First Entire Zagreb Index of Fuzzy Transformation Graphs

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Abstract: ***Topological indices play vital role in mathematical chemistry. The first entire Zagreb index (FEZI) is one among them. This parameter has proven to be essential in various real-life scenarios, such as networking businesses and traffic management on roads. In this paper,explore the first entire Zagreb index for various class of generalized fuzzy transformation graphs. This research paper is also deals with the bounds for FEZI for generalized fuzzy transformation graphs in terms of elements of a fuzzy graph G.***

Keywords: First Zagreb index; Second Zagreb index; First Entire Zagreb Index.

***AMS* Subject Classification:** 05C09; 05C72

1. Introduction

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph Model'. Nowadays, due to various applications of fuzzy graph theory (FGT), a huge number of researchers are working on topological indices (TIs). In 1965, the idea of fuzzy set (FS) was first introduced by Zadeh [22]. Motivated by this, in 1975, Rosenfeld [14] defined the fuzzy graph (FG). At the same time, Yeh et al. [21] also introduced the FG independently and defined some connectivity parameters of a FG and their applications. In [19, 20], Sunitha et al. studied the fuzzy block (FB), fuzzy bridge (FBg), fuzzy cut vertex, fuzzy tree (FT), fuzzy forest, partial fuzzy subgraph (PFSG), fuzzy subgraph (FSG), complete fuzzy graph (CFG), etc. Samanta and Pal discussed a fuzzy planar graph in [18-36]. In [16, 17] Sahoo and Pal provided a product of intuitionistic fuzzy graphs.

In the field of molecular chemistry, TIs are molecular descriptors which are calculated on the molecular graph (MG) of a chemical compound. These TIs are numerical quantities of a graph which describes its topology. Zagreb index $M_1(G)$ is one such TIs which is degree-based TI and introduced by Gutman and Trinajstic in 1972 [4] and this TI is used to calculate π-electron energy of a conjugate system. One of the most useful topological indices are the Zagreb indices which are defined as:

$$M_1(G) = \sum_{i=1}^{n} d_G(v_i)^2$$

and

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$$

where M_1 and M_2 are the first and second Zagreb indices respectively.

First entire Zagreb index (FEZI) [9] have an important role for finding strength of vertices. The strength of vertices is important in fuzzy graph theory. Thus, in this section the FEZI of a fuzzy graph is initiated. Various properties and an application of FEZI for fuzzy graphs is given.

Definition 1. Let G be an FG. Then, the FEZI of G is indicated by M_1^Z and is defined by:

$$M_1^z = \sum_{v\in V(G)} (\sigma(u)d_G(u))^2 + \sum_{e\in E(G)} (\mu(u)d_G(e))^2$$

The total degree of an FG with respect to vertices is indicated by $T_v(G)$ and is defined by $T_v(G)\sum_{v\in V(G)} d_G(v).$

Additionally, the total degree of an FG with respect to edge is indicated by $T_e(G)\sum_{e\in E(G)} d_G(e).$

Example 1. Let G be an FG with vertex set $V=\{v_1,v_2,v_3,v_4\}$ such that $\sigma(v_1)=0.8, \sigma(v_2)=0.7, \sigma(v_3)=0.6, \sigma(v_4)=0.4$, whereas $\mu(v_1v_2) = 0.6, \mu(v_2v_3)=0.5$ and $\mu(v_3v_4)=0.4$. Then $d_G(v_1)=0.6$, $d_G(v_2)=1.1$, $d_G(v_3)=0.9$ and $d_G(v_4)=0.4$. Further, $d_G(e_1)=1.1$, $d_G(2_2)=1.0$ and $d_G(e_3)=0.5$. The total vertex and edge degrees of G are $T_v(G)=0.6+1.1+0.9+0.4=3.0$ and $T_e(G)=1.1+1+0.5=2.6$. Hence

$$M_1^z(G) = \sum_{v\in V(G)} (\sigma(u)d_G(u))^2 + \sum_{e\in E(G)} (\mu(u)d_G(e))^2$$

$=[(0.8)\times(0.6)]^2+[(0.7)\times(1.1)]^2$
$+[(0.6)\times(0.9)]^2+[(0.6)\times(1.1)]^2+[(0.5)\times(1.0)]^2+[(0.4)\times(0.5)]^2$
$=1.8405.$

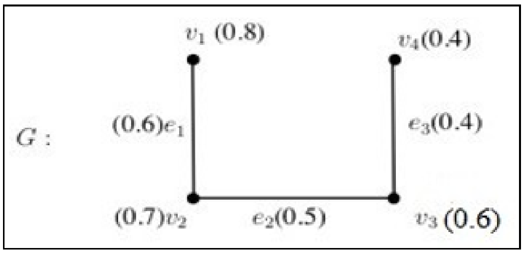

Figure 1: A fuzzy graph

In this article, the *F*-index for fuzzy graphs is introduced which is a more generalization of the first Zagreb index for crisp graphs and it is extensively used in spectral fuzzy graph theory, fuzzy network theory, fuzzy graph theory and several fields of fuzzy mathematics and chemistry.

Preliminaries

Here, we provide some fundamental definitions and theorems which are crucial to devel-oping the later sections. A fuzzy graph G=(V,σ,μ) corresponding to the crisp graph G^* is a non-empty set V together with a pair of functions σ: V → [0, 1] and μ: V×V → [0, 1] such that for all $a, b \in V$, $\mu(a, b) \leq min\{\sigma(a),\sigma(b)\}$

We assume that S is finite and nonempty,μ is reflexive and symmetric. In all the examples σ is chosen suitably. Also, we denote the underlying graph by G^*: (σ^*,μ^*) where $\sigma^*=\{u\in S: \sigma(u)>0\}$ and $\mu^*=\{(u,v)\in S\times S: \mu(u,v)>0\}$. A fuzzy graph H: $(\tau,\sqsubseteq)$ is called a partial fuzzy subgraph of G: (σ,μ) if $\tau(u)\leq\sigma(u)$ for every u and $\sqsubseteq(u,v)\leq\mu(u,v)$ for every u,v. In particular we call a partial fuzzy subgraph H: $(\tau,\sqsubseteq)$ a fuzzy subgraph of G: (σ,μ) if $\tau(u)=\sigma(u)$ for every $u\in\tau^*$ and $\sqsubseteq(u,v)=\mu(u,v)$ for every arc $u,v\in\sqsubseteq^*$. Now a fuzzy subgraph H: $(\tau,\sqsubseteq)$ spans the fuzzy graph G: (σ,μ) if $\tau(u)=\sigma(u)$. A connected f-graph G: (σ,μ) is a fuzzy tree(f-tree) if it has a fuzzy spanning subgraph F: $(\tau,\sqsubseteq)$ which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$) is a weakest arc of G. Here P_1: x, w,v is an α-strongest u-v path whereas P_2: x, u,v is a β-strong x-v path.

Fuzzy Transformation Graphs

Let $G=(V,\sigma,\mu)$ be a fuzzy graph with underlying crisp graph $G=(V,\sigma^*,\mu^*)$. Then fuzzy transformation graph $G^{ab}=(X,\sigma^{ab},\mu^{ab})$, where $X=V\cup E$. Let α and β be any two elements of G^{ab} then we say that the associativity of $\alpha$and β is+if they are adjacent or incident in G otherwise. Let a and b be two permutation of the set {+.-}. For more details on transformation graphs refer [2, 3, 12, 15]. Then we define the four kinds of fuzzy transformation graphs based on the permutations of a and b. These fuzzy transformation graphs are defined as follows:

Definition 2. *Let $G=(V,\sigma,\mu)$ be a fuzzy graph with underlying crisp graph $G=(V,\sigma^*,\mu^*)$. Then the vertex set of fuzzy transformation graph $V(G^{++})=V\cup E$. The adjacency and incidence relations and their values are defined as follows:*

$$\sigma^{++}(u)=\begin{cases}\sigma(u) & if \ \ u\in\sigma\\ \mu(u) & if \ \ u\in\mu\\ 0 & \text{otherwise}\end{cases}$$

$$\&\mu^{++}(v_i,v_j)=\begin{cases}\sigma(v_i)\wedge\sigma(v_j) & if \ \ v_i,v_j\in\sigma \quad \text{and are adjacent in G}\\ 0 & \text{otherwise}\end{cases}$$

$$\mu^{++}(e_i,e_j)=\{0 \quad if \ e_i,e_j\in\mu$$

$$\mu^{++}(v_i,e_j)=\begin{cases}\mu(e_j) & if \ v_i\in\sigma \quad \text{lies on the edge} \quad e_j\in\mu\\ 0 & \text{otherwise}\end{cases}$$

Definition 3. *Let $G=(V,\sigma,\mu)$ be a fuzzy graph with underlying crisp graph $G=(V,\sigma^*,\mu^*)$. Then the vertex set of fuzzy transformation graph $V(G^{+-})=V\cup E$. The adjacency and incidence relations and their values are defined as follows:*

$$\sigma^{+-}(u)=\begin{cases}\sigma(u) & \text{if } u\in\sigma\\ \mu(u) & \text{if } u\in\mu\\ 0 & \text{otherwise}\end{cases} \quad \&\mu^{+-}(v_i,v_j)=\begin{cases}\sigma(v_i)\wedge\sigma(v_j) & \text{if } v_i,v_j\in\sigma \quad \text{and are adjacent in G}\\ 0 & \text{otherwise}\end{cases}$$

$$\mu^{+-}(e_i,e_j)=\{0 \quad \text{if } e_i,e_j\in\mu \qquad \&\mu^{+-}(v_i,e_j)=\begin{cases}0 & \text{otherwise}\\ \mu(e_j) & \text{if } v_i\in\sigma \quad \text{lies on the edge} \quad e_j\in\mu\end{cases}$$

Definition 4. *Let $G=(V,\sigma,\mu)$ be a fuzzy graph with underlying crisp graph $G=(V,\sigma^*,\mu^*)$. Then the vertex set of fuzzy transformation graph $V(G^{-+})=V\cup E$. The adjacency and incidence relations and their values are defined as follows:*

$$\sigma^{-+}(u)=\begin{cases}\sigma(u) & \text{if } u\in\sigma\\ \mu(u) & \text{if } u\in\mu\\ 0 & \text{otherwise}\end{cases} \quad \&\mu^{-+}(v_i,v_j)=\begin{cases}0 & \text{if } \mu(v_i,v_j)>0\\ \sigma(v_i)\wedge\sigma(v_j) & \text{otherwise}\end{cases}$$

$$\mu^{-+}(e_i,e_j)=\{0 \quad \text{if } e_i,e_j\in\mu \qquad \&\mu^{-+}(v_i,e_j)=\begin{cases}\mu(e_j) & \text{if } v_i\in\sigma \quad \text{lies on the edge} \quad e_j\in\mu\\ 0 & \text{otherwise}\end{cases}$$

Definition 5. *Let $G=(V,\sigma,\mu)$ be a fuzzy graph with underlying crisp graph $G=(V,\sigma^*,\mu^*)$. Then the vertex set of fuzzy transformation graph $V(G^{--})=V\cup E$. The adjacency and incidence relations and their values are defined as follows:*

$$\sigma^{--}(u)=\begin{cases}\sigma(u) & \text{if } u\in\sigma\\ \mu(u) & \text{if } u\in\mu\\ 0 & \text{otherwise}\end{cases} \quad \&\mu^{--}(v_i,v_j)=\begin{cases}0 & \text{if } \mu(v_i,v_j)>0\\ \sigma(v_i)\wedge\sigma(v_j) & \text{otherwise}\end{cases}$$

$$\mu^{--}(e_i,e_j)=\{0 \quad \text{if } e_i,e_j\in\mu \qquad \&\mu^{--}(v_i,e_j)=\begin{cases}0 & \text{if } v_i\in\sigma \quad \text{lies on the edge} \quad e_j\in\mu\\ \mu(e_j) & \text{otherwise}\end{cases}$$

Example for G^{++}: Consider a graph G in Figure 1. Then according Definition 1, the fuzzy transformation graph G^{++}shown in Figure 2 has vertex set $V(G^{++})=\{v_1,v_2,v_3,v_4,e_1,e_2,e_3\}$ such that $\sigma(v_1)=0.8,\sigma(v_2)=0.7,\sigma(v_3)=0.6,\sigma(v_4)=0.4,\sigma(e_1)=0.6,\sigma(e_2)=0.5$ and $\sigma(e_3)=0.4$.Further,$\mu(v_1,v_2)=0.6,\mu(v_2,v_3)=0.5,\mu(v_3,v_4)=0.4,\mu(v_1,e_1)=0.6,\mu(v_2,e_1)=0.6,\mu(v_2,e_2)=0.5,\mu(v_3,e_2)=0.5,\mu(v_3,e_3)=0.4$

and $\mu(v_4,e_3)$=0.4. By Lemma 1,

$$d_G^{++}(u)=2\sum_{u\neq v}\mu(u,v) \text{ and } d_G^{++}(e)=2\mu(u,v).$$

Therefore, we have vertex and edge degrees as follows:

$$d_G(v_1)=\sum_{v_1\neq v_2}\mu(v_1,v_2)=2(0.6)=1.2$$

$d_G(v_2)$=2(1.1)=2.2, $d_G(v_3)$=2(0.9)=1.8, $d_G(v_4)$=2(0.4)=0.8, $d_G(e_1)=2\mu(v_1v_2)$=2(0.6)=1.2, $d_G(e_2)=2\mu(v_2v_3)$=2(0.5)=1, $d_G(e_3)=2\mu(v_3v_4)$=2(0.4)=0.8, $d_G(v_1v_2)$=2.2, $d_G(v_2v_3)$=3, $d_G(v_3v_4)$=1.8, $d_G(v_4e_3)$=0.8, $d_G(e_3v_3)$=1.8, $d_G(e_2v_3)$=1.8, $d_G(e_2v_2)$=2.2, $d_G(e_1v_2)$=2.2, $d_G(e_1v_1)$=1.2, Therefore, the M_1^z of $G^{++}$is given by

$$M_1^z(G^{++})=\sum_{v\in V(G^{++})}(\sigma(u)d_{G^{++}}(u))^2+\sum_{e\in E(G^{++})}(\mu(u)d_{G^{++}}(e))^2$$

=[(0.8)×(1.2)]^2+[(0.7)×(2.2)]^2+[(0.6)×(1.8)]^2+[(0.4)×(0.8)]^2 +[(0.6)×(1.2)]^2+[(0.5)×(1.0)]^2+[(0.4)×(0.8)]^2+[(0.6)×(2.2)]^2 +[(0.5)×(3.0)]^2+[(0.4)×(1.8)]^2+[(0.4)×(0.8)]^2+[(0.4)×(1.8)]^2 +[(0.5)×(0.8)]^2+[(0.5)×(2.2)]^2+[(0.6)×(2.2)]^2+[(0.6)×(1.2)]^2 =14.8452.

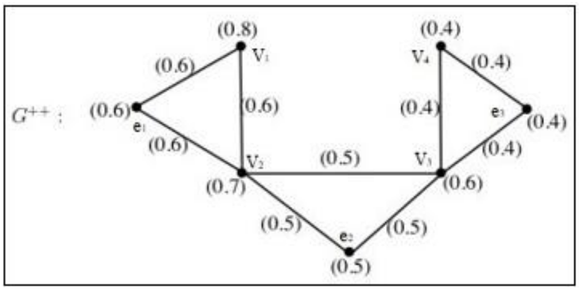

Figure 3: The Transformation Graph G $^{++}$

Example for G^{+-}: Consider a graph G in Figure 1. Then according Definition 2, the fuzzy transformation graph G^{+-} shown in Figure 3 has vertex set $V(G^{+-})=\{v_1,v_2,v_3,v_4,e_1,e_2,e_3\}$ such that $\sigma(v_1)$=0.8,$\sigma(v_2)$=0.7,$\sigma(v_3)$=0.6,$\sigma(v_4)$=0.4,$\sigma(e_1)$ 0.6,$\sigma(e_2)$= 0.5 and $\sigma(e_3)$=0.4. Further, $\mu(v_1,v_2)$=0.6, $\mu(v_2,v_3)$=0.5, $\mu(v_3,v_4)$=0.4, $\mu(v_1,e_3)$=0.4, $\mu(v_2,e_3)$=0.4, $\mu(v_3,e_1)$=0.6, $\mu(v_4,e_1)$=0.6, $\mu(v_1,e_2)$=0.5 and $\mu(v_4,e_2)$=0.5. By Lemma 1,

$$d_G^{+-}(u)=2\sum_{u,v\in VXV}\mu(u,v) \quad \text{and} \quad d^{+-}(e)=(|V|-2)\mu(u,v).$$

Therefore, we have $d_G(v_1)=d_G(v_2)=d_G(v_3)=d_G(v_4)$=1.5, $d_G(e_1)=(|V|-2)\mu(u,v)$=(4-2)(0.6)=1.2, $d_G(e_2)$=(4-2)(0.5)=1, $d_G(e_3)$=(4-2)(0.4)=0.8. Therefore, the M_1^z of $G^{+-}$is given by

$$M_1^z(G^{+-})=\sum_{v\in V(G^{+-})}(\sigma(u)d_{G^{+-}}(u))^2+\sum_{e\in E(G^{+-})}(\mu(u)d_{G^{+-}}(e))^2$$

=[(0.8)×(1.5)]^2+[(0.7)×(1.5)]^2+[(0.6)×(1.5)]^2+[(0.4)×(1.5)]^2 +[(0.6)×(1.2)]^2+[(0.5)×(1.0)]^2+[(0.4)×(0.8)]^2+[(0.6)×(1.8)]^2 +[(0.5)×(2.0)]^2+[(0.4)×(2.2)]^2+[(0.5)×(1.5)]^2+[(0.5)×(1.5)]^2 +[(0.4)×(1.5)]^2+[(0.4)×(1.5)]^2+[(0.6)×(1.5)]^2+[(0.6)×(1.5)]^2 =10.9891.

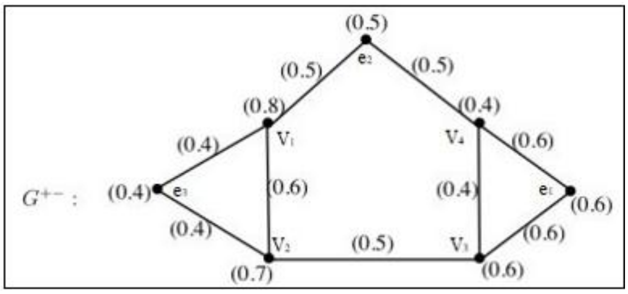

Figure 4: The Transformation Graph G $^{+-}$

Example for G^{-+}: Consider a graph G in Figure 1. Then according Definition 3, the fuzzy transformation graph G^{-+}shown in Figure 4 has vertex set $V(G^{-+})=\{v_1,v_2,v_3,v_4,e_1,e_2,e_3\}$ such that $\sigma(v_1)$=0.8, $\sigma(v_2)$=0.7, $\sigma(v_3)$=0.6, $\sigma(v_4)$=0.4, $\sigma(e_1)$=0.6, $\sigma(e_2)$=0.5 and $\sigma(e_3)$=0.4. Further, $\mu(v_1,v_3)$=0.6, $\mu(v_1,v_4)$=0.4, $\mu(v_2,v_4)$=0.4, $\mu(v_1,e_1)$=0.6, $\mu(v_2,e_1)$=0.6, $\mu(v_2,e_2)$=0.5, $\mu(v_3,e_2)$=0.5,$\mu(v_3,e_3)$=0.4 and $\mu(v_4,e_2)$=0.4. By Lemma 1,

$$d_G^{-+}(u)=2\sum_{uv\notin E}(\sigma(u)\wedge\sigma(v))+\sum_{uv\notin E}\mu(u,v) \text{ and } d_G^{-+}(e)=2\mu(u,v).$$

Therefore, we have
$d_G(v_1)$=0.4+0.6+0.6=1.6,$d_G(v_2)$=0.5+0.6+0.4=1.5, $d_G(v_3)$=0.4+0.4+0.5=1.3, $d_G(v_4)$=0.4+0.4+0.4=1.2, $d_G(e_1)=2\mu(v_1v_2)$=2(0.6)=1.2, $d_G(e_2)=2\mu(v_2v_3)$=2(0.5)=1, $d_G(e_3)=2\mu(v_3v_4)$=2(0.4)=0.8.
Therefore, the M_1^z of $G^{-+}$is given by

$$M_1^z(G^{-+})=\sum_{v\in V(G^{-+})}(\sigma(u)d_{G^{-+}}(u))^2+\sum_{e\in E(G^{-+})}(\mu(u)d_{G^{-+}}(e))^2$$

=[(0.8)×(1.6)]^2+[(0.7)×(1.5)]^2+[(0.6)×(1.4)]^2+[(0.6)×(1.2)]^2 +[(0.5)×(1.0)]^2+[(0.4)×(0.8)]^2+[(0.4)×(2.0)]^2+[(0.6)×(1.8)]^2 +[(0.4)×(1.9)]^2+[(0.4)×(1.4)]^2+[(0.4)×(1.2)]^2+[(0.6)×(1.6)]^2 +[(0.6)×(1.5)]^2+[(0.5)×(1.5)]^2+[(0.6)×(1.4)]^2=10.2598.

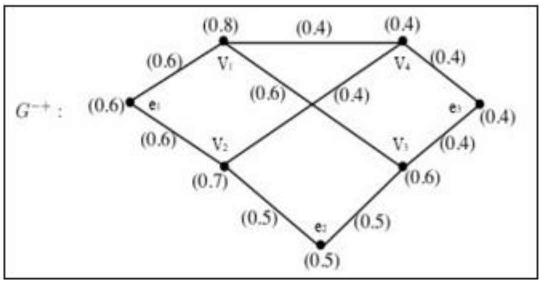

Figure 4: The Transformation Graph G $^{-+}$

Example for G^{--}: Consider a graph G in Figure 1. Then according Definition 4, the fuzzy transformation graph G^{--} shown in Figure 5 has vertex set $V(G^{--})=\{v_1,v_2,v_3,v_4,e_1,e_2,e_3\}$ such that $\sigma(v_1)$=0.8,$\sigma(v_2)$=0.7, $\sigma(v_3)$=0.6,$\sigma(v_4)$=0.4,$\sigma(e_1)$=0.6,$\sigma(e_2)$=0.5 and $\sigma(e_3)$=0.4. Further,$\mu(v_1,v_3)$=0.6,$\mu(v_1,v_4)$=0.4,$\mu(v_2,v_4)$=0.4,$\mu(v_1,e_3)$=0.4,$\mu(v_2,e_3)$=0.4,$\mu(v_3,e_1)$=0.6,$\mu(v_4,e_1)$=0.6,$\mu(v_1,e_2)$=0.5 and $\mu(v_4,e_2)$=0.5. By Lemma 1,

$$d_G^{--}(u)=2\sum_{uv\notin E}(\sigma(u)\wedge\sigma(v))+\sum_{uv\notin E}\mu(u,v) \text{ and } d_G^{--}(e)=(|V|-2)\mu(u,v).$$

Therefore, we have $d_G(v_1)$=0.4+0.6+0.5+0.4=1.9, $d_G(v_2)$=0.4+0.4=0.8, $d_G(v_3)$=0.6+0.6=1.2, $d_G(v_4)$=0.4+0.4+0.6+0.5=1.9, $d_G(e_1)=2\mu(v_1v_2)$=2(0.6)=1.2, $d_G(e_2)=2\mu(v_2v_3)$=2(0.5)=1, $d_G(e_3)=2\mu(v_3v_4)$=2(0.4)=0.8.
Therefore, the M_1^z of $G^{--}$is given by

$$M_1^z(G^{--})=\sum_{v\in V(G^{--})}(\sigma(u)d_{G^{--}}(u))^2+\sum_{e\in E(G^{--})}(\mu(u)d_{G^{--}}(e))^2$$

=[(0.8)×(1.9)]^2+[(0.7)×(0.8)]^2+[(0.6)×(1.2)]^2+[(0.4)×(1.9)]^2 +[(0.6)×(1.2)]^2+[(0.5)×(1.0)]^2+[(0.4)×(0.8)]^2+[(0.5)×(1.9)]^2 +[(0.6)×(1.9)]^2+[(0.6)×(1.2)]^2+[(0.6)×(1.9)]^2+[(0.4)×(3.0)]^2 +[(0.4)×(0.8)]^2+[(0.4)×(1.9)]^2=11.5854.

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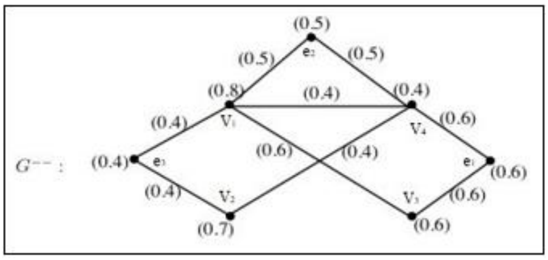

Figure 5: The Transformation Graph G⁻⁻

2. Results

Lemma 1. *Let G=(V,σ,μ) be a fuzzy graph and H=G^{ab}=(V∪E,σ*,μ*) be fuzzy transformation graph. Then the degree of point vertex and line vertex are given by*

$$d_G^{++}(u)=2\sum_{u\neq v}\mu(u,v) \text{ and } d_G^{++}(e)=2\mu(u,v).$$

$$d_G^{+-}(u)=2\sum_{u,v\in VXV}\mu(u,v) \text{ and } d_G^{+-}(e)=(|V|-2)\mu(u,v).$$

$$d_G^{-+}(u)=2\sum_{uv\notin E}\left(\sigma(u)\wedge\sigma(v)\right)+\sum_{uv\notin E}\mu(u,v) \text{ and } d_G^{-+}(e)=2\mu(u,v).$$

$$d_G^{--}(u)=2\sum_{uv\notin E}\left(\sigma(u)\wedge\sigma(v)\right)+\sum_{uv\notin E}\mu(u,v) \text{ and } d_G^{--}(e)=(|V|-2)\mu(u,v).$$

Theorem 1. *Let G be n-vertex fuzzy graph with m-edges. Then*

$$M_1^z(G^{++})\leq n^2T_v^2(G)+m^2T_1^2(G)+m^2T_e^2(G)+4m^2T_2^2(G)$$

Proof. Now, the FEZI of $G^{++}$is given by

$$M_1^z(G^{++})=\sum_{v\in V(G^{++})}(\sigma(u)d_{G^{++}}(u))^2+\sum_{e\in E(G^{++})}(\mu(u)d_{G^{++}}(e))^2$$

$$=\sum_{v\in V(G^{++})\cap V(G)}(\sigma(u)d_{G^{++}}(u))^2+\sum_{v\in V(G^{++})\cap E(G)}(\sigma(u)d_{G^{++}}(u))^2+\sum_{e\in E(G^{++})\cap E(G)}(\mu(u)d_{G^{++}}(e))^2+\sum_{e\in E(G^{++})-E(G)}(\mu(u)d_{G^{++}}(e))^2$$

Since $\sum(pq)^2\leq\sum p^2\sum q^2$, we have $\sum_{u,v\in E(G)}\mu(u,v)=\sum_{e\in E(G)}\mu(e)$

Since 0≤σ≤1 and 0≤μ≤1. Therefore, we have

$$M_1^z(G^{++})\leq n^2T_v^2(G)+m^2T_1^2(G)+m^2T_e^2(G)+4m^2T_2^2(G)$$

where $T_1(G)=\sum_{v\in V(G^{++})\cap E(G)}(d_{G^{++}}(u))^2$ and $T_2(G)=\sum_{e\in E(G^{++})-E(G)}(d_{G^{++}}(e))^2$. ■

Theorem 2. *Let G be n-vertex fuzzy graph with m-edges. Then*

$$M_1^z(G^{+-})\leq n^2T_v^2(G)+m^2T_1^2(G)+m^2T_e^2(G)+(m(n-2))^2T_2^2(G)$$

Proof. Now, the FEZI of $G^{+-}$is given by

$$M_1^z(G^{+-})=\sum_{v\in V(G^{+-})}(\sigma(u)d_{G^{+-}}(u))^2+\sum_{e\in E(G^{+-})}(\mu(u)d_{G^{+-}}(e))^2$$

$$=\sum_{v\in V(G^{+-})\cap V(G)}(\sigma(u)d_{G^{+-}}(u))^2+\sum_{v\in V(G^{+-})\cap E(G)}(\sigma(u)d_{G^{+-}}(u))^2+\sum_{e\in E(G^{+-})\cap E(G)}(\mu(u)d_{G^{+-}}(e))^2+\sum_{e\in E(G^{+-})-E(G)}(\mu(u)d_{G^{+-}}(e))^2$$

Since $\sum(pq)^2\leq\sum p^2\sum q^2$, we have

$$M_1^z(G^{+-})\leq\sum_{v\in V(G^{+-})\cap V(G)}(\sigma(u))^2\sum_{v\in V(G^{+-})\cap V(G)}(d_{G^{+-}}(u))^2+\sum_{v\in V(G^{+-})\cap E(G)}(\sigma(u))^2\sum_{v\in V(G^{+-})\cap E(G)}(d_{G^{+-}}(u))^2$$

$$+\sum_{e\in E(G^{+-})\cap E(G)}(\mu(u))^2\sum_{e\in E(G^{+-})\cap E(G)}(d_{G^{+-}}(e))^2+\sum_{e\in E(G^{+-})-E(G)}(\mu(u))^2\sum_{e\in E(G^{+-})-E(G)}(d_{G^{+-}}(e))^2$$

$$\leq\sum_{v\in V(G^{+-})\cap V(G)}(\sigma(u))^2T_v^2(G)+\sum_{v\in V(G^{+-})\cap E(G)}(\sigma(u))^2\sum_{v\in V(G^{+-})\cap E(G)}(\sigma(u))^2+\sum_{e\in E(G^{+-})\cap E(G)}(\mu(u))^2T_e^2(G)$$

$$+\sum_{e\in E(G^{+-})-E(G)}(\mu(u))^2\sum_{e\in E(G^{+-})-E(G)}(d_{G^{+-}}(e))^2$$

Since 0≤σ≤1 and 0≤μ≤1. Therefore, we have

$$M_1^z(G^{+-})\leq n^2T_v^2(G)+m^2T_1^2(G)+m^2T_e^2(G)+(m(n-2))^2T_2^2(G)$$

Lemma 2. [*Handshaking Lemma For Fuzzy Graphs*] The sum of degrees of all vertices in a fuzzy graph *G=(V,σ,μ)* is equal to twice the sum of membership values of all edges in *G. i.e* $\sum_{u\in V(G)}d_G(u)=2\sum_{e\in E(G)}\mu(e)$.

Observation 1. *For every fuzzy graph G=(V,σ,μ) the following are true:*

1) $\sigma(u)\leq1$
2) $\sum_{u\in V(G)}\sigma(u)\leq n$
3) $\mu(u,v)=\wedge\ \sigma(u).\sigma(v)\leq1$
4) $\sum_{u,v\in E(G)}\mu(u,v)=\sum_{e\in E(G)}\mu(e)$

where $T_1(G)=\sum_{v\in V(G^{+-})\cap E(G)} (d_{G^{+-}}(u))^2$ and $T_2(G)=\sum_{e\in E(G^{+-})-E(G)} (d_{G^{+-}}(e))^2$.

Theorem 3. *Let G be n-vertex fuzzy graph with m-edges. Then*

$$M_1^z(G^{-+})\le n^2T_v^2(\bar{G})+m^2T_1^2(G)+m^2T_e^2(G)+(\frac{n(n-2)}{2})^2T_2^2(G)$$

Proof. Now, the FEZI of G^{-+} is given by

$$M_1^z(G^{-+})=\sum_{v\in V(G^{-+})}(\sigma(u)d_{G^{-+}}(u))^2+\sum_{e\in E(G^{-+})}(\mu(u)d_{G^{-+}}(e))^2$$

$$=\sum_{v\in V(G^{-+})\cap V(G)}(\sigma(u)d_{G^{-+}}(u))^2+\sum_{v\in V(G^{-+})\cap E(G)}(\sigma(u)d_{G^{-+}}(u))^2+\sum_{e\in E(G^{-+})\cap E(G)}(\mu(u)d_{G^{-+}}(e))^2+\sum_{e\in E(G^{-+})-E(G)}(\mu(u)d_{G^{-+}}(e))^2$$

Since $\sum(pq)^2\le\sum p^2\sum q^2$, we have

$$M_1^z(G^{-+})\le\sum_{v\in V(G^{-+})\cap V(G)}(\sigma(u))^2\sum_{v\in V(G^{-+})\cap V(G)}(d_{G^{-+}}(u))^2+\sum_{v\in V(G^{-+})\cap E(G)}(\sigma(u))^2\sum_{v\in V(G^{-+})\cap E(G)}(d_{G^{-+}}(u))^2$$

$$+\sum_{e\in E(G^{-+})\cap E(G)}(\mu(u))^2\sum_{e\in E(G^{-+})\cap E(G)}(d_{G^{-+}}(e))^2+\sum_{e\in E(G^{-+})-E(G)}(\mu(u))^2\sum_{e\in E(G^{-+})-E(G)}(d_{G^{-+}}(e))^2$$

$$\le\sum_{v\in V(G^{-+})\cap V(G)}(\sigma(u))^2T_v^2(G)+\sum_{v\in V(G^{-+})\cap E(G)}(\sigma(u))^2\sum_{v\in V(G^{-+})\cap E(G)}(\sigma(u))^2+\sum_{e\in E(G^{-+})\cap E(G)}(\mu(u))^2T_e^2(G)$$

$$+\sum_{e\in E(G^{-+})-E(G)}(\mu(u))^2\sum_{e\in E(G^{-+})-E(G)}(d_{G^{-+}}(e))^2$$

Since $0\le\sigma\le1$ and $0\le\mu\le1$. Therefore, we have

$$M_1^z(G^{-+})\le n^2T_v^2(\bar{G})+m^2T_1^2(G)+m^2T_e^2(G)+(\frac{n(n-2)}{2})^2T_2^2(G)$$

where $T_1(G)=\sum_{v\in V(G^{-+})\cap E(G)} (d_{G^{-+}}(u))^2$ and $T_2(G)=\sum_{e\in E(G^{-+})-E(G)} (d_{G^{-+}}(e))^2$. ■

Theorem 4. *Let G be n-vertex fuzzy graph with m-edges. Then*

$$M_1^z(G^{--})\le n^2T_v^2(\bar{G})+m^2T_1^2(G)+m^2T_e^2(G)+(\frac{n(n-2)}{2}+m(n-2)-m)^2T_2^2(G)$$

Proof. Now, the FEZI of G^{--} is given by

$$M_1^z(G^{--})=\sum_{v\in V(G^{--})}(\sigma(u)d_{G^{--}}(u))^2+\sum_{e\in E(G^{--})}(\mu(u)d_{G^{--}}(e))^2$$

$$=\sum_{v\in V(G^{--})\cap V(G)}(\sigma(u)d_{G^{--}}(u))^2+\sum_{v\in V(G^{--})\cap E(G)}(\sigma(u)d_{G^{--}}(u))^2+\sum_{e\in E(G^{--})\cap E(G)}(\mu(u)d_{G^{--}}(e))^2+\sum_{e\in E(G^{--})-E(G)}(\mu(u)d_{G^{--}}(e))^2$$

Since $\sum(pq)^2\le\sum p^2\sum q^2$, we have

$$M_1^z(G^{--})\le\sum_{v\in V(G^{--})\cap V(G)}(\sigma(u))^2\sum_{v\in V(G^{--})\cap V(G)}(d_{G^{--}}(u))^2+\sum_{v\in V(G^{--})\cap E(G)}(\sigma(u))^2\sum_{v\in V(G^{--})\cap E(G)}(d_{G^{--}}(u))^2$$

$$+\sum_{e\in E(G^{--})\cap E(G)}(\mu(u))^2\sum_{e\in E(G^{--})\cap E(G)}(d_{G^{--}}(e))^2+\sum_{e\in E(G^{--})-E(G)}(\mu(u))^2\sum_{e\in E(G^{--})-E(G)}(d_{G^{--}}(e))^2$$

$$\le\sum_{v\in V(G^{--})\cap V(G)}(\sigma(u))^2T_v^2(G)+\sum_{v\in V(G^{--})\cap E(G)}(\sigma(u))^2\sum_{v\in V(G^{--})\cap E(G)}(\sigma(u))^2+\sum_{e\in E(G^{--})\cap E(G)}(\mu(u))^2T_e^2(G)$$

$$+\sum_{e\in E(G^{--})-E(G)}(\mu(u))^2\sum_{e\in E(G^{--})-E(G)}(d_{G^{--}}(e))^2$$

Since $0\le\sigma\le1$ and $0\le\mu\le1$. Therefore, we have

$$M_1^z(G^{--})\le n^2T_v^2(\bar{G})+m^2T_1^2(G)+m^2T_e^2(G)+(\frac{n(n-2)}{2}+m(n-2)-m)^2T_2^2(G)$$

where $T_1(G)=\sum_{v\in V(G^{--})\cap E(G)} (d_{G^{--}}(u))^2$ and $T_2(G)=\sum_{e\in E(G^{--})-E(G)} (d_{G^{--}}(e))^2$. ■

3. Conclusion

In this paper we have studied the first entire Zagreb index of generalized fuzzy transformation graphs and obtained some upper bounds. The research on this topic is new and many properties of generalized fuzzy transformation graphs can be considered for future study.

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