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Degree Distance of Grid Graphs

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Abstract: ***The degree distance index is a distance-based topological invariant that extends the Wiener index by incorporating vertex degrees. In this paper, we determine the degree distance of grid graphs ($P_m \times P_n$). A closed-form formula for the degree distance is established by using the transmission of vertices and the degree classification of grid vertices.***

Keywords: Transmission of a vertex, Degree Distance, Grid Graph

1. Introduction

Graph theory has a wide range of real-world applications [6]. It provides different techniques for solving problems involving connectivity, optimization, and graph structure. Graph invariants of molecular graphs are useful in analyzing the physicochemical properties of chemical compounds. A topological index is a numerical value assigned to a graph according to a specific rule that remains unchanged under graph isomorphism. It characterizes structural features of molecular topology and plays an important role in chemical applications, particularly in quantitative structure–property relationship (QSPR) and quantitative structure–activity relationship (QSAR) studies [7, 8].

In recent years, various classes of graph indices, including degree-based and distance-based indices, have been extensively studied. Distance-based topological indices are of particular interest in graph theory and chemical graph theory because they describe structural information through vertex distances. The degree distance index, introduced by Dobrynin and Kochetova [9], is a degree-weighted analogue of the Wiener index and has been widely studied because of its theoretical significance and applications.

Several studies have investigated degree distance and related indices for trees, composite graphs, and graph transformations [1–5]. These results motivate the study of explicit degree-distance formulae for structured graph families. In this paper, we derive a closed-form expression for the degree distance of the rectangular grid graph $P_m \times P_n$.

2. Preliminaries

Let $G = (V, E)$ be a simple connected graph. The degree $d(v)$ of a vertex v is the number of edges incident with v. The distance $d(u, v)$ between vertices u and v is the length of a shortest u–v path. The *transmission* (or status) of a vertex v is the sum of its distances to all other vertices. The degree distance index combines the degree of each vertex with its transmission, thereby incorporating both local and global structural information of the graph.

Definition 2.1
The transmission of a vertex v in G is

$$T(v) = \sum_{\{u,v\}\in V} d(u,v)$$

Definition 2.2.
Degree distance of a connected graph G is

$$\mathrm{DD}(G) = \sum_{v\in V(G)} d(v)T(v)$$

where $d(v)$ is degree of vertex v.

Definition 2.3. Grid graph $P_m \times P_n$
An $m \times n$ Grid graph is cartesian product of P_m and P_n for $m, n \geq 2$. Its vertex set is $V(P_m \times P_n) = \{(i, j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Figure 1 shows the Grid graph $P_5 \times P_5$.

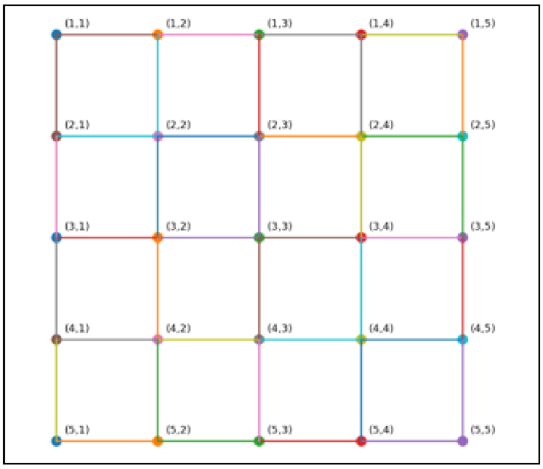

Figure 1: Grid graph $P_5 \times P_5$

In the $m \times n$ grid graph, vertices can naturally be classified according to their locations. In an $m \times n$ grid graph, Vertices are classified according to their positions. The four corners have degree 2, the boundary vertices excluding corners have degree 3, and the internal vertices have degree 4. These vertex types can be divided as in the following table 1.

Since the degree of each vertex depends only on its position in the grid, it is convenient to determine the total transmission of each vertex class separately. Accordingly, we first compute the transmission of boundary vertices, followed by that of corner vertices, then that of internal vertices and finally sum these results to obtain the degree distance index of the grid graph.

Table 1: *Classification of vertices in Grid graph $P_m \times P_n$*

Vertex types	Number of vertices
Corner vertices	4
Boundary vertices excluding corners	$2(m + n - 4)$
Internal vertices	$(m - 2)(n - 2)$

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In this paper, the degree distance index of Grid graphs is determined by finding transmission of three types of vertices - i. e., corner vertices, boundary vertices excluding the corners and internal vertices.

3. Transmission of Vertex *v* in a Grid Graph

Let $G = P_m \times P_n$. Then each vertex v in G is (i, j) for $1 \le i \le m$, $1 \le j \le n$. Transmission of each vertex types are determined. For vertices (i, j) and (k, l), the distance in the Grid graph is the Manhattan distance

$$d((i, j), (k, l)) = |i - k| + |j - l|.$$

3.1 Transmission of Boundary vertices

In a Grid graph $P_m \times P_n$, there are $2(n-2)$ boundary vertices on the top and bottom sides and $2(m-2)$ boundary vertices on the left and right sides. Hence the total number of boundary vertices is $2(n-2) + 2(m-2) = 2(m + n - 4)$.

For a top boundary vertex $u = (1, j)$,

$$T(u) = \frac{m(m-1)}{2} + \frac{j(j-1)}{2} + \frac{(n-j)(n-j+1)}{2}.$$

Summing over $j = 2 \dots n-1$ gives

$$\sum_{j=2}^{n-1} T(1, j) = (n-2)\left(\frac{m(m-1)}{2}\right) + \sum_{j=2}^{n-1}\left\{\frac{j(j-1)}{2} + \frac{(n-j)(n-j+1)}{2}\right\}$$

By symmetry, the same sum is obtained for the bottom side. Similarly, the two vertical sides contribute the analogous expression with m and n interchanged. Hence the total transmission of all boundary vertices is

$$\sum_{Boundary} T(u) = \frac{2m^3 + 3m^2 n - 12m^2 + 3mn^2 - 6mn + 10m + 2n^3 - 12n^2 + 10n}{3} \quad 3.2$$

Transmission of Corner vertices

The four corner vertices of grid graph $P_m \times P_n$ are (1, 1), (1, n), (m, 1), (m, n).

Transmission of corner vertex (1, 1) is

$$T(1,1) = \sum_{i=1}^{m}(i-1) + \sum_{i=1}^{m}(j-1) = \frac{m^2 + n^2 - m - n}{2}.$$

Similarly, the transmissions of the other three corner vertices can be calculated. Thus, sum of the transmissions of all four corner vertices is $2(m^2 + n^2 - m - n)$.

$$\sum_{corner} T(u) = 2(m^2 + n^2 - m - n)$$

3.3 Transmission of Internal Vertices

All internal vertices of a Grid graph $P_m \times P_n$ are $u = (i, j)$, where $2 \le i \le m-1$ and $2 \le j \le n-1$. Transmission of all these internal vertices is $\sum_{Internal} T(u) = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} T(u)$

$$= \frac{(m-2)(n-2)(m^2 - m + n^2 - n)}{3}.$$

Theorem 3.1. For the grid graph $G = P_m \times P_n$ $(m, n \ge 2)$, the degree distance index of G is

$$DD(G) = \frac{mn}{3}(m^2 + n^2 - 2) + \frac{2}{3}(m^3 + n^3) - 2(m^2 + n^2).$$

Proof: We use the degree classification: corner vertices have degree 2, boundary vertices excluding corners have degree 3, and internal vertices have degree 4. Hence

$$DD(G) = \sum_u d(u)T(u)$$

$$= 3\sum_{Boundary} T(u) + 2\sum_{corner} T(u) + 4\sum_{Internal} T(u)$$

$$= 3.\left[\frac{2m^3 + 3m^2 n - 12m^2 + 3mn^2 - 6mn + 10m + 2n^3 - 12n^2 + 10n}{3}\right] + 2.\frac{m^2 + n^2 - m - n}{2} + 4.\frac{(m-2)(n-2)(m^2 - m + n^2 - n)}{3}$$

$$DD(G) = \frac{mn}{3}(m^2 + n^2 - 2) + \frac{2}{3}(m^3 + n^3) - 2(m^2 + n^2).$$

Thus, the degree distance of grid graph $P_m \times P_n$ is $\frac{mn}{3}(m^2 + n^2 - 2) + \frac{2}{3}(m^3 + n^3) - 2(m^2 + n^2)$.

4. Conclusion

In this study, the degree distance of grid graphs has been investigated using the structural and distance-related properties of these graphs. The regular and systematic nature of grid graphs makes them an important class for studying degree distance and developing efficient computational approaches. The results obtained provide useful insights into the relationship between the degrees of vertices and the distances between them, and they contribute to a better understanding of the degree distance of grid graphs.

The study also highlights the importance of degree distance as a graph invariant in analyzing the structural characteristics of complex networks. The methods used for grid graphs may be extended to other families of graphs and can provide a foundation for deriving general formulas and computational techniques. Further research may focus on higher-dimensional grid graphs, different boundary conditions, and other graph classes for which the degree distance is not yet fully characterized.

References

[1] Y. Alizadeh, A. Iranmanesh, and T. Došlić, “On some degree-and-distance-based graph invariants of trees,” *MATCH Communications in Mathematical and in Computer Chem-istry*, vol. 70, no. 3, pp. 859–874, 2013.

[2] T. Vetrík, “General degree distance of graphs,” *Mathematica Slovaca*, vol. 65, no. 5, pp. 991–1006, 2015.

[3] H. Hua, “On the degree distance of some composite graphs,” *Discrete Applied Mathematics*, vol. 160, no. 18, pp. 2731–2740, 2012.

[4] M. Zhang and S. Li, “Extremal properties of distance-based graph invariants for *k*-trees,” *Applied Mathematics and Computation*, vol. 306, pp. 236–245, 2017.

[5] S. Brezovnik and N. Tratnik, “New methods for

calculating the degree distance and the Gutman index," *Discrete Applied Mathematics*, vol. 268, pp. 93–104, 2019.

[6] Prathik, K. Uma, and J. Anuradha, "An overview of application of graph theory," *International Journal of ChemTech Research*, vol. 9, no. 2, pp. 242–248, 2016.

[7] P. C. Costa, J. S. Evangelista, I. Leal, and P. C. Miranda, "Chemical graph theory for property modeling in QSAR and QSPR | Charming QSAR & QSPR," *Mathematics*, vol. 9, no. 1, p. 60, 2020, doi: 10.3390/math9010060.

[8] E. Estrada and E. Molina, "Novel local (fragment-based) topological molecular descrip-tors for QSPR/QSAR and molecular design," *Journal of Molecular Graphics and Mod-elling*, vol. 20, no. 1, pp. 54–64, 2001.

[9] Dobrynin and A. A. Kochetova, "Degree distance of a graph: A degree analog of the Wiener index," *Journal of Chemical Information and Computer Sciences*, vol. 34, no. 5, pp. 1082–1086, 1994.

[10] Harshitha, S. D'Souza, and P. G. Bhat, "Degree distance index of class of graphs," *Communications in Combinatorics and Optimization*, vol. 10, no. 2, pp. 453–462, 2025.

[11] S. V. Patil *et al.*, "Generalized degree distance indices under graph operations," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 25, no. 6, pp. 1681–1694, 2022.

[12] Z. U. Khan and Q. Iqbal, "On extremal values of the *NkN* -degree distance index in trees," *Mathematics*, vol. 13, no. 14, p. 2284, 2025.

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