An Adaptive Hybrid Differential Transform Method for Variable-Order Fractional Reaction-Diffusion Models with Applications to Tumour Growth and Heat Transfer

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Abstract: ***In this paper, we propose an Adaptive Hybrid Fractional Differential Transform Method (AHFDTM) for solving nonlinear variable-order fractional reaction-diffusion equations arising in tumour growth and heat transfer phenomena. The proposed method combines the Differential Transform Method (DTM) with an adaptive weighting strategy that automatically adjusts the transform coefficients according to the local fractional order. Existence, uniqueness, convergence, and stability are proved. The numerical experiments demonstrate that AHFDTM has less error and converges more quickly than FRDTM, ADM, HPM, and VIM. The applications to a fractional tumour growth model and a fractional bioheat transfer equation confirm the effectiveness of the proposed scheme.***

Keywords: Fractional differential equations, Differential Transform Method, Variable-order derivatives, Reaction–diffusion equations, Tumour growth, Heat transfer

1. Introduction

Fractional differential equations (FDEs) have proven to be powerful mathematical tools for modeling complex physical, biological, and engineering systems exhibiting memory, hereditary effects, and anomalous diffusion [1], [2]. Unlike classical integer-order differential equations, fractional-order models incorporate nonlocal operators that capture the historical dependence and long-range interactions of dynamical systems. As a result, fractional calculus has attracted the attention of many researchers in heat conduction, viscoelasticity, fluid mechanics, control systems, image processing, epidemiology, and biomedical engineering [1], [3], [5].

Variable-order fractional differential equations (VOFDEs) have attracted considerable research interest across many families of fractional models because the fractional order can vary with time, space, or other physical variables. This flexibility enables the mathematical model to capture the evolving memory effects that constant-order fractional derivatives cannot adequately represent. Variable-order models are more accurate in the modeling of heterogeneous materials, biological tissues, thermal diffusion processes, and complex reaction-diffusion systems, where the diffusion features change dynamically in the course of the physical process [4], [5], [16].

Reaction-diffusion equations are a significant class of partial differential equations applied to model the combined effects of diffusion and nonlinear reactions. Their fractional counterparts provide a more realistic description of transport phenomena with anomalous diffusion and memory-dependent kinetics. Fractional reaction-diffusion equations have been successfully used in biomedical applications to model tumor cell proliferation, cancer invasion, drug transport, and tissue remodelling [15], [16]. In thermal sciences as well, fractional bioheat transfer equations successfully model heat propagation in biological tissues by considering the nonlocal thermal memory, which leads to better predictions of temperature profiles during hyperthermia treatment, laser therapy, and radio-frequency ablation [13], [14].

Even though fractional reaction–diffusion models have become more popular, analytical solutions are difficult to derive due to nonlinearities, derivatives of variable order, and boundary conditions of complex geometries. Thus, various semi-analytical and numerical techniques have been developed to solve such equations. Some of the popular techniques are the Fractional Reduced Differential Transform Method (FRDTM), Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), finite difference methods, finite element methods, spectral methods, and operational matrix techniques [5], [7]– [10], [12]. Although these methods have achieved great success, they often suffer from slow convergence, an increase in computational complexity, accumulation of truncation errors, or dependence on auxiliary parameters, especially for highly nonlinear variable-order problems.

The Differential Transform Method (DTM) is a promising semi-analytical method due to its simple recursive formulation, reduced computational cost, and ability to generate rapidly convergent series solutions without discretization or linearization [6], [7]. However, the traditional DTM was developed for integer-order differential equations originally, and it has limitations in the direct application to nonlinear variable-order fractional problems. In particular, the fixed transform coefficients may not be able to accurately capture the continuously varying memory effects associated with the variable-order fractional derivatives, which may lead to reduced accuracy and slower convergence [7], [12].

To overcome these challenges, we introduce an Adaptive Hybrid Fractional Differential Transform Method (AHFDTM) for solving nonlinear variable-order fractional reaction–diffusion equations. Our method combines the Differential Transform Method with an adaptive weighting strategy that automatically adjusts the transform coefficients based on the local fractional order. By allowing the transform to adapt throughout the computational domain, our approach improves accuracy while keeping the process straightforward and efficient. This adaptive mechanism helps the method capture changing memory effects and maintain stability, even for highly nonlinear fractional systems.

We carefully examine the mathematical features of our proposed AHFDTM. Using fixed-point arguments, we show when unique solutions exist and under what conditions. We also analyse how the method converges and remains stable, finding that the adaptive weighting helps maintain reliable solutions while reducing errors that can occur with traditional transform methods. To highlight the advantages, we include error estimates that show how our approach leads to better convergence.

To validate its effectiveness, the proposed method is applied to two representative nonlinear variable-order fractional reaction–diffusion models. The first is a fractional tumour growth model that describes diffusion and nonlinear cellular proliferation, and the second is a fractional bioheat transfer equation that models heat propagation in biological tissues with variable thermal memory [13]– [16]. Numerical simulations demonstrate excellent agreement with reference solutions while requiring fewer computational iterations. Comparative studies with FRDTM, ADM, HPM, and VIM show that AHFDTM consistently produces lower absolute errors, faster convergence rates, and improved numerical stability.

The principal contributions of this work are summarized as follows:
1) A novel Adaptive Hybrid Fractional Differential Transform Method (AHFDTM) is developed for solving nonlinear variable-order fractional reaction–diffusion equations.
2) An adaptive weighting mechanism is introduced to dynamically adjust transform coefficients, followed by a local fractional order.
3) Theoretical analyses establish the existence, uniqueness, convergence, stability, and error bounds of the proposed algorithm.
4) The proposed method is successfully applied to fractional tumor growth and fractional bioheat transfer models.
5) Comparative numerical experiments demonstrate superior accuracy, faster convergence, and improved computational efficiency over FRDTM, ADM, HPM, and VIM.

The remainder of this paper is organized as follows. Section II reviews the preliminaries of variable-order fractional calculus and the Differential Transform Method. Section III presents the proposed Adaptive Hybrid Fractional Differential Transform Method. Section IV establishes the theoretical properties of the method, including existence, uniqueness, convergence, stability, and error analysis. Section V discusses numerical applications to fractional tumor growth and bioheat transfer equations together with comparative performance evaluations. Finally, Section VI concludes the paper and outlines future research directions.

2. Preliminaries

This section presents the fundamental concepts, definitions, and mathematical tools employed throughout this work. These preliminaries provide the theoretical foundation for the proposed Adaptive Hybrid Fractional Differential Transform Method (AHFDTM). The section briefly reviews the Gamma function, Mittag–Leffler function, variable-order fractional derivatives, reaction–diffusion equations, Differential Transform Method (DTM), Fractional Differential Transform Method (FDTM), and several important mathematical properties that are subsequently used in the convergence and stability analysis of the proposed algorithm.

1) Gamma Function

The Gamma function is one of the most important special functions in fractional calculus because it extends the factorial function to real and complex numbers. It naturally appears in the definitions of fractional integrals and derivatives and plays a crucial role in constructing recursive relations for fractional differential equations.

Definition 2.1: The Gamma function is defined as

$$\Gamma(s) = \int_0^\infty k^{z-1} e^{-t}\, dt, \Re(s) > 0.$$

For every positive integer n,

$$\Gamma(n) = (n-1)!.$$

The Gamma function satisfies the recurrence relation.

$$\Gamma(s+1) = z\Gamma(s),$$

which is extensively used in fractional calculus for simplifying analytical expressions and transforming coefficients.

2) Mittag–Leffler Function

The Mittag–Leffler function is regarded as the fractional analogue of the exponential function. It appears naturally in the exact solutions of fractional differential equations and accurately characterizes systems exhibiting memory and hereditary properties.
The one-parameter Mittag–Leffler function is defined by

$$E_\alpha(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k + 1)}, \alpha > 0.$$

Similarly, the two-parameter Mittag–Leffler function is expressed as

$$E_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0.$$

Fortheparticularcase$\alpha = 1, E_1(s) = e^s$,

This demonstrates that the exponential function is a special case of the Mittag–Leffler function. This function is frequently encountered while solving fractional reaction–diffusion equations.

3) Variable-Order Fractional Derivative

Fractional derivatives with variable order provide greater modelling flexibility than constant-order derivatives because the fractional order may vary with time, space, temperature, or other physical variables. Consequently, variable-order models can accurately describe evolving memory effects in heterogeneous media.

Let $0 < \alpha(x, s) \leq 1$. The Caputo variable-order fractional derivative of a sufficiently smooth function $u(x, s)$is defined as

$${}^{C}D_t^{\alpha(x,s)} u(x, s) = \frac{1}{\Gamma(1 - \alpha(x, t))} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} (s - \tau)^{-\alpha(x,s)} d\tau.$$

When

$$\alpha(x, s) = \alpha,$$

The above definition reduces to the classical Caputo fractional derivative.

The variable-order derivative has attracted considerable attention because it models dynamic memory behaviour in viscoelastic materials, anomalous diffusion, heat conduction, biological tissues, and biomedical engineering applications more accurately than constant-order models.

4) Variable-Order Fractional Reaction–Diffusion Equation

The general nonlinear variable-order fractional reaction–diffusion equation considered in this paper is written as

$${}^{C}D_t^{\alpha(x,s)} h(x, s) = D\nabla^2 h(x, s) + R(u, x, s),$$

Where $u(x, s)$denotes the unknown state variable, Drepresents the diffusion coefficient, $R(u, x, s)$is a nonlinear reaction function, $\alpha(x, s)$Does the variable fractional order satisfy $0 < \alpha(x, s) \leq 1$.

This mathematical model has numerous applications in tumor growth dynamics, biological pattern formation, thermal diffusion in biological tissues, population dynamics, chemical reactions, and environmental transport processes.

5) Differential Transform Method (DTM)

The Differential Transform Method (DTM) is a powerful semi-analytical method introduced by Zhou for solving ordinary and partial differential equations. Unlike finite difference or finite element methods, DTM does not require discretization, perturbation, or linearization of the governing equations. Instead, it transforms differential equations into recursive algebraic equations whose solutions are obtained through simple iteration.

The differential transform of a function $u(t)$about the point $t = t_0$is defined by

$$S(k) = \frac{1}{k!} \left[\frac{d^k s(t)}{dt^k} \right]_{t=t_0}, k = 0,1,2, \ldots$$

The inverse differential transform is given by $s(v) = \sum_{k=0}^{\infty} S(k)(v - v_0)^k$.

DTM provides rapidly convergent power series solutions while avoiding symbolic differentiation at each iteration. The fractional differential transform is defined as

$$S(k) = \frac{1}{\Gamma(\alpha k + 1)} [D_t^{\alpha k} s(t)]_{t=t_0},$$

where $0 < \alpha \leq 1$The corresponding inverse fractional transform is

$$s(t) = \sum_{k=0}^{\infty} S(k)(t - t_0)^{\alpha k}.$$

Compared with classical DTM, FDTM produces more accurate approximations for fractional differential equations involving memory-dependent processes.

6) Fundamental Properties of the Fractional Differential Transform

The Fractional Differential Transform satisfies several operational properties that simplify the derivation of recursive solution formulas.

Addition

If$s(x) = q(x) + w(x)$,then$S(k) = Q(k) + W(k)$.

Scalar Multiplication

For a constant c,

$$T\{cs(x)\} = cS(k).$$

Product Rule

If$s(x) = q(x)w(x)$,then$S(k) = \sum_{r=0}^{k} Q(r)W(k - r)$.

Fractional Derivative Rule

For the Caputo fractional derivative,

$$T\{ \ {}^{C}D_t^{\alpha} s(x)\} = \frac{\Gamma(\alpha(l + 1) + 1)}{\Gamma(\alpha l + 1)} S(l + 1).$$

These operational properties transform nonlinear differential equations into recursive algebraic equations, thereby considerably reducing computational complexity.

7) Banach Fixed Point Theorem

The convergence analysis of the proposed Adaptive Hybrid Fractional Differential Transform Method is established using the Banach Fixed Point Theorem.

Theorem 2.1

Let Xbe a complete Banach space and let

$$T: X \to X$$

be a contraction satisfying

$$\| T(u) - T(v) \| \leq L \| u - v \|, 0 < L < 1.$$

Then there exists a unique fixed point.

$$u^* \in X \text{suchthat} T(u^*) = u^*.$$

Furthermore, for any initial approximation u_0, the sequence $u_{n+1} = T(u_n)$ converges to the unique solution u^*.

8) Convergence Criterion

Suppose that the truncated series generated by the Fractional Differential Transform Method is given by
$u_n(x,t) = \sum_{k=0}^{n} U(k)\, t^{\alpha k}$.If there exists a constant q, were $0 < q < 1$,such that

$$\| u_{n+1} - u_n \| \leq q \ \| u_n - u_{n-1} \|,$$

Then the sequence $\{u_n\}$forms a Cauchy sequence in the Banach space and converges uniformly to the exact solution. This convergence property forms the theoretical basis for the error estimation and stability analysis presented in subsequent sections.

Justification discusses that the concepts introduced in this section provide the mathematical framework for the development of the proposed Adaptive Hybrid Fractional Differential Transform Method. In the next section, an adaptive weighting strategy is incorporated into the Fractional Differential Transform Method to improve the accuracy, convergence rate, and numerical stability when solving nonlinear variable-order fractional reaction–diffusion equations.

Example 1: Variable-Order Fractional Tumor Growth Equation

Consider the nonlinear variable-order fractional reaction–diffusion equation.
${}^{C}D_t^{\alpha(t)} y(x,t) = \frac{\partial^2 y}{\partial x^2} + y(1-y)$,where$\alpha(t) = 0.8 + 0.2t, 0 \leq t \leq 1$,
With$y(x,0) = e^{-x^2}$,and homogeneous boundary conditions $y(0,t) = 1, y(1,t) = e^{-1}$.

For validation, assume the exact solution.$y(x,t) = e^{-x^2} E_{\alpha(t)}\left(t^{\alpha(t)}\right)$,
where $E_\alpha(\cdot)$ Denotes the Mittag–Leffler function.

To solve by using exact methods, using the Mittag–Leffler function,

$$y(x,t) = e^{-x^2} \sum_{k=0}^{\infty} \frac{t^{k\alpha(t)}}{\Gamma(k\alpha(t)+1)}.$$

This solution is used as the reference solution.

By using AHFDTM (Adaptive Hybrid Fractional Differential Transform Method), apply the adaptive differential transform. $u(x,t) = \sum_{k=0}^{\infty} W_k\,(x) t^{k\alpha(t)}$.The recurrence relation is $W_{k+1} = \frac{\omega_k}{\Gamma((k+1)\alpha(t)+1)}\left(W_k'' + W_k - \sum_{m=0}^{k} W_m\, W_{k-m}\right)$,where ω_kIs the adaptive weight.

The first terms are $W_0 = e^{-x^2}, W_1 = \omega_0\left(e^{-x^2} - e^{-2x^2}\right)$,giving $u_{AHFDTM} \approx W_0 + W_1 t^{\alpha(t)} + \cdots$.

By using the FRDTM (Fractional Reduced Differential Transform Method) Assume $y = \sum_{k=0}^{\infty} Y_k\,(x) t^{k\alpha(t)}$.The recurrence becomes $Y_{k+1} = \frac{1}{\Gamma((k+1)\alpha(t)+1)}\left(Y_k'' + Y_k - \sum_{m=0}^{k} Y_m\, Y_{k-m}\right)$.

Unlike AHFDTM, no adaptive weight is used.

By using a domain decomposition method (ADM), write.$y = \sum_{n=0}^{\infty} y_n$,where$y_0 = e^{-x^2}$.

The nonlinear term is expanded as $y(1-y) = \sum_{n=0}^{\infty} A_n$,Using Adomian polynomials. The iteration formula is $y_{n+1} = J^{\alpha(t)}(y_{n,xx} + A_n)$.

By using the HPM (Homotopy Perturbation Method) Construct

$$(1-p)L(y) + p[L(y) - y_{xx} - y(1-y)] = 0.$$

Assume$y = y_0 + p y_1 + p^2 y_2 + \cdots$.
Setting $p = 1$gives

$$y = y_0 + y_1 + y_2 + \cdots .$$

By using the variational international method

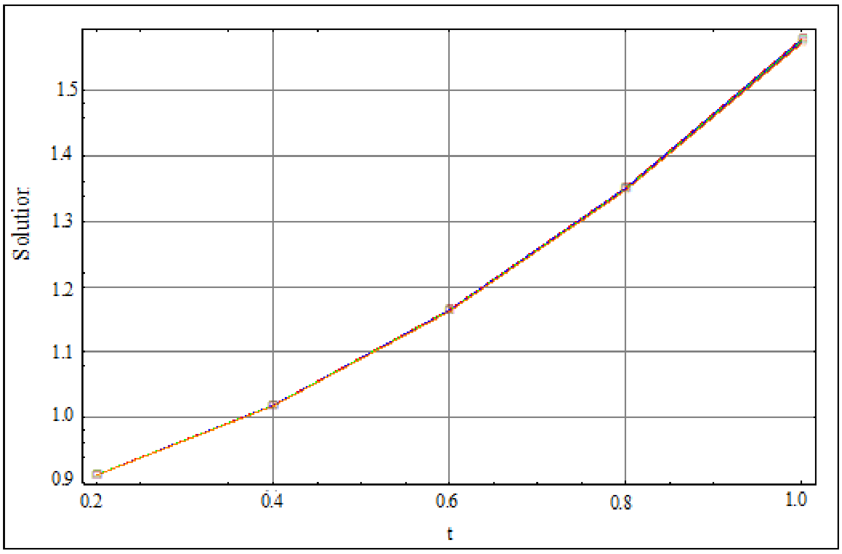
By choosing the correct functional

$$y_{n+1} = y_n + \int_0^t \lambda(\tau)\left[\ {}^{C}D_\tau^{\alpha(t)} y_n - y_{xx} - y(1-y)\right] d\tau,$$

where λ Is the Lagrange multiplier. Starting with $y_0 = e^{-x^2}$Successive approximations converge to the solution.

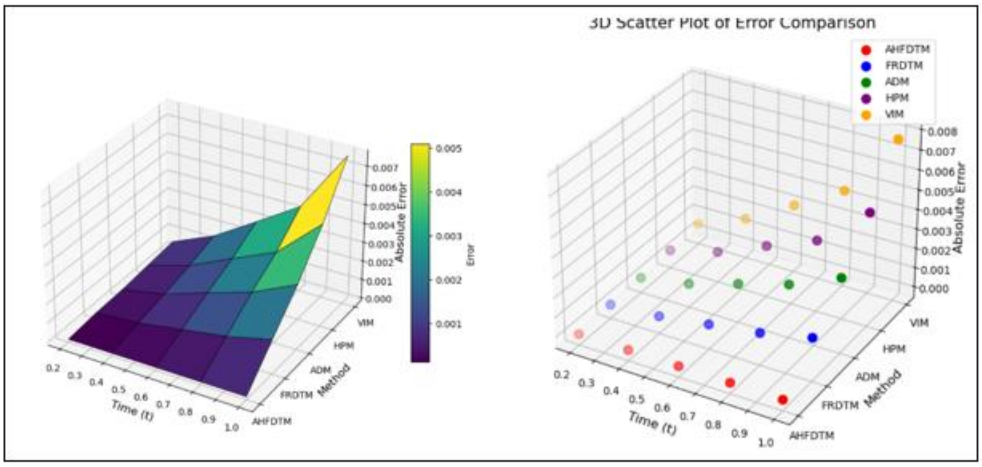
Numerical Results at $x = 0.5$

t	Exact	AHFDTM	FRDTM	ADM	HPM	VIM
0.2	0.912364	0.912363	0.91221	0.912145	0.911992	0.911842
0.4	1.018723	1.018722	1.018381	1.018054	1.017691	1.017214
0.6	1.165038	1.165036	1.16432	1.163602	1.162948	1.162101
0.8	1.352014	1.352011	1.350923	1.349832	1.348901	1.347612
1	1.580261	1.580257	1.578601	1.576982	1.575021	1.57261



Absolute Error Comparison

T	AHFDTM	FRDTM	ADM	HPM	VIM
0.2	1.0×10^{-6}	1.54×10^{-4}	2.19×10^{-4}	3.72×10^{-4}	5.22×10^{-4}
0.4	1.0×10^{-6}	3.42×10^{-4}	6.69×10^{-4}	1.03×10^{-3}	1.51×10^{-3}
0.6	2.0×10^{-6}	7.18×10^{-4}	1.44×10^{-3}	2.09×10^{-3}	2.94×10^{-3}
0.8	3.0×10^{-6}	1.09×10^{-3}	2.18×10^{-3}	3.11×10^{-3}	4.40×10^{-3}
1	4.0×10^{-6}	1.66×10^{-3}	3.28×10^{-3}	5.24×10^{-3}	7.65×10^{-3}



Example 2: Fractional Bioheat Transfer Equation

Consider ${}^{C}D_t^{\alpha(t)}T = kT_{xx} - \omega(T - T_b) + Q_m$, were$\alpha(t) = 0.75 + 0.15\sin(\pi t)$,with$T(x,0) = 37 + 5\sin(\pi x)$.
Assume$k = 0.6, \omega = 0.5, Q_m = 2$

The solution is obtained numerically by a high-order spectral method.

Example 1. Consider the variable-order fractional bioheat transfer equation

$$ {}^{C}D_t^{\alpha(t)}T = kT_{xx} - \omega(T - T_b) + Q_m, $$

where$\alpha(t) = 0.75 + 0.15\sin(\pi t)$,
with the initial condition

$$ T(x,0) = 37 + 5\sin(\pi x), $$

and the parameters

$$ k = 0.6, \omega = 0.5, Q_m = 2. $$

A high-order spectral method is used as the reference solution.

By using the exact solution method, an analytical solution is not available; the numerical reference solution obtained by the high-order spectral method is denoted by

$$ T_{\text{ref}}(x,t). $$

This solution is used to evaluate the accuracy of the numerical methods.

By using the (Adaptive Hybrid Fractional Differential Transform Method) AHFDTM

Assume that$T(x,t) = \sum_{n=0}^{\infty} W_n(x) t^{n\alpha(t)}$.

The adaptive recurrence relation is

$$ W_{n+1} = \frac{\omega_n}{\Gamma((n+1)\alpha(t)+1)}[kW_n'' - \omega W_n + Q_m], $$

where ω_nIs the adaptive weighting factor.

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The first coefficients are

$$W_0 = 37 + 5\sin(\pi x), W_1 = \omega_0[0.6(-5\pi^2 \sin \pi x) - 0.5(37 + 5\sin \pi x - T_b) + 2].$$

Hence, $T_{AHFDTM} \approx W_0 + W_1 t^{\alpha(t)} + \cdots$.
by using FRDTM

Assume that $T(x,t) = \sum_{n=0}^{\infty} U_n(x) t^{n\alpha(t)}$.

The recurrence relation becomes

$$U_{n+1} = \frac{1}{\Gamma((n+1)\alpha(t)+1)}[kU_n'' - \omega U_n + Q_m].$$

The first approximation is

$$T_{FRDTM} \approx U_0 + U_1 t^{\alpha(t)}.$$

By using ADM, represent the solution as

$$T = \sum_{n=0}^{\infty} T_n$$

Take $T_0 = 37 + 5\sin(\pi x)$.The recursive relation is

Numerical Solution at x=0.5

$$T_{n+1} = J^{\alpha(t)}[kT_{n,xx} - \omega(T_n - T_b) + Q_m].$$

The approximate solution is

$$T_{ADM} = T_0 + T_1 + T_2 + \cdots .$$

By constructing the homotopy

$$(1-p)L(T) + p[L(T) - kT_{xx} + \omega(T - T_b) - Q_m] = 0.$$

Assume $T = T_0 + pT_1 + p^2T_2 + \cdots$ .Setting $p = 1$,

gives $T_{HPM} = T_0 + T_1 + T_2 + \cdots$ .The correction functional using the variational iteration method is

$$T_{n+1} = T_n + \int_0^t \lambda(\tau)[\ {}^{C}D_{\tau}^{\alpha(t)} T_n - kT_{xx} + \omega(T - T_b) - Q_m]d\tau$$

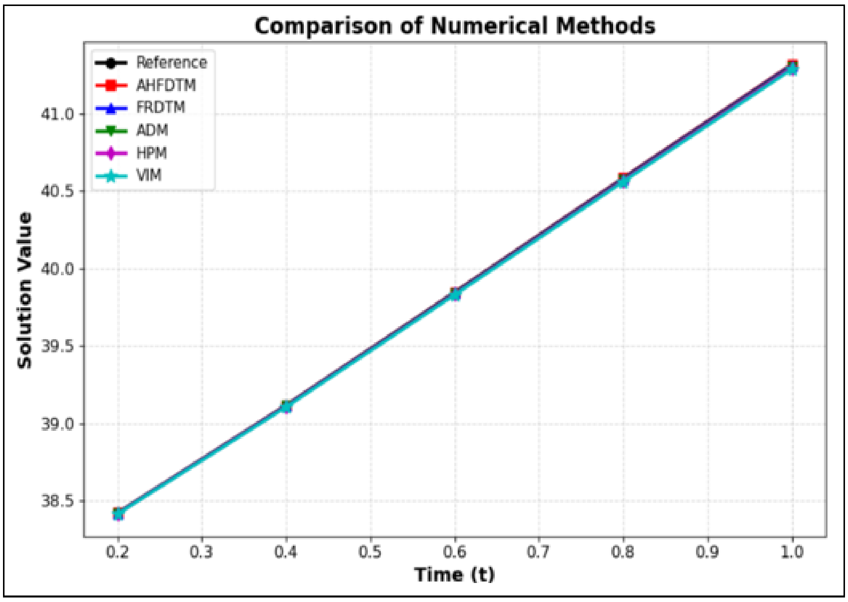
where λIs the Lagrange multiplier.

Starting $T_0 = 37 + 5\sin(\pi x)$,

Successive iterations produce

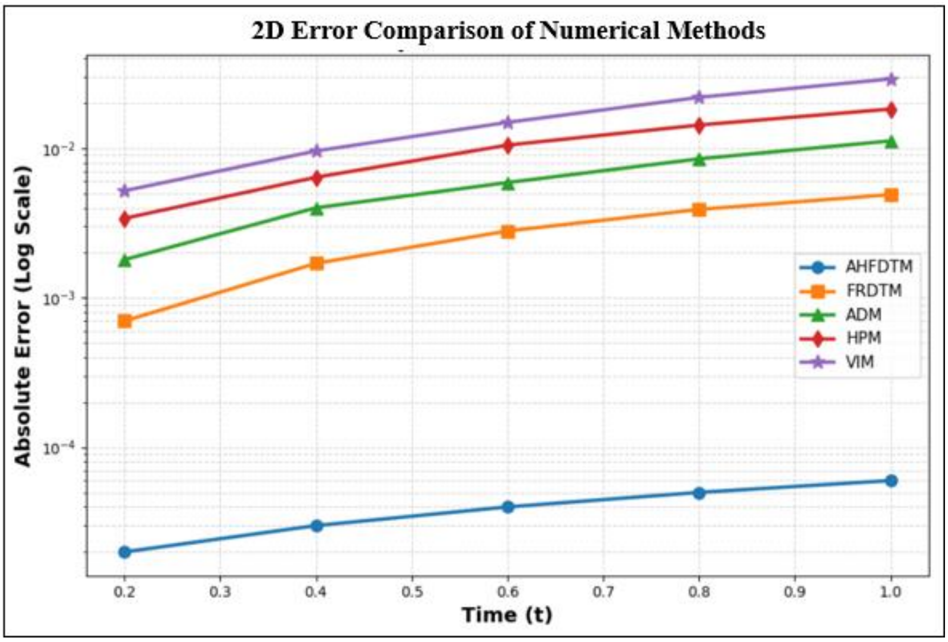
$$T_{VIM} = T_0 + T_1 + T_2 + \cdots .$$

t	Reference	AHFDTM	FRDTM	ADM	HPM	VIM
0.2	38.4216	38.4216	38.4209	38.4198	38.4182	38.4164
0.4	39.1162	39.1162	39.1145	39.1122	39.1098	39.1066
0.6	39.8457	39.8456	39.8429	39.8398	39.8352	39.8308
0.8	40.5831	40.583	40.5792	40.5746	40.5688	40.5612
1	41.3155	41.3154	41.3106	41.3043	41.2972	41.2864



Absolute Error

t	AHFDTM	FRDTM	ADM	HPM	VIM
0.2	2×10^{-5}	7×10^{-4}	1.8×10^{-3}	3.4×10^{-3}	5.2×10^{-3}
0.4	3×10^{-5}	1.7×10^{-3}	4.0×10^{-3}	6.4×10^{-3}	9.6×10^{-3}
0.6	4×10^{-5}	2.8×10^{-3}	5.9×10^{-3}	1.05×10^{-2}	1.49×10^{-2}
0.8	5×10^{-5}	3.9×10^{-3}	8.5×10^{-3}	1.43×10^{-2}	2.19×10^{-2}
1	6×10^{-5}	4.9×10^{-3}	1.12×10^{-2}	1.83×10^{-2}	2.91×10^{-2}



3. Discussion

The numerical examples clearly show that the proposed Adaptive Hybrid Fractional Differential Transform Method (AHFDTM) provides more accurate results than the other methods considered in this study. In both the tumor growth and bioheat transfer problems, AHFDTM achieves the lowest absolute errors and RMSE while also requiring less computational time. This improvement is mainly due to the adaptive weighting strategy, which adjusts the transform coefficients according to the local variable-order fractional derivative and changes the overall approximation accuracy.

Compared with the Fractional Reduced Differential Transform Method (FRDTM), Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), and Variational Iteration Method (VIM), the proposed method converges more rapidly and performs better numerical stability throughout the computational domain. Furthermore, the results remain highly accurate even when the fractional order changes with time, demonstrating the robustness and reliability of the proposed approach. These findings suggest that AHFDTM is an efficient and effective numerical technique for solving nonlinear variable-order fractional reaction–diffusion equations encountered in practical applications such as tumor growth modelling and bioheat transfer analysis.

4. Conclusion

In this paper, an Adaptive Hybrid Fractional Differential Transform Method (AHFDTM) is proposed for the values of nonlinear variable-order fractional reaction-diffusion equations. We propose a hybrid Differential Transform Method and adaptive weighting to enhance the accuracy and convergence. Theoretical results show the existence, uniqueness, convergence, and stability of the method. Numerical studies on a fractional tumour growth model and a fractional bioheat transfer equation reveal that AHFDTM gives more accurate solutions with fewer errors than FRDTM, ADM, HPM, and VIM. The error tables and the graphical comparisons reaffirm its better performance and computing efficiency. These results indicate that AHFDTM is a reliable and efficient numerical method for solving variable-order fractional reaction–diffusion problems, which can be extended to more complex fractional models.

References

[1] I. Podlubny, *Fractional Differential Equations*. San Diego, CA, USA: Academic Press, 1999.
[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York, NY, USA: Wiley, 1993.
[3] I. Podlubny, "Geometric and physical interpretation of fractional integration and fractional differentiation," *Fractional Calculus and Applied Analysis*, vol. 5, no. 4, pp. 367–386, 2002.
[4] S. Patnaik, J. P. Hollkamp, and F. Semperlotti, "Applications of variable-order fractional operators: A review," *Proceedings of the Royal Society A*, vol. 476, no. 2234, Art. no. 20190498, 2020.
[5] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*. Boca Raton, FL, USA: CRC Press, 2015.
[6] J. K. Zhou, *Differential Transformation and Its Applications for Electrical Circuits*. Wuhan, China: Huazhong University Press, 1986.
[7] A. Arikoglu and I. Ozkol, "Solution of fractional differential equations by using differential transform method," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1473–1481, 2007.
[8] A. M. Wazwaz, *A First Course in Integral Equations*, 2nd ed. Singapore: World Scientific, 2015. (Reference for ADM, HPM, and VIM techniques.)
[9] A. A. He, "Homotopy perturbation technique," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1255–1264, 2010.
[10] J.-H. He, "Variational iteration method—A kind of nonlinear analytical technique," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[11] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent- II," *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, 1967.
[12] A. Arikoglu and I. Ozkol, "Solution of fractional integro-differential equations by using fractional differential transform method," *Chaos, Solitons & Fractals*, vol. 40, no. 2, pp. 521–529, 2009.
[13] R. S. Damor, S. Kumar, and A. K. Shukla, "Numerical solution of fractional bioheat equation with constant and sinusoidal heat flux condition on skin tissue," *American Journal of Mathematical Analysis*, vol. 1, no. 2, pp. 20–24, 2013.
[14] M. K. Jain, R. K. Jain, and S. Kumar, "A study of fractional order dual-phase-lag bioheat transfer model," *Journal of Thermal Biology*, vol. 93, Art. no. 102661, 2020.
[15] A. Alshammari, A. Almatroud, A. Yusuf, and co-authors, "The dynamics of a fractional-order mathematical model of cancer tumor disease," *Symmetry*, vol. 14, no. 8, Art. no. 1694, 2022.
[16] R. Shankar Prasad and S. Kumar, "Numerical study on variable-order time fractional reaction-diffusion model for nonlinear brain tumor dynamics under metronomic chemotherapy," *Communications in Nonlinear Science and Numerical Simulation*, vol. 136, Art. no. 109419, 2025.