# Recurrence Relations for Marginal and Joint Moment Generating Functions of Generalized Order Statistics from Two-Parameter Exponential Distribution

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**Abstract:** ***In this paper we give some recurrence relations satisfied by marginal and joint moment generating functions of generalized order statistics from a two-parameter exponential distribution. These recurrence relations would hence enable one to compute all the higher order moments of the generalized order statistics from those of the lower order in a simple recursive way.***

**Keywords and Phrases:** Order statistics; record values; generalized order statistics; moment generating function; marginal moment generating function; joint moment generating function; single moments; product moments; recurrence relations; two-parameter exponential distribution

## 1. Introduction

A random variable X is said to have a two-parameter exponential distribution (Exp (μ, σ)) if its probability density function (pdf) is of the form

$$f(x) = \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, \quad \mu \le x < \infty, \ \sigma > 0 \quad (1.1)$$

and the cumulative distribution function is of the form

$$F(x) = 1 - e^{-\frac{(x-\mu)}{\sigma}}, \quad \mu \le x < \infty, \ \sigma > 0 \quad (1.2)$$

Note that for two- parameter exponential distribution we have

$$f(x) = \frac{1}{\sigma}(1 - F(x)), \quad \mu \le x < \infty, \quad \sigma > 0. \quad (1.3)$$

Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables with cdf F(x) and pdf f(x). Let $X_{j:n}$ denote the j-th order statistic of a sample ($X_1$, $X_2, \ldots, X_n$ ). Assume that $k > 0$, $n \in N$, $m \in Z$ and $\gamma_r = k + (n-r)(m+1) > 0$. If the random variables X(r,n,m,k), r = 1, 2, …,n, possess a joint pdf of the form

$$f^{X(1,n,m,k),\ldots,X(n,n,m,k)}(x_1, x_2, \ldots, x_n) = k\left(\prod_{r=1}^{n-1}\gamma_r\right)\left(\prod_{j=1}^{n-1}(1-F(x_j))^m f(x_j)\right)(1-F(x_n))^{k-1} f(x_n),$$
$$F^{-1}(0+) < x_1 \le \ldots \le x_n < F^{-1}(1), \quad (1.4)$$

then they are called generalized order statistics of a sample from a distribution with cdf F(x), (cf. Kamps (1995 a)).

For convenience, let us define X (0, n, m, k) = 0. It can be seen that for the case when m = 0, k = 1, we have the joint pdf of the ordinary order statistics, and when m = −1 we obtain the joint pdf of the k-th upper record values (cf. Dziubdziela and Kopociński (1976)). The pdf of the r-th generalized order statistic obtained on using (1.4) is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r - 1} f(x) \ g_m^{r-1}(F(x)), \ x \in R \quad (1.5)$$

Where

$$c_{r-1} = \prod_{j=1}^{r}\gamma_j, \qquad r = 1,2,\ldots,n,$$
$$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0,1),$$
$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & , m \ne -1 \\ -\log(1-x) & , m = -1 \end{cases}$$

The joint pdf of X (r, n, m, k) and X(s, n, m, k), r < s, is given by

$$f^{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}(1-F(x))^m f(x) \ g_m^{r-1}(F(x)) \ [h_m(F(y)) - h_m(F(x))]^{s-r-1}$$
$$.(1-F(y))^{\gamma_s - 1} f(y), \qquad x < y, \quad (1.6)$$

(cf. Kamps (1995 a,b)).

## 2. Relations for Marginal Moment Generating Functions

Let us denote the marginal moment generating function of $X(r,n,m,k)$ by $M_{X(r,n,m,k)}(t)$ and its j-th derivative by $M^{(j)}_{X(r,n,m,k)}(t)$ . The relation in (1.3) shall be employed to derive the recurrence relations for marginal moment generating functions of generalized order statistics from a two-parameter exponential distribution.

We shall first establish the existence of $M_{X(r,n,m,k)}(t)$ . Using (1.5), we get

$$M_{X(r,n,m,k)}(t)=\frac{c_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}e^{tx}(1-F(x))^{\gamma_r-1}\,f(x)\;g_m^{r-1}(F(x))\,dx\;. \tag{2.1}$$

Further, on using (1.1) and (1.2) in (2.1), we obtain

$$M_{X(r,n,m,k)}(t)=\frac{c_{r-1}}{(r-1)!}\frac{1}{\sigma}\left(\frac{-1}{m+1}\right)^{r-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\int_{\mu}^{\infty}e^{tx}\,e^{-\{(m+1)(r-1-j)+\gamma_r\}\frac{(x-\mu)}{\sigma}}\,dx$$

$$=\frac{c_{r-1}}{(r-1)!}\left(\frac{-1}{m+1}\right)^{r-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}e^{t\mu}\int_{0}^{\infty}e^{-z\,\{(m+1)(r-1-j)+\gamma_r-t\sigma\}}\,dz$$

$$=\frac{c_{r-1}}{(r-1)!}\left(\frac{-1}{m+1}\right)^{r-1}e^{t\mu}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\frac{1}{\{(m+1)(r-1-j)+\gamma_r-t\sigma\}}$$

$$=\frac{c_{r-1}}{(r-1)!}\left(\frac{-1}{m+1}\right)^{r-1}e^{t\mu}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\frac{1}{\{(m+1)(r-1-j)+\gamma_r\}}\cdot\left\{1-\frac{t\sigma}{(m+1)(r-1-j)+\gamma_r}\right\}^{-1},\qquad t\sigma<(m+1)(r-1-j)+\gamma_r\,.$$

It can easily be seen that $M_{X(r,n,m,k)}(t)$ exists and is analytic in t, $-\infty<t<\infty$ where $t\sigma<(m+1)(r-1-j)+\gamma_r$ , for j = 0, 1, 2,…., r–1.

**Theorem 2.1:** Fix a positive integer k. For n $\in N$ , m $\in Z$ , $1\le r\le n$ and j =1,2,…,

$$(\gamma_{r+1}-t\,\sigma)\,M^{(j)}_{X(r+1,n,m,k)}(t)=j\,\sigma M^{(j-1)}_{X(r+1,n,m,k)}(t)+\gamma_{r+1}\,M^{(j)}_{X(r,n,m,k)}(t). \tag{2.2}$$

**Proof:** For n $\in N$ , m $\in Z$ , $1\le r\le n$, it follows from (1.5) that

$$M_{X(r+1,n,m,k)}(t)=\frac{c_r}{r\,!}\int_{-\infty}^{\infty}e^{tx}(1-F(x))^{\gamma_{r+1}-1}f(x)\;g_m^r(F(x))\,dx \tag{2.3}$$

Integrating (2.3) by parts taking $\{(1-F(x))^{\gamma_{r+1}-1}f(x)\}$ as the part to be integrated, we get

$$M_{X(r+1,n,m,k)}(t)=M_{X(r,n,m,k)}(t)+\frac{c_r}{r!}\frac{t}{\gamma_{r+1}}\int_{-\infty}^{\infty}e^{tx}\,(1-F(x))^{\gamma_{r+1}}\;g_m^r(F(x))\,dx\;.$$

Upon using (1.3), we obtain

$$M_{X(r+1,n,m,k)}(t)=M_{X(r,n,m,k)}(t)+\frac{c_r}{r!}\frac{t\sigma}{\gamma_{r+1}}\int_{\mu}^{\infty}e^{tx}\,(1-F(x))^{\gamma_{r+1}-1}g_m^r(F(x))\,f(x)dx$$

$$=M_{X(r,n,m,k)}(t)+\frac{t\sigma}{\gamma_{r+1}}M_{X(r+1,n,m,k)}(t)\;. \tag{2.4}$$

Differentiating both sides of (2.4) j times with respect to t, we obtain for $1\le r\le n$, n $\in N$ , m $\in Z$ that

$$M^{(j)}_{X(r+1,n,m,k)}(t) = M^{(j)}_{X(r,n,m,k)}(t) + \frac{t\sigma}{\gamma_{r+1}} M^{(j)}_{X(r+1r,n,m,k)}(t) + \frac{\sigma}{\gamma_{r+1}} M^{(j-1)}_{X(r+1,n,m,k)}(t),$$

Which, when rewritten, gives the recurrence relation in equation (2.2).

**Remark 2.1:** Putting t = 0 in (2.2), we shall deduce the following recurrence relation for the single moments of generalized order statistics from a two-parameter exponential distribution**:**

$$E\left[X^j(r+1,n,m,k)\right] = \frac{j\sigma}{\gamma_{r+1}} E\left[X^{j-1}(r+1,n,m,k)\right] + E\left[X^j(r,n,m,k)\right]. \quad (2.5)$$

**Remark 2.2:** On setting m = –1 and t = 0 in (2.2), we shall obtain the following recurrence relation for single moments of k-th upper record values from a two-parameter exponential distribution:

$$E\left(Y_n^{(k)}\right)^j = \frac{j\sigma}{k} E\left(Y_n^{(k)}\right)^{j-1} + E\left(Y_{n-1}^{(k)}\right)^j$$

## 3. Recursive Computational Algorithm for Evaluation of Single Moments

We shall first obtain an exact expression for E[X(r,n,m,k)]. On using (1.5), we get

$$E[X(r,n,m,k)] = \frac{c_{r-1}}{(r-1)!}\int_{\mu}^{\infty} x(1-F(x))^{\gamma_r-1} f(x)\; g_m^{r-1}(F(x))\; dx, \quad (3.1)$$

which on using (1.2) and (1.3) and simplifying gives

$$E[X(r,n,m,k)] = \frac{c_{r-1}}{(r-1)!}\frac{1}{\sigma}\int_{\mu}^{\infty} x\, e^{-\frac{(x-\mu)}{\sigma}\gamma_r}\left[\frac{-(1-F(x))^{m+1}}{m+1} + \frac{1}{m+1}\right]^{r-1} dx$$

Now, expanding $\left[\frac{-(1-F(x))^{m+1}}{m+1} + \frac{1}{m+1}\right]^{r-1}$ binomially and using (1.2), we get

$$E[X(r,n,m,k)] = \frac{c_{r-1}}{(r-1)!}\frac{1}{\sigma}\left(\frac{1}{m+1}\right)^{r-1}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\int_{\mu}^{\infty} x\; e^{-\{(m+1)i+\gamma_r\}\frac{(x-\mu)}{\sigma}}\, dx,$$

which on further simplification reduces to

$$E[X(r,n,m,k)] = \frac{c_{r-1}}{(r-1)!}\left(\frac{1}{m+1}\right)^{r-1}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left[\frac{\mu}{((m+1)i+\gamma_r)} + \frac{\sigma}{((m+1)i+\gamma_r)^2}\right]$$

(3.2)

Starting with the means of generalized order statistics obtained from their exact and explicit expression given in (3.2), the recurrence relation given in (2.5) could be used in a simple recursive way in order to compute the higher order single moments of generalized order statistics from a two-parameter exponential distribution.

## 4. Relations For Joint Moment Generating Functions

Let $M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)$ denote the joint moment generating function of $X(r,n,m,k)$ and $X(s,n,m,k)$ and $M^{(i,j)}_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)$ denote its i,j-th partial derivative**.** The relation in (1.3) shall be employed to derive the recurrence relations for joint moment generating functions of generalized order statistics from a two-parameter exponential distribution.

**Theorem 4.1:** For $1 \le r \le s-1 \le n$, $n \in N$, $m \in Z$, i , j = 0, 1 ,2,……, and a fixed positive integer $k \ge 1$ ,

$$\{\gamma_{s+1} - t_2\sigma\} M^{(i,j+1)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2) = (j+1)\sigma M^{(i,j)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2) + \gamma_{s+1} M^{(i,j+1)}_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2), \quad (4.1)$$

and, for $1 \le r \le n$

$$\{\gamma_{r+1} - t_2\sigma\} M^{(i,j+1)}_{X(r,n,m,k),X(r+1,n,m,k)}(t_1,t_2) = (j+1)\sigma M^{(i,j)}_{X(r,n,m,k),X(r+1,n,m,k)}(t_1,t_2) + \gamma_{r+1} M^{(i,j+1)}_{X(r,n,m,k)}(t_1+t_2).$$


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(4.2)

**Proof:** Using (1.6), the joint moment generating function of X (r, n, m, k) and X (s+1, n, m, k) is given by

$$M_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2)=\frac{c_s}{(r-1)!(s-r)!}\iint_{x<y} e^{t_1x+t_2y}[1-F(x)]^m\; g_m^{r-1}(F(x))\,[h_m(F(y))-h_m(F(x))]^{s-r}$$
$$.[1-F(y)]^{\gamma_{s+1}-1} f(y)\, f(x)\; dy\; dx$$

$$=\frac{c_s}{(r-1)!(s-r)!}\int_{-\infty}^{\infty}[1-F(x)]^m\; f(x)\quad g_m^{r-1}(F(x))\; I(x)\; dx \quad ,\ (4.3)$$

where

$$I(x)=\int_x^{\infty} e^{t_1x+t_2y}[h_m(F(y))-h_m(F(x))]^{s-r}\,[1-F(y)]^{\gamma_{s+1}-1} f(y)\; dy$$

Solving the integral in I(x) by parts and substituting the resulting expression in (4.3), we get

$$M_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2)=M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)$$
$$+\frac{t_2\, c_s}{(r-1)!(s-r)!\gamma_{s+1}}\iint_{x<y} e^{t_1x+t_2y}[1-F(x)]^m f(x)\; g_m^{r-1}(F(x))$$
$$.[h_m(F(y))-h_m(F(x))]^{s-r}[1-F(y)]^{\gamma_{s+1}}\; dy\; dx\,.$$

On using the relation (1.3), we obtain

$$M_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2)=M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)+\frac{t_2\,\sigma}{\gamma_{s+1}}M_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2). \quad (4.4)$$

Differentiating both sides of (4.4) i times with respect to $t_1$ and then (j+1) times with respect to $t_2$, we get

$$M^{(i,j+1)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2)=M^{(i,j+1)}_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)+\frac{t_2\,\sigma}{\gamma_{s+1}}M^{(i,j+1)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2)$$
$$+\frac{(j+1)\sigma}{\gamma_{s+1}}M^{(i,j)}_{X(r,n,m,k),X(s+1,n,m,k)}(t_1,t_2)\,,$$

which, when rewritten gives the recurrence relation in (4.1). Proceeding in a similar manner for the case s = r, the recurrence relation given in (4.2) can easily be established. One can also note that Theorem 2.1 can be deduced from Theorem 4.1 by letting $t_1$ tend to zero.

**Remark 4.1:** On setting $t_1 = t_2 = 0$ in (4.1) and (4.2) we shall obtain the following recurrence relations for product moments of generalized order statistics from a two-parameter exponential distribution for the cases $1\le r \le s-1 \le n$ and $1 \le r \le n$, respectively.

$$\gamma_{s+1}E[X^i(r,n,m,k)X^{j+1}(s+1,n,m,k)]=(j+1)\sigma\, E[X^i(r,n,m,k)X^j(s+1,n,m,k)]$$
$$+\gamma_{s+1}E[X^i(r,n,m,k)X^{j+1}(s,n,m,k)]\,, \quad (4.5)$$

For $n\in N$, $m\in Z$, $1\le r\le s-1 \le n$ and i,j = 0,1,2,…,
and, for $1 \le r\le n$, i, j =0,1,2,...,

$$\gamma_{r+1}E[X^i(r,n,m,k)X^{j+1}(r+1,n,m,k)]=(j+1)\sigma E[X^i(r,n,m,k)X^j(r+1,n,m,k)]$$
$$+\gamma_{r+1}E[X^{i+j+1}(r,n,m,k)]. \quad (4.6)$$

**Remark 4.2:** Putting m = –1 and $t_1 = t_2 = 0$ in (4.1) and (4.2), we shall obtain the recurrence relations for product moments of k-th upper record values from a two-parameter exponential distribution:

For $1 \le p \le n-1$, i, j = 0,1,2,….,

$$E\left[\left(Y_p^{(k)}\right)^i\left(Y_{n+1}^{(k)}\right)^{j+1}\right]=\frac{(j+1)\sigma}{k}E\left[\left(Y_p^{(k)}\right)^i\left(Y_{n+1}^{(k)}\right)^j\right]+E\left[\left(Y_p^{(k)}\right)^i\left(Y_n^{(k)}\right)^{j+1}\right],$$

and, for $p \ge 1$, i, j =0,1,2,...,

$$E\left[\left(Y_p^{(k)}\right)^i\left(Y_{p+1}^{(k)}\right)^{j+1}\right]=\frac{(j+1)\sigma}{k}E\left[\left(Y_p^{(k)}\right)^i\left(Y_{p+1}^{(k)}\right)^j\right]+E\left[\left(Y_p^{(k)}\right)\right]^{i+j+1}.$$

## 5. Recursive Computational Algorithm for Evaluation Product Moments

On using (1.6), we obtain

$$E[X(r,n,m,k)X(s,n,m,k)] = \frac{c_{s-1}}{(s-r-1)!(r-1)!}\int_{\mu}^{\infty}\int_{x}^{\infty} x\,y\,(1-F(y))^{\gamma_s-1} g_m^{r-1}(F(x))\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1} .(1-F(x))^m f(y)\,f(x)\,dydx\,.$$

Further,

$$E[X(r,n,m,k)X(s,n,m,k)] = \frac{c_{s-1}}{(s-r-1)!(r-1)!}\int_{\mu}^{\infty}\int_{x}^{\infty} x\,y\,(1-F(y))^{\gamma_s}\left[\frac{-(1-F(y))^{m+1}}{m+1}+\frac{(1-F(x))^{m+1}}{m+1}\right]^{s-r-1} .\left[\frac{-(1-F(x))^{m+1}}{m+1}+\frac{1}{m+1}\right]^{r-1}(1-F(x))^{m+1}\,dydx,$$

on using (1.3).

On expanding $\left[\frac{-(1-F(y))^{m+1}}{m+1}+\frac{(1-F(x))^{m+1}}{m+1}\right]^{s-r-1}$ and $\left[\frac{-(1-F(x))^{m+1}}{m+1}+\frac{1}{m+1}\right]^{r-1}$ binomially and using (1.2), we obtain
where

$$E[X(r,n,m,k)X(s,n,m,k)] = \frac{c_{s-1}}{(s-r-1)!(r-1)!}\frac{1}{\sigma^2}\left(\frac{1}{m+1}\right)^{s-2}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}\int_{\mu}^{\infty}\int_{x}^{\infty} xy .(1-F(y))^{(\gamma_s+(m+1)j)}(1-F(x))^{(m+1)(s-r+i-j)}\,dydx$$

$$= \frac{c_{s-1}}{(s-r-1)!(r-1)!}\frac{1}{\sigma^2}\left(\frac{1}{m+1}\right)^{s-2}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}\int_{\mu}^{\infty} x\,G(x) .e^{-(m+1)(s-r+i-j)\frac{(x-\mu)}{\sigma}}\,dx, \qquad (5.1)$$

$$G(x) = \int_{x}^{\infty} y\;e^{-\{\gamma_s+(m+1)j\}\frac{(y-\mu)}{\sigma}}\,dy$$

On solving the integral in G(x), we have

$$G(x) = \frac{\sigma x\;e^{-\{\gamma_s+(m+1)j\}\frac{(x-\mu)}{\sigma}}}{\{\gamma_s+(m+1)j\}}+\frac{\sigma^2\;e^{-\{\gamma_s+(m+1)j\}\frac{(x-\mu)}{\sigma}}}{\{\gamma_s+(m+1)j\}^2}$$

Substituting the above expression for G(x) in (5.1), we obtain

$$E[X(r,n,m,k)X(s,n,m,k)] = \frac{c_{s-1}}{(s-r-1)!(r-1)!}\frac{1}{\sigma^2}\left(\frac{1}{m+1}\right)^{s-2}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}H(i,j), \qquad (5.2)$$

Where

$$H(i,j) = \frac{\sigma}{\{\gamma_s+(m+1)j\}}\int_{\mu}^{\infty} x^2 e^{-\{(m+1)(s-r+i)+\gamma_r\}\frac{(x-\mu)}{\sigma}}\,dx+\frac{\sigma^2}{\{\gamma_s+(m+1)j\}^2}\int_{\mu}^{\infty} x\,e^{-\{(m+1)(s-r+i)+\gamma_r\}\frac{(x-\mu)}{\sigma}}\,dx$$

On solving and simplifying the integrals in the above expression for H(i,j), we get

$$H(i,j)=\frac{\mu^2\sigma^2}{\{\gamma_s+(m+1)j\}\{\gamma_s+(m+1)(s-r+i)\}}+\frac{2\,\mu\sigma^3}{\{\gamma_s+(m+1)j\}\{\gamma_s+(m+1)(s-r+i)\}^2}+$$
$$\frac{2\sigma^4}{\{\gamma_s+(m+1)j\}\{\gamma_s+(m+1)(s-r+i)\}^3}+\frac{\mu\sigma^3}{\{\gamma_s+(m+1)j\}^2\{\gamma_s+(m+1)(s-r+i)\}}+$$
$$\frac{\sigma^4}{\{\gamma_s+(m+1)j\}^2\{\gamma_s+(m+1)(s-r+i)\}^2}.$$

Further, on substituting for H(i,j) in (5.2), an exact and explicit expression for the product moments can be obtained.

Next, by using the exact and explicit expression for the product moments obtained in (5.2) and also the recurrence relations given in equations (4.5) and (4.6) in a simple recursive way one can easily obtain the variances, covariances and higher order product moments of all generalized order statistics from a two-parameter exponential distribution.

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