A Cone-Increasing Function Approach to Optimality Conditions in Composite Vector Optimization

Mritunjay Kumar[1], Pooja Louhan[2]

[1]Department of Mathematics, L. S. College, B.R.A. Bihar University, Muzaffarpur, Bihar- 842001, India

[2]Department of Mathematics, R. D. S. College, B.R.A. Bihar University, Muzaffarpur, Bihar- 842001, India

Abstract: ***In this paper we establish optimality conditions for a composite vector optimization problem over cones where the objective function and constraint function both are composition of two vector functions. We have introduced an auxiliary problem and use the notion of cone-increasing functions to derive a Fritz John type necessary condition for a weak minimum of the composite problem. Under a Slater-type constraint qualification we strengthen this to develop Karush Kuhn Tucker type necessary condition.***

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1. Introduction

Composite Vector Optimization problem over Cones are those vector optimization problems where the objective functions and/or constraint functions are taken as composition of functions. Many optimization problems arising from various directions can be formulated as optimization of some compositions of functions, such as, minimax problem, penalty method for constrained optimization problem, etc.

In 1991 Jeyakumar [3] initiated the study of composite optimization problems. He took up a nonsmooth optimization problem where the objective function and the constraint functions are compositions of locally Lipschitz and Gˆateaux differentiable functions and presented necessary and sufficient optimality conditions. Jeyakumar and Yang [4] considered multi-objective optimization problems where the objective and the constraints are compositions of convex and locally Lipschitz functions and Gˆateaux differentiable functions and established optimality and duality results. Jeyakumar and Yang [5] presented second-order optimality conditions for convex composite minimization problems in which the objective function is a composition of a lower semicontinuous convex function and a C1,1 function. Yang and Jeyakumar [9] established first-order optimality conditions for a weakly efficient solution of a convex composite multi- objective optimization problem subject to a closed convex constraint set via scalarization and then extended these conditions to derive second-order optimality conditions. Reddy and Mukherjee [6] proved generalized Karush-Kuhn-Tucker sufficient optimality conditions and duality results for a non- smooth composite multi- objective programs using V-ρ-invexity conditions. Bot¸ et. al. [1] considered a convex composite optimization problem where the objective function is the composition of a cone-increasing convex function and a cone-convex function and the constraint is taken to be a cone-convex function. They established optimality conditions and duality results for this problem using Fenchel-Lagrangian duality results. Then Bot¸ et. al. [2] studied a convex multi- objective optimization problem where each component of the objective function is the composition of a cone-increasing convex function and a cone-convex function and the constraint is taken to be a cone-convex function. They proved necessary and sufficient optimality conditions for the weakly efficient solutions of this problem and gave duality results for a multi- objective dual. Recently, Tang and Zhao [8] considered a class of composite multi-objective non- smooth optimization problems with cone constraints and established necessary and sufficient optimality conditions under cone-generalized invexity and generalized null space condition.

In this paper we have considered a composite vector optimization problem where both, the objective and the constraint function, are composition of two vector functions, the same problem was considered by Tang and Zhao [8]. Rather than following their approach, we establish necessary optimality conditions for a point to be a weak minimum of this problem using the notion of cone-increasing functions. We relate the composite problem to an auxiliary non-composite problem and prove the Fritz John type conditions and under a Slater-type constraint qualification we further obtain a Karush Kuhn Tucker type necessary condition.

2. Notations and Definitions

Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed ($K \cap (-K) = \{0\}$) cone such that int $K \neq \phi$, where int K denotes the interior of K. The positive dual cone K^+ and strict positive dual cone K^{s+} of K are defined as follows:

$$K^+ = \{\lambda \in \mathbb{R}^m : \lambda^T x \geq 0, \text{ for all } x \in K\},$$

$$K^{s+} = \{\lambda \in \mathbb{R}^m : \lambda^T x > 0, \text{ for all } x \in K \setminus \{0\}\}$$

In this section, we recall some of the basic definitions and results, which are to be used throughout the paper. Let $f: \mathbb{R}^n \to \mathbb{R}^m$, where $f = (f_1, f_2, \ldots, f_m)^T$ and $h: \mathbb{R}^n \to \mathbb{R}$.

The function h is said to be locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ if there exists a nonnegative constant L and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for all $x, y \in N(\bar{x})$, we have

$$\| h(x) - h(y) \| \leq L \| x - y \|$$

Also the function h is said to be locally Lipschitz on $\mathbb{R}^n$, if it is locally Lipschitz at every $\bar{x} \in \mathbb{R}^n$.

The vector-valued function f is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ if for each i = 1, 2,...,m, f_i is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ and f is called locally Lipschitz on $\mathbb{R}^n$, if it is locally Lipschitz at every $\bar{x} \in \mathbb{R}^n$.

Now we briefly describe the Clarke's [11] notion of generalized directional derivative and subdifferential of a locally Lipschitz function.

The Clarke generalized directional derivative of a locally Lipschitz function h at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is given as

$$h^{\circ}(\bar{x}; d) = \limsup_{y \to \bar{x}, t \to 0^+} \frac{h(y+td) - h(y)}{t},$$

where $y \in \mathbb{R}^n$ and $t > 0$.

The Clarke generalized gradient or the Clarke subdifferential of a locally Lipschitz function h at $\bar{x} \in \mathbb{R}^n$ is given as

$$\partial^c h(\bar{x}) = \left\{ \xi \in \mathbb{R}^n : h^{\circ}(\bar{x}, d) \geq \xi^T d, \quad \text{for all } d \in \mathbb{R}^n \right\}$$

Furthermore, $\partial^c h(\bar{x})$ is a nonempty convex compact subset of $\mathbb{R}^n$ and

$h^{\circ}(\bar{x}; d) = \max \{\xi^T d : \xi \in \partial^c h(\bar{x})\}$.

The Clarke generalized directional derivative of the vector-valued locally Lipschitz function f: $\mathbb{R}^n \to \mathbb{R}^m$ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is given by,

$$f^{\circ}(\bar{x}; d) = (f_1^o(\bar{x}; d), f_2^o(\bar{x}; d), \ldots, f_m^o(\bar{x}; d))^T$$

The Clarke generalized jacobian of a vector-valued locally Lipschitz function f at $\bar{x} \in \mathbb{R}^n$, denoted by $\partial^c f(\bar{x})$, is a nonempty convex compact subset of $\mathbb{R}^{m+n}$, the space of $m \times n$ matrices and it is defined as

$\partial^c f(\bar{x}) = \partial^c f_1(\bar{x}) \times \partial^c f_2(\bar{x}) \times \ldots \times \partial^c f_m(\bar{x})$,

where the latter denotes the set of all matrices whose i^{th} row belongs to the set $\partial^c f_i(\bar{x})$ for each i = 1,2,...,m

Definition 2.1: The function f is said to be $(P, K)-$ increasing (or nondecreasing on $\mathbb{R}^n$ *with respect to (P, K)* [7, Definition 4.1]), if for all $x, u \in \mathbb{R}^n$,

$x - u \in P \Longrightarrow f(x) - f(u) \in K$

For all the below definitions we assume that f is a locally Lipschitz function and $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.2: [12] The function f is said to be K- generalized invex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$ and $A \in \partial^c f(\bar{x})$,

$$f(x) - f(\bar{x}) - A_{\eta}(x, \bar{x}) \in K$$

Definition 2.3: [12] The function f is said to be K- non-smooth invex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$,

$$f(x) - f(\bar{x}) - f^{\circ}(\bar{x}; \eta(x, \bar{x})) \in K.$$

Definition 2.4: [12] The function f is said to be $K-$nonsmoth pseudoinvex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$,

$$-f^{o}(\bar{x}; \eta(x, \bar{x})) \notin \text{int}K \Rightarrow -[f(x) - f(\bar{x})] \notin \text{int}K.$$

Definition 2.5: [12] The function f is said to be $K-$nonsmoth quasiinvex at $\bar{x} \in \mathbb{R}^n$, if there exists η such that for every $x \in \mathbb{R}^n$

$$f(x) - f(\bar{x}) \notin \text{intK} \Rightarrow -f^{o}(\bar{x}; \eta(x, \bar{x})) \in K$$

3. Necessary Optimalty Conditions

We consider the following composite vector optimization problem studied by Tang and Zhao [8]

(CVP) K-Minimize $(f^{o} F)(x) = ((f_1 \circ F)(x), (f_2 \circ F)(x), \ldots, (f_m \circ F)(x))^T$

Subject to

$-(g^{o} G)(x) = -((g_1 \circ G)(x), (g_2 \circ G)(x), \ldots, (g_p \circ G)(x))^T \in Q,$

where $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz vector-valued functions such that $f = (f_1, f_2, \ldots, f_m)^T$, $g = (g_1, g_2, \ldots, g_p)^T$. $F: \mathbb{R}^s \to \mathbb{R}^n$ and $G: \mathbb{R}^s \to \mathbb{R}^n$ are differentiable functions such that $F = (F_1, F_2, \ldots, F_n)^T$ and $G = (G_1, G_2, \ldots, G_n)^T$. $K \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^p$ are closed convex pointed cones with nonempty interiors. The feasible set of (CVP) is given by

$S_0 = \{x \in S : -(g \circ G)(x) \in Q\}$.

Definition 3.1: A point $\bar{x} \in S_0$ is called *a*

1) Weak minimum of (CVP), if for all $x \in S_0$,

$(f \circ F)(\bar{x}) - (f \circ F)(x) \notin \text{intK}$

2) Minimum of (*CVP*), if for all $x \in S_0$,

$(f \circ F)(\bar{x}) - (f \circ F)(x) \notin K \setminus \{0\}$

In order to establish necessary optimality conditions we consider the following problem:

(CVP)[1] K- Minimize $f(y) = (f_1(y), f_2(y), \ldots, f_m(y))^T$

Subject to
$-g(z) = -(g_1(z), g_2(z), \ldots, g_p(z))^T \in Q,$
$-(F(x) - y) = -(F_1(x) - y_1, F_2(x) - y_2, \ldots, F_n(x) - y_n)^T \in P_1,$
$-(G(x) - z) = -(G_1(x) - z_1, G_2(x) - z_2, \ldots, G_n(x) - z_n)^T \in P_2,$
$x \in \mathbb{R}^s,\ y, z \in \mathbb{R}^n$

where P_1, $P_2 \subseteq \mathbb{R}^n$ are closed convex pointed cones with nonempty interiors.

The following lemma shows that if a weak minimum of the problem (CVP) exists, then a weak minimum of $(CVP)^1$ also exists

Lemma 3.1: Let $\bar{x}$ is a weak minimum of (CVP), f be (P_1, K)− increasing and g is (P_2, Q)− increasing. Then there exist $\bar{y}, \bar{z} \in \mathbb{R}^n$ such that $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimum of $(CVP)^1$.

Proof: Suppose $\bar{x}$ is a weak minimum of (CVP). Now we will prove that $(\bar{x}, \bar{y} = F(\bar{x}), \bar{z} = G(\bar{x}))$ is a weak minimum of $(CVP)^1$.

Clearly, $(\bar{x}, \bar{y}, \bar{z})$ is feasible for $(CVP)^1$.

Now, suppose $(\bar{x}, \bar{y}, \bar{z})$ is not a weak minimum of $(CVP)^1$, then there exists (x, y, z) feasible for $(CVP)^1$ such that

$$f(\bar{y}) - f(y) \in \text{int}K \qquad (1)$$

Since (x, y, z) is feasible for $(CVP)^1$, we have

$z - G(x) \in P_2$
and
$-g(z) \in Q$
Using the fact that g is (P_2, Q)- increasing, we get

$g(z) - g(G(x)) \in Q$

Adding above two relations, we get
$-g(G(x)) \in Q$

that is, x is feasible for (CVP)

Also, due to feasibility of (x, y, z) for $(CVP)^1$, we have

$y - F(x) \in P_1$.

Now, using the fact that f is (P_1, K)- increasing, we get

$$f(y) - f(F(x)) \in K \qquad (2)$$

Adding (1) and (2), we have that

$f(\bar{y}) - f(F(x)) = f(F(\bar{x})) - f(F(x)) \in intK,$

which is a contradiction to the fact that $\bar{x}$ is a weak minimum of (CVP)

The next lemma will be used in the proof of Fritz John type necessary optimality conditions for a point to be a weak minimum of (CVP) and it is proved on the lines of Craven [10].

Lemma 3.2: *Consider the problem*

K-Minimize $\Phi(x)$

subject to

$-\Psi(x) \in Q,$

where $\Phi: \mathbb{R}^n \to \mathbb{R}^m$ *and* $\Psi: \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz functions. Let $\bar{x}$ be a weak minimum of this problem, then there exists $(0,0) \neq (\lambda, \mu) \in K^+ \times Q^+$ such that

$0 \in \partial^c (\lambda^T \Phi + \mu^T \Psi (\bar{x})$

$(\mu^T \Psi (\bar{x}) = 0.$

Proof: This Lemma can be proved by taking $F = \Phi$, $G = \Psi$, $Q = K$, $S = Q$ and $T = \phi$ (the empty set) in the problem (P) considered by Craven [10] and applying Theorem 2 given by him.

Below we have the Fritz John type necessary optimality conditions for the problem (CVP).

Theorem 3.1: Let $\bar{x}$ be a weak minimum of (CVP). If f is (P_1, K)− increasing and g is (P_2, Q)- increasing, then there exist $\bar{\lambda} \in K^+$, $\bar{\mu} \in Q+$ with $(\bar{\lambda}, \bar{\mu}) \neq 0$, such that

$$0 \in \nabla F(\bar{x})^T \cdot \partial^c(\bar{\lambda}^T f)(F(\bar{x})) + \nabla G(\bar{x})^T \cdot \partial^c(\bar{\mu}^T g)(G(\bar{x})) \qquad (3)$$

$$(\bar{\mu}^T g)(G(\bar{x})) = 0 \qquad (4)$$

Note: $\nabla F(\bar{x}) = [\nabla F_1(\bar{x}), \nabla F_2(\bar{x}), \ldots, \nabla F_n(\bar{x})]^T$ is the $n \times s$ Jacobian matrix of F at $\bar{x}$ and for each i = $1,2,\ldots,n$,

$$\nabla F_i(\bar{x}) = \left(\frac{\partial F_i}{\partial x_1}, \frac{\partial F_i}{\partial x_2}, \ldots, \frac{\partial F_i}{\partial x_s}\right)^T$$

is the $s \times 1$ Gradient vector of F_i at $\bar{x}$. Also

$$\nabla F(\bar{x}) \cdot \partial^c(\bar{\lambda}^T f)(F(\bar{x})) \text{ denotes the set } \{\nabla F(\bar{x})^T \xi : \xi \in \partial^c(\bar{\lambda}^T f)(F(\bar{x}))\}.$$

Proof: Since $\bar{x}$ is a weak minimum of (CVP), therefore by Lemma 3.1, $(\bar{x}, \bar{y} = F(\bar{x}), \bar{z} = G(\bar{x}))$ is a weak minimum of $(CVP)^1$ and hence by Lemma 3.2 there exist $\bar{\lambda} \in K^+$, $\bar{\mu} \in Q^+$, $\bar{\alpha} \in P_1^+$ and $\bar{\beta} \in P_2^+$ with $(\bar{\lambda}, \bar{\mu}, \bar{\alpha}, \bar{\beta}) \neq 0$ Such that

$$0 \in \partial^c_{(x,y,z)} \left\{ (\bar{\lambda}^T f)(\bar{y}) + (\bar{\mu}^T g)(\bar{z}) + \bar{\alpha}^T (F(\bar{x}) - \bar{y}) + \bar{\beta}^T (G(\bar{x}) - \bar{z}) \right\} \qquad (5)$$

$$(\bar{\mu}^T g)(\bar{z}) = 0 \qquad (6)$$

$$\bar{\alpha}^T (F(\bar{x}) - \bar{y}) = 0$$

And

$$\bar{\beta}^T (G(\bar{x}) - \bar{z}) = 0$$

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By Clarke [11], we have that

$$\partial^c_{(x,y,z)}\left\{(\overline{\lambda}^T f)(\overline{y})+(\overline{\mu}^T g)(\overline{z})+\overline{\alpha}^T(F(\overline{x})-\overline{y})+\overline{\beta}^T(G(\overline{x})-\overline{z})\right\}\subseteq$$

$$\partial^c_x\left\{(\overline{\alpha}^T F)(\overline{x})+(\overline{\beta}^T G)(\overline{x})\right\}\times\partial^c_y\left\{(\overline{\lambda}^T f)(\overline{y})-\overline{\alpha}^T\overline{y}\right\}\times\partial^c_z\left\{(\overline{\mu}^T g)(\overline{z})-\overline{\beta}^T\overline{z}\right\}$$

Hence (5) can be written as

$$\begin{aligned}0 &\in \partial^c\left\{(\overline{\lambda}^T f)(\overline{y})-\overline{\alpha}^T\overline{y}\right\}\subseteq\partial^c(\overline{\lambda}^T f)(\overline{y})-\partial^c\left\{\overline{\alpha}^T\overline{y}\right\}\\ 0 &\in \partial^c\left\{(\overline{\mu}^T g)(\overline{z})-\overline{\beta}^T\overline{z}\right\}\subseteq\partial^c(\overline{\mu}^T g)(\overline{z})-\partial^c\left\{\overline{\beta}^T\overline{z}\right\}\\ 0 &\in \partial^c\left\{(\overline{\alpha}^T F)(\overline{x})+(\overline{\beta}^T G)(\overline{x})\right\}\subseteq\partial^c\left\{(\overline{\alpha}^T F)(\overline{x})\right\}+\partial^c\left\{(\overline{\beta}^T G)(\overline{x})\right\}\end{aligned}$$

That is,

$$\nabla(\overline{\alpha}^T F)(\overline{x})+\nabla(\overline{\beta}^T G)(\overline{x})=0$$

or,

$$\nabla F(\overline{x})^T\overline{\alpha}+\nabla G(\overline{x})^T\overline{\beta}=0 \qquad (7)$$

$$0\in\partial^c(\overline{\lambda}^T f)(F(\overline{x}))-\{\overline{\alpha}\}$$

$$0\in\partial^c(\overline{\mu}^T g)(G(\overline{x}))-\{\overline{\beta}\}$$

So,
$\overline{\alpha}\in\partial^c(\overline{\lambda}^T f)(F(\bar{x}))$ and $\overline{\beta}\in\partial^c(\overline{\mu}^T g)(G(\bar{x}))$ (8)

and hence using (7), we have

$0=\nabla F(\bar{x})^T\overline{\alpha}+\nabla G(\bar{x})^T\overline{\beta}\in\nabla F(\bar{x})^T\cdot\partial^c(\overline{\lambda}^T f)(F(\bar{x}))+\nabla G(\bar{x})^T\cdot\partial^c(\overline{\mu}^T g)(G(\bar{x}))$.

Also, from (6), we have

$(\overline{\mu}^T g)(G(\bar{x}))=0$

Now it only remains to show that $(\overline{\lambda},\overline{\mu})\neq 0$. Let, if possible, $\overline{\lambda}=0$ and $\overline{\mu}=0$. Then from (8) we get that $\overline{\alpha}=0=\overline{\beta}$, which is a contradiction to the fact that $(\overline{\lambda},\overline{\mu},\overline{\alpha},\overline{\beta})\neq 0$. Hence $(\overline{\lambda},\overline{\mu})\neq 0$.

Definition 3.2: ***(Slater-type Constraint Qualification)*** The function g is said to satisfy Slater-type Constraint Qualification at $\bar{x}$ if for every $B\in\partial^c g(G(\bar{x}))$, there exists $x^*\in\mathbb{R}^s$ such that $-B\nabla G(\bar{x})x^*\in\operatorname{int}Q$.

Theorem 3.2: Let $\bar{x}$ be a weak minimum of (CVP). If f is (P_1, K) - increasing, g is (P_2, Q)- increasing and g satisfies Slater-type Constraint Qualification at x, then there exist $\overline{\lambda}\in K^+\backslash\{0\}$ and $\overline{\mu}\in Q^+$, such that (3) and (4) hold.

Proof: Since $\bar{x}$ is a weak minimum of (CVP), f is (P_1, K) − increasing and g is (P_2, Q)−increasing, therefore by Theorem 3.1 there exist $\overline{\lambda}\in K^+$, $\overline{\mu}\in Q^+$ with $(\overline{\lambda},\overline{\mu})\neq 0$, such that (3) and (4) hold

Now the only thing remained to be proved is that if g satisfies Slater-type Constraint Qualification at $\bar{x}$, then
$\overline{\lambda}\neq 0$.

Let, if possible, $\overline{\lambda}=0$. Then from (3), we have

$0\in\nabla G(\bar{x})^T\cdot\partial^c(\overline{\mu}^T g)(G(\bar{x}))$.

So, there exists $\overline{\zeta}\in\partial^c(\overline{\mu}^T g)(G(\bar{x}))$, such that

$$\nabla G(\bar{x})^T\overline{\zeta}=0 \qquad (9)$$

Now from the chain rule for subdifferentials in [11], we have

$\partial^c(\mu^T g)(G(u))=\mu^T\partial^c g(G(u))=\{\mu^T B: B\in\partial^c g(G(u))\}$

Hence we can say that there exists $\overline{B}\in\partial^c g(G(\bar{x}))$, such that $\overline{\zeta}=(\overline{\mu}^T\overline{B})^T$ and substituting it in (9), we get

$$\overline{\mu}^T\overline{B}\nabla G(\bar{x})x^*=0 \qquad (10)$$

Since g satisfies Slater-type Constraint Qualification at $\bar{x}$, so there exists $x^*\in\mathbb{R}^s$ such that $-\overline{B}\nabla G(\bar{x})x^*\in\operatorname{int}Q$, which implies that
$\overline{\mu}^T\overline{B}\nabla G(\bar{x})x^*<0$.

But this not possible as from (10) we have that

$\overline{\mu}^T\overline{B}\nabla G(\bar{x})x^*=0$.

Therefore $\overline{\lambda}\neq 0$.

4. Concluding Remarks

We have established Fritz John type (Theorem 3.1) and under a Slater-type constraint qualification, we have established Karush Kuhn Tucker type (Theorem 3.2) necessary optimality conditions for a point to be weak minimum of the composite vector optimization problem (CVP) using the notion of cone-increasing functions and an auxiliary problem $(CVP)_1$.

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