcal{F}$;
- (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, where $A^c = \Omega/A$ denotes the complement of A ;
- (iii) $A_j \in \mathcal{F}, j \geq 1, \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$.

The pair (Ω, \mathcal{F}) is called a measurable space.

Let \mathcal{C} be the collection of subsets of Ω . The smallest σ -algebra containing \mathcal{C} is called the σ -algebra generated by \mathcal{C} , and is denoted by $\sigma(\mathcal{C})$. The σ -algebra generated by all open set is called the Borel σ -algebra.

2.1 Theorem

Let \mathcal{G} be a sub- σ -algebra of \mathcal{F} , $g(x, y)$ a non negative Borel function on \mathcal{R}^2 and X a \mathcal{G} -measurable random variable. Then for any random variable Y , we have

$$\mathbb{E}[g(X, Y) | \mathcal{G}] = \mathbb{E}[g(x, Y) | \mathcal{G}] |_{x=X} \quad (2.1)$$

Proof: If A and B are two Borel sets and $g(\mathbb{1}_A(x)\mathbb{1}_B(y))$, then

$$\mathbb{E}[(X, Y) | \mathcal{G}] = Y\mathbb{E}[(X) | \mathcal{G}],$$

If Y is independent of \mathcal{G} , then

$$\mathbb{E}[g(X, Y) | \mathcal{G}] = \mathbb{E}[g(x, Y) | \mathcal{G}] |_{x=X}$$

2.2 Theorem

Let Q be a probability measure equivalent to P and \mathcal{G} a sub σ -algebra of \mathcal{F} . We put $\xi = \frac{dQ}{dP}$; $\eta = [\xi | \mathcal{G}]$. Then for a Q -integrable random variable X , we have

$$\mathbb{E}_Q = [X | \mathcal{G}] = \eta^{-1}\mathbb{E}[X\xi | \mathcal{G}] \quad (2.2)$$

Proof: For any $A \in \mathcal{G}$, we have

$$\begin{aligned} \mathbb{E}[X\xi \mathbb{1}_A] &= \mathbb{E}_Q[X\mathbb{1}_A] = \mathbb{E}_Q[\mathbb{E}_Q[X | \mathcal{G}]\mathbb{1}_A] = \\ \mathbb{E}[\mathbb{E}_Q[X | \mathcal{G}]\xi \mathbb{1}_A] &= \mathbb{E}[\mathbb{E}_Q[X | \mathcal{G}]\eta \mathbb{1}_A], \end{aligned}$$

Since $\mathbb{E}[\mathbb{E}_Q[X | \mathcal{G}]\eta \mathbb{1}_A]$ is \mathcal{G} -measurable, we get

$$\begin{aligned} \mathbb{E}[X\xi | \mathcal{G}] &= \mathbb{E}_Q[X | \mathcal{G}]\eta \\ \mathbb{E}_Q[X | \mathcal{G}] &= \eta^{-1}\mathbb{E}[X\xi | \mathcal{G}] \end{aligned}$$

2.1. Dynamics of underlying securities

Consider a market in which the securities are riskless asset (saving account) and risky asset (stock). Assume that the unit stock price S is a stochastic process on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and that

$$dS_t = S_t(\mu dt + \sigma dW_t), S(0) = S_0 \quad (2.3)$$

Where W is a Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, and μ, σ are constant. Equation (2.3) gives the stock price at time t as a solution of an SDE, and has the following explicit solution

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}, t \in [0, T] \quad (2.4)$$

Showing that S_t is lognormal.

The value process B_t of the saving account is assumed to satisfies

$$dB_t = rB_t dt, B_0 = 1 \quad (2.5)$$

Whence

$$B_t = e^{rt}$$

2.2. Self-financing portfolios and hedging

A portfolio (α, θ) is a pair of adapted processes such that α_t (resp. θ_t) is the number of shares of the saving account (resp. of the asset) owned by an investor. The time t of the portfolio is $V_t = \alpha_t B_t + \theta_t S_t$. The portfolio defines an hedging strategy for the contingent claim H if its terminal value is equal to H :

$$\alpha_T B_T + \theta_T S_T = H$$

The contingent claim H is of the form $H = h(S_T) = (S_T - K)^+$ for a call.

A portfolio is self financing if its changes in value are due to changes of prices, not the rebalancing of the portfolio, equivalently if one has

$$dV_t = \alpha_t dB_t + \theta_t dS_t \quad (2.6)$$

Black-Scholes methodology is to find a hedging strategy for the contingent claim assuming the value of the hedging portfolio is a smooth function of time and the underlying. A portfolio made up of π_t shares of the underlying asset which hedges the derivatives is constructed.

Definition: Suppose that some contingent security is represented by a stochastic process E on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Then a portfolio $\pi = (\pi_\alpha, \pi_\theta)$ is said to replicate E if at any time $t \in [0, T]$, we have

$$E_t = \pi_{\alpha_t} B_t + \pi_{\theta_t} S_t$$

with probability one.

Definition: A hedging strategy is self financing portfolio that replicates some specified contingent security.

Remark: For a European call, a hedging strategy is always available in an ideal condition.

Since the call price process $C(t, S_t)$ depend on S_t and t . Then setting $C(t, S_t) = V_t$

$$\begin{aligned} dV_t &= rV_t dt + \pi_t(dS_t - rS_t dt) \\ &= [rV_t + (\mu - r)\pi_t S_t]dt + \sigma S_t dW_t \\ &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial x} dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S_t^2 \sigma^2 dt \\ &= \left[\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S_t^2 \sigma^2 \right] dt + \frac{\partial C}{\partial x} \sigma S_t dW_t \end{aligned}$$

By identification, we have

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S_t^2 \sigma^2 = rV_t + (\mu - r)\pi_t S_t$$

since

$$\frac{\partial C}{\partial x} \sigma S_t = \pi_t \sigma S_t, \text{ and } \pi_t = \frac{\partial C}{\partial x}$$

Therefore

$$\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} S_t^2 \sigma^2 = rC - rS_t \frac{\partial C}{\partial x}$$

Hence, the price of an European option is $C(t, S_t)$ where C is the solution of

$$\frac{\partial C}{\partial t}(t, x) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 C}{\partial x^2}(t, x) + rx \frac{\partial C}{\partial x}(t, x) = rC(t, x)$$

which satisfies the terminal condition

$$C(T, x) = (x - K)^+$$

2.3. Martingale approach and Pricing by Arbitrage

The process S is not a martingale. Hence Setting $k = \frac{u-r}{\sigma}$, the process

$$\begin{aligned} S_t \exp\left(-rt - kW_t - \frac{1}{2}k^2 t\right) &= S_t e^{-rt} L_t = \\ x \exp\left((\sigma - k)W_t - \frac{1}{2}(\sigma - k)^2 t\right) \end{aligned}$$

is a martingale where $L_t = \left(-kW_t - \frac{1}{2}k^2 t\right)$. The choice $H_e = e^{-rt} L_t$ as a multiplier is such that the price of the risky and riskless asset multiplied by this factor is a martingale. We now want show that if V is the value of a self financing portfolio, the process $V_t e^{-rt} L_t$ is a martingale.

2.3.1 Lemma (Discounted processes)

A trading strategy (α, θ) is self financing if and only if its discounted value process \tilde{V}_t satisfies

$$d\tilde{V}_t = \theta_t e^{-rt} d\tilde{S}_t \quad (2.7)$$

Proof: Assume (α, θ) is a self financing. Since $\tilde{V}_t = e^{-rt}V_t$, by eqn(2.5) and using integration by parts formula

$$\begin{aligned} d\tilde{V}_t &= e^{-rt}dV_t - re^{-rt}V_t dt \\ &= e^{-rt}[\alpha_t d\beta_t + \theta_t dS_t] - [\alpha_t \beta_t + \theta_t S_t] dt \\ &= e^{-rt}[\alpha_t r \beta_t + \theta_t dS_t] - re^{-rt}[\alpha_t \beta_t + \theta_t S_t] dt \\ &= \theta_t [e^{-rt} dS_t - re^{-rt} S_t dt] \\ &= \theta_t d(e^{-rt} S_t) \\ &= \theta_t e^{-rt} d\tilde{S}_t \end{aligned}$$

Similarly, we can also used the Girsanov's theorem to establish the above result.

2.3.2 Girsanov's theorem

A change of probability changes the law of the variable or of the process.

Proof: Now we show that there exist a unique probability measure P^* equivalent to P such that the \tilde{S}_t is a P^* -martingale. In fact, we can rewriter equation (2.3) as

$$d\tilde{S}_t = \tilde{S}_t(\mu - r)dt + \sigma dW_t$$

Consequently, if we put $\frac{dP^*}{dP} \Big|_{\mathcal{F}_t} = \xi(-k.W)_T$ with $k(t) = k = (\mu - r)/\sigma$, then by the Girsanov's theorem and setting

$W_t^* = W_t + kt$, where $k = \frac{\mu-r}{\sigma}$ the dynamics of S may therefore be written as

$$dS_t = S_t(rdt + \sigma dW_t^*)$$

or ,in an equivalent form the dynamics of the discounted price $\tilde{S}_t = S_t e^{-rt}$ are

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^* \tag{2.8}$$

which solution is

$$\tilde{S}_t = S_0 \exp\left(\sigma W_t^* - \frac{\sigma^2 t}{2}\right).$$

Thus, the discounted price $(\tilde{S}_t, t \geq 0)$ is a martingale under the risk neutral probability P^* as soon as W^* is a Brownian motion under the probability P^* . The Girsanov's theorem states that there exist a probability measure P^* , equivalent to P , such that , under P^* , the process $(W_t^*, t \geq 0)$ is a Brownian motion. The probability measure P^* is defined by its Radon-Nykodym density: $dP^* = L_t dP$ on the σ -algebra \mathcal{F}_t with $L_t = \exp\left(-kW_t - \frac{1}{2}k^2 t\right)$. The discounted value of any self financing portfolio is a martingale. Since

$$V_t e^{-rt} = V_0 + \int_0^t \sigma \theta_s \tilde{S}_s dW_s^* \tag{2.9}$$

It follows that

$$V_t e^{-rt} = \mathbb{E}^*(V_T e^{-rT} | \mathcal{F}_t) = \mathbb{E}^*(H e^{-rT} | \mathcal{F}_t) \tag{2.10}$$

and $V_0 = \mathbb{E}^*(H e^{-rT})$. The hedging portfolio is obtained from the fact that a martingale with respect to a Brownian filtration may be written as a stochastic integral with respect to the Brownian motion, hence

$$\mathbb{E}^*(H e^{-rT} | \mathcal{F}_t) = V_0 + \int_0^t \psi_s W_t^* \tag{2.11}$$

2.4. Pricing of European option.

Now we consider the trading of European call or put option. Then the pay-off of a call option is $h(S_T)$ with $h(x) = (x - K)^+$, we have

$$C = \mathbb{E}^*(e^{-rT}(S_T - K)^+) = \mathbb{E}^*(e^{-rT} S_T \mathbb{1}_{S_T \geq K}) - K e^{-rT} \mathbb{E}^*(\mathbb{1}_{S_T \geq K})$$

As $S_T e^{-rT} = S_0 \exp\left(\sigma W_T^* - \frac{\sigma^2 T}{2}\right)$, we get immediately the following Black-Scholes formula

$$C = S_0 N(d_1(S_0, T)) - K e^{-rT} N(d_2(S_0, T))$$

With

$$d_1(x, T) = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x}{K e^{-rT}}\right) + \frac{\sigma\sqrt{T}}{2}, d_2(x, T) = d_1(x, T) - \sigma\sqrt{T}$$

The price of the call at time t equals $C(t, S_t) = \mathbb{E}^*(e^{-rT}(S_T - K)^+ | \mathcal{F}_t)$ with

$$C(t, x) = xN(d_1(x, T - t)) - K e^{-r(T-t)} N(d_2(x, T - t))$$

The time t value of the hedging portfolio may be written in the form $H(t, S_t)$, apply Ito's formula to get

$$d\tilde{V}_t = e^{-rt} \left(\frac{\partial H}{\partial t} + r S_t \frac{\partial H}{\partial x} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 H}{\partial x^2} - rH \right) (t, S_t) dt + e^{-rt} \sigma S_t \frac{\partial H}{\partial x} (t, S_t) dW_t^*$$

As $e^{-rt} H(t, S_t) = \tilde{V}_t$ is a martingale, the drift term is equal to zero,

$$\frac{\partial H}{\partial t} + r x \frac{\partial H}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 H}{\partial x^2} - rH = 0, \forall t \geq 0, \forall x \geq 0.$$

Therefore,

$$d\tilde{V}_t = e^{-rt} \sigma S_t \frac{\partial H}{\partial x} (t, S_t) dW_t^*$$

3. The Ito's process model

Let the time horizon be $[0, T]$. Let $W = (W^1, \dots, W^d)$ be a Brownian motion on a complete probability space (Ω, \mathcal{F}, P) . We denote by \mathcal{F}_t the natural filtration of W_t and L the set of measurable \mathcal{F}_t -adapted processes. We consider financial markets which consist of $m + 1$ asset. The price process S_t^i

of each asset i is assumed to be strictly positive Ito process, we represent S_t^i as

$$dS_t^i = [S_t^i \mu^i(t) dt + S_t^i \sigma^i(t) dW_t^i], S_t^i = P_i, 0 < i < k \tag{3.1}$$

Where $\mu = (\mu^0, \dots, \mu^k)$ is called the expected rate return vector and σ the volatility matrix.

We specify asset 0 as the numeraire asset and set $\gamma_t = (S_t^0)^{-1}$ and call γ_t the deflate at time t . By Ito formula, we have

$$d\gamma_t = -\gamma_t [\mu^0(t) - |\sigma^0(t)|^2 + \sigma^0(t) dW_t^0] dt \tag{3.2}$$

Setting $S_t = (S_t^1, \dots, S_t^k)$ and $\tilde{S}_t = (\tilde{S}_t^1, \dots, \tilde{S}_t^k)$, where $\tilde{S}_t^i = \gamma_t S_t^i$. Then we have

$$d\tilde{S}_t^i = [\tilde{S}_t^i \theta^i(t) dt + \tilde{S}_t^i \alpha^i(t) dW_t^i], 1 \leq i < k \tag{3.3}$$

Where

$$\theta^i(t) = \mu^i(t) - \mu^0(t) + |\sigma^0(t)|^2 - \sigma^i(t) \cdot \sigma^0(t); \alpha^i(t) = \sigma^i(t) - \sigma^0(t).$$

In particular, if asset 0 is a saving account with interest rate process $r(t)$, then

$$\theta^i(t) = \mu^i(t) - r(t); \alpha^i(t) = \sigma^i(t).$$

3.1. Fundamental question

What condition should we impose on coefficients α and θ of the price process \tilde{S}_t such that market is fair?. The following theorem provides a partial answer to this question.

3.1.1 Theorem

If the market is fair, the linear equation

$$\theta(t)\psi(t) = \alpha(t), dt \times dP \text{ on } [0, T] \times \Omega \tag{3.4}$$

Has a solution $\psi \in (L^2)^d$. Conversely, if

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^T |\theta^i(t)|^2 dt \right\} \right] < \infty, 1 \leq i \leq k \tag{3.5}$$

And equation (3.4) has a solution $\psi \in (L^2)^d$ satisfying

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^T |\psi(t)|^2 dt \right\} \right] < \infty \tag{3.6}$$

then the probability measure P^* with Radon-Nikodym derivative $\frac{dP^*}{dP} = h(-\psi \cdot W)_T$ is an equivalent martingale measure.

The next theorem provides a sufficient condition for the existence of a unique martingale measures.

3.1.2 Theorem

Assumed that $m \geq d$, a satisfies equation (3.5) and $\theta^T(t)\theta(t)$ are non degenerated for all t , where $\theta^T(t)$ stands for the transpose of $\theta(t)$. Setting $\psi(t) = (\theta^T(t)\theta(t))^{-1}\theta^T(t)\alpha(t)$. If ψ satisfies equation (3.4) and (3.6), then there exist a unique equivalent martingale measure P^* for the market. We have

$$\mathbb{E} \left[\frac{dP^*}{dP} \mid \mathcal{F}_t \right] = \exp \left\{ - \int_0^t \psi(s) dW_s - \frac{1}{2} \int_0^t |\psi(s)|^2 ds \right\}, 0 \leq t \leq T$$

Proof Under the assumption of the theorem, the market is standard, so by theorem (3.1.1), there exist an equivalent martingale measure. To prove the uniqueness, let P^* be an equivalent martingale measure. There exist a $\varphi \in (L^2)^d$ such that

$$\frac{dP^*}{dP} = h(\varphi \cdot W)_T. \text{ By theorem (3.1.1), we have}$$

$$\theta(t)\varphi(t) = \alpha(t). \tag{3.7}$$

Now applying $(\theta^T(t)\theta(t))^{-1}\theta^T(t)$ to both sides of the equation (3.7), we have

$$\theta(t)\varphi(t) (\theta^T(t)\theta(t))^{-1}\theta^T(t) = (\theta^T(t)\theta(t))^{-1}\theta^T(t)\alpha(t)$$

Since $\psi(t) = (\theta^T(t)\theta(t))^{-1}\theta^T(t)\alpha(t)$, we get $\varphi(t) = \psi(t)$ which proved the uniqueness.

Remark: if $m = d$, then ψ satisfies equation (3.4) automatically.

3.2 Pricing and hedging of European contingent claims

We present the pricing and hedging of European contingent claims. We assume that the market fair. Let h be a contingent claim. Assume that $\gamma_T h$ is P^* -integrable for some $P^* \in \mathcal{M}^0$. Setting

$$V_t = \gamma_t^{-1} \mathbb{E}^* [\gamma_T h \mid \mathcal{F}_t] \tag{3.8}$$

and letting V_t be the price process of an asset, we want to show that for a replicate contingent claims the fair price is unique.

3.2.1 Theorem

Let $P^*, P \in \mathcal{M}^0$ and h be a P^* and P -replicable contingent claim. Let V_t (resp. U_t) be the wealth process of a P^* -

(resp. P -) hedging strategy for h . Then V_t and U_t are the same.

Proof Setting $\tilde{V}_t = \gamma_t V_t \tilde{U}_t = \gamma_t U_t$. Then \tilde{V}_t is a P^* -martingale and a P -super martingale and \tilde{U}_t is a P -martingale and a P^* -super martingale. Since $U_T = V_T = H$, we have

$\mathbb{E}^*[\tilde{V}_T | \mathcal{F}_t] = \tilde{V}_t \geq \mathbb{E}_p[\tilde{V}_T | \mathcal{F}_t] = \tilde{U}_t$ We have $V_t \geq U_t$, a.s.. Similarly, we have $U_t \geq V_t$, a.s.. Hence $V = U$.

Remark According to theorem (3.2.1), for a P^* -replicable contingent claim h it is natural to define its "fair" price at time t by equation (3.8). we call this method arbitrage pricing. The next theorem shows that the replicability of a contingent claim and the arbitrage pricing of replicable contingent claim are independent of the choice of numeraire.

3.2.2 Theorem

Let $P^* \in \mathcal{M}^0$ and h be a P^* -replicable contingent claim and φ be a fair hedging strategy for h . Then for any $0 \leq j \leq m$, h is a $k_j(P^*)$ -replicable contingent claim and its "fair" price process remains the same.

Proof Keeping the notations of the proof of the previous theorem and by equation (3.8), we have

$$\mathbb{E}_p[(\gamma'_T)^{-1}h] = \mathbb{E}^*[M_T(\gamma'_T)^{-1}h] = \frac{S_0^0}{S_0^j} \mathbb{E}^*[\gamma_T h] = (S_0^j)^{-1}V_0$$

This implies that a P^* -hedging strategy for h is also a P -hedging strategy for h . So h is a P -replicable contingent claim. By the Bayes rule, we have

$$(\gamma'_t)^{-1} \mathbb{E}_p[\gamma'_T h | \mathcal{F}_t] = (\gamma'_t)^{-1} M_t^{-1} \mathbb{E}^*[M_T \gamma'_T h | \mathcal{F}_t] = \gamma_t^{-1} \mathbb{E}^*[\gamma_T h | \mathcal{F}_t]$$

This proves that the "fair" price process of h is independent of the choice of numeraire.

4. Pricing of Exotic Options

We present the pricing of path-dependent exotic option. These are Asian options, lookback options and barrier option. Asian options have a payoff equal to $(\frac{1}{T} \int_0^T S_u du - K)^+$ and depend the average of S over the time interval $[0, T]$. Lookback options have a payoff equal to $(\max_{0 \leq t \leq T} S_t - K)^+$ and depend on the maximum of S over the time interval. Barrier options are options which disappear when the underlying asset hit a pre specified barrier. In an ideal market situation we apply the martingale approach to price those options under the risk neutral probability measure.

4.1 Asian Options

An Asian option is an option whose payoff depends on a suitably defined average of the asset over a certain time period. We consider the geometric average rate call option whose payoff is given by

$$h_1 = \left(\exp \left\{ \frac{1}{2} \int_0^T \log(S_u) du \right\} - K \right)^+$$

Let C_t^1 denote the price at time t . Letting P^* be the equivalent martingale measure of \tilde{S} . By equation (2.10) we have

$$C_t^{(1)} = e^{-r(T-t)} \mathbb{E}^*[h_1 | \mathcal{F}_t] \tag{4.1}$$

Letting

$$I_t = \int_0^T \log(S_u) du \tag{4.2}$$

Then

$$h_1 = \left(\exp \left\{ \frac{1}{T} I_t + \frac{1}{T} \int_0^T \log(S_u S_u^{-1}) du + \frac{T-t}{T} \log S_t \right\} - K \right)^+ = (X_t Y_t - K)^+$$

Where

$$X_t = e^{\frac{I_t}{T}} S_t^{\left(\frac{T-t}{T}\right)}, Y_t = \frac{1}{T} \int_0^T \log(S_u S_u^{-1}) du \tag{4.3}$$

recall that

$$S_t = S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t^* \right\},$$

we have

$$Y_t = \exp \left\{ \frac{1}{T} \int_t^T \left[\left(r - \frac{\sigma^2}{2} \right) (u - t) + \sigma (W_u^* - W_t^*) \right] du \right\} = e^{r^*(T-t) + Z_t}$$

with

$$r^* = \left(r - \frac{\sigma^2}{2} \right) \frac{T-t}{2T}, \quad Z_t = \frac{1}{T} \int_t^T \sigma (W_u^* - W_t^*) du \tag{4.4}$$

Since Z_t is independent of \mathcal{F}_t and X_t is \mathcal{F}_t -measurable, by theorem (2.1) we have $C_t^{(1)} = e^{-r(T-t)} F(t, X_t)$, where

$$F(t, x) = \mathbb{E}^* \left[\left(x e^{r^*(T-t) + Z_t} - K \right)^+ \right]$$

Z_t is a Gaussian random variable with mean zero and variance $\sigma^2(T - t)$ with

$$\sigma^2 = \frac{\sigma^2(T-t)^2}{3T} \tag{4.5}$$

We now have

$$F(t, x) = e^{xr^*(T-t)} \int_{-\infty}^{\infty} \left(e^{\sigma^* \sqrt{T-ty}} - Ke^{-r^*(T-t)} \right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \tag{4.6}$$

$$= xe^{(r^* + \frac{\sigma^{*2}}{2})(T-t)} N(d_1^*) - KN(d_2^*)$$

Where

$$d_1^* = \frac{\log(x/K) + (r^* + \frac{\sigma^{*2}}{2})(T-t)}{\sigma^* \sqrt{T-t}}, \quad d_2^* = \frac{\log(x/K) + r^*(T-t)}{\sigma^* \sqrt{T-t}} \tag{4.7}$$

We turn to the pricing of an arithmetic average rate call option whose payoff is given by

$$h_2 = \left(\frac{1}{T} \int_0^T S_u du - K \right)^+$$

Letting $C_t^{(2)}$ denote the price at time t . By equation (2.10) we have

$$C_t^{(2)} = \mathbb{E}^* \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \mid \mathcal{F}_t \right] \tag{4.8}$$

From [8; 9], we have

$$M_t = \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \mid \mathcal{F}_t \right] \tag{4.9}$$

Since $\int_t^T S_u^{-1} S_u du$ is independent of \mathcal{F}_t , we have

$$M_t = S_t \mathbb{E}^* \left[\left(\int_t^T S_t^{-1} S_u du - S_t^{-1} (TK - \int_0^t S_u du) \right)^+ \mid \mathcal{F}_t \right] = S_t f(t, Y_t)$$

Where

$$f(t, x) = \mathbb{E}^* \left[\left(\int_t^T S_t^{-1} S_u du - x \right)^+ \mid \mathcal{F}_t \right], Y_t = S_t^{-1} (TK - \int_0^t S_u du) \tag{4.10}$$

Since

$$dS_t^{-1} = -S_t^{-2} dS_t + S_t^{-3} d\langle S, S \rangle_t = S_t^{-1} [(\sigma^2 - r)dt - \sigma dW_t^*]$$

We have

$$dY_t = \left(TK - \int_0^t S_u du \right) dS_t^{-1} - dt = Y_t [(\sigma^2 - \sigma)dW_t^*] - dt$$

By Ito's formula, we get

$$dM_t = S_t df(t, Y_t) + f(t, Y_t) dS_t + d\langle S, (\cdot, Y_t) \rangle$$

$$= S_t \left[f_t(t, Y_t) dt + f_x(t, Y_t) dY_t + \frac{1}{2} f_{xx}(t, Y_t) d\langle Y, Y \rangle_t \right] + f(t, Y_t) dS_t + d\langle S, f(\cdot, Y_t) \rangle_t$$

where $d\langle Y, Y \rangle_t = Y_t^2 \sigma^2 dt$;

$$d\langle S, f(\cdot, Y_t) \rangle_t = f_x(t, Y_t) d\langle S, Y \rangle_t = -f_x(t, Y_t) S_t Y_t \sigma^2 dt$$

$$\sim S_t \left(f_t - (1 + rY_t) f_x + \frac{1}{2} \sigma^2 Y_t^2 f_{xx} + rf \right) (t, Y_t) dt.$$

Since M_t is a martingale under P^* , the right hand side of the above term will vanish which lead to the following PDE

$$f_t - (1 + rx) f_x + \frac{\sigma^2 x^2}{2} f_{xx} + rf = 0, \quad x \geq 0 \tag{4.11}$$

Satisfying the following boundary condition:

$$f(T, x) = 0 \quad \text{and} \quad f(t, 0) = r^{-1} (e^{r(T-t)} - 1),$$

Since

$$\mathbb{E}^* [S_t^{-1} S_u] = \mathbb{E}^* \left[\exp \left\{ \sigma (S_u^* - W_t^*) - \left(\frac{\sigma^2}{2} - r \right) (u - t) \right\} \right] = e^{r(u-t)}$$

and

$$f_x(t, x) = -\mathbf{P}^* \left(\int_t^T S_t^{-1} S_u du \geq x \right),$$

We have the next boundary condition:

$$\lim_{x \rightarrow \infty} f_x(t, x) = 0.$$

4.2 Lookback option

Lookback options are options whose payoff at expiration depends on the maximum or minimum realized asset over the option's life. We consider the lookback rate call option whose payoff is defined by

$$h_1 = \max_{0 \leq s \leq T} (S_s - K)^+$$

Letting C_t denote its price at time t . By equation (2.10) we have

$$C_t = e^{-r(T-t)} \mathbb{E}^* [h_1 \mid \mathcal{F}_t] \tag{4.12}$$

We let $\lambda = r - \frac{1}{2} \sigma^2$, and set $M_t = \max_{0 \leq s \leq t} S_s$; $L_t = \max_{t \leq s \leq T} S_s$

then M_t is \mathcal{F}_t -measurable. Since

$$S_t^{-1} L_t = \exp \{ \max_{t \leq s \leq T} (\sigma (W_s^* - W_t^*)) + \lambda (s - t) \}$$

Since $S_t^{-1} L_t$ is independent of \mathcal{F}_t . By using these notation and letting $K_t = \max(M_t, K)$, we have

$$C_t = e^{-r(T-t)} \mathbb{E}^* [h_1 \mid \mathcal{F}_t]$$

$$C_t = e^{-r(T-t)} \mathbb{E}^* [\max(M_t, K) - K \mid \mathcal{F}_t]$$

$$\begin{aligned}
 &= e^{-r(T-t)} \mathbb{E}^* [\max(K_t, L_t) - K_t \mid \mathcal{F}_t] + K_t - K \\
 &= e^{-r(T-t)} \mathbb{E}^* [\max(L_t, K_t)^+ \mid \mathcal{F}_t] + K_t - K \\
 &= e^{-r(T-t)} S_t \mathbb{E}^* [\max(S_t^{-1} L_t, S_t^{-1} K_t)^+ \mid \mathcal{F}_t] + K_t - K \\
 &= e^{-r(T-t)} S_t \mathbb{E}^* [(S_t^{-1} L_t - y)^+] \Big|_{y=S_t^{-1} K_t} + K_t - K \\
 &= e^{-r(T-t)} S_t \mathbb{E}^* \left[\exp \left\{ (\max_{t \leq s \leq T} (\sigma(W_s^* - W_t^*) + \lambda(s-t) - y)^+) \right\} \right] + K_t - K \\
 &= e^{-r(T-t)} S_t \mathbb{E} \left[\exp \left\{ (\max_{0 \leq s \leq T-t} (\sigma W_s + \lambda_s - y)^+) \right\} \right] + K_t - K \\
 &= -y(1 - F_{T-t}(\log y)) + \int_{\log y}^{\infty} e^x F'_{T-t}(x) dx
 \end{aligned}$$

See [8; 10] for detail computation, we get the following formula

$$C_t = S_t \left(N(d_3) \left(1 + \frac{\sigma^2}{2r} \right) \right) + K_t e^{-r(T-t)} \left(N(d_1) - \frac{\sigma^2}{2r} (S_t^{-1} K_t)^{\left(\frac{2r}{\sigma^2} \right)^{-1}} N(d_2) \right) - e^{-r(T-t)} K \quad (4.13)$$

Where

$$\begin{aligned}
 d_1 &= \frac{\log\left(\frac{K_t}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
 d_2 &= \frac{-\log\left(\frac{K_t}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
 d_3 &= \frac{-\log\left(\frac{K_t}{S_t}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}
 \end{aligned}$$

5. Conclusion

In this paper, we propose the martingale approach to the pricing of contingent claims in a Black-Scholes model setting. The pricing formulas for the celebrated Black-Scholes equation were presented. Explicit formulas for the valuation of European, Asian and Look back option using martingale approach in an ideal market situation were derived.

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