Euler Hyperbolic Directrix Sombor and Modified Euler Hyperbolic Directrix Sombor Indices of Certain Chemical Structures

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Abstract: ***In this paper, we introduce the Euler hyperbolic directrix Sombor and the modified Euler hyperbolic directrix Sombor indices and their exponentials of a graph. We compute these newly defined the Euler hyperbolic directrix Sombor indices for certain chemical structures.***

Keywords: Euler hyperbolic directrix Sombor index, modified Euler hyperbolic directrix Sombor index, chemical structure.

1. Introduction

The simple graphs which are finite, undirected, connected graphs without loops and multiple edges are considered. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u.

We introduce the Euler hyperbolic directrix Sombor index of a graph G and it is defined as

$$EHDSO(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2 + d_G(u)d_G(v)}}{\min\{d_G(u)^2, d_G(v)^2\}}.$$

Considering the Euler hyperbolic directrix Sombor index, we introduce the Euler hyperbolic directrix Sombor exponential of a graph G and defined it as

$$EHDSO(G,x) = \sum_{uv\in E(G)} x^{\frac{\sqrt{d_G(u)^2 + d_G(v)^2 + d_G(u)d_G(v)}}{\min\{d_G(u)^2, d_G(v)^2\}}}.$$

Recently, some hyperbolic Sombor indices were studied in [1, 2, 3, 4, 5, 6, 7].

We define the modified Euler hyperbolic directrix Sombor index of a graph G as

$$^{m}EHDSO(G) = \sum_{uv\in E(G)} \frac{\min\{d_G(u)^2, d_G(v)^2\}}{\sqrt{d_G(u)^2 + d_G(v)^2 + d_G(u)d_G(v)}}.$$

Considering the modified Euler hyperbolic directrix Sombor index, we introduce the modified Euler hyperbolic directrix Sombor exponential of a graph G and defined it as

$$^{m}EHDSO(G,x) = \sum_{uv\in E(G)} x^{\frac{\min\{d_G(u)^2, d_G(v)^2\}}{\sqrt{d_G(u)^2 + d_G(v)^2 + d_G(u)d_G(v)}}}.$$

In this research, we compute the Euler hyperbolic directrix Sombor and modified Euler hyperbolic directrix Sombor indices for some standard graphs and certain chemical structures.

2. Results for Some Standard Graphs

Proposition 1. If G is an r-regular graph with n vertices, $r \geq 3$, then

$$EHDSO(G) = \frac{\sqrt{3}n}{2}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 3$. Then $|E(G)| = \frac{nr}{2}$. For any vertex u in $V(G)$, $d_G(u) = r$. Then

$$EHDSO(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2 + d_G(u)d_G(v)}}{\min\{d_G(u)^2, d_G(v)^2\}}$$

$$= \frac{nr}{2}\frac{\sqrt{r^2 + r^2 + rr}}{r^2}$$

$$= \frac{\sqrt{3}n}{2}.$$

Proposition 2. Let $K_{m,n}$ be a complete bipartite graph with $1 \leq m \leq n$ *and* $n \geq 2$. Then

$$EHDSO(K_{m,n}) = \frac{n\sqrt{m^2 + n^2 + mn}}{m}.$$

Proof: Let $K_{m,n}$ be a complete bipartite graph with $m + n$ vertices and mn *edges* such that $|V_1| = m$, $|V_2| = n$, $V(K_{r,s}) = V_1 \cup V_2$ for $1 \leq m \leq n$, and $n \geq 2$. Every vertex of V_1 is incident with n edges and every vertex of V_2 is incident with m edges.

$$EHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}$$
$$=mn\frac{\sqrt{m^2+n^2+mn}}{\min\{m^2,n^2\}}$$
$$=\frac{n\sqrt{m^2+n^2+mn}}{m}.$$

Corollary 2.1. Let $K_{n,n}$ be a complete bipartite graph with $n\geq 2$. Then

$$EHDSO(K_{n,n})=n\sqrt{3}.$$

Corollary 2.2. Let $K_{1,n}$ be a star with $n\geq 2$. Then

$$DRF(K_{1,n})=n\sqrt{n^2+n+1}.$$

3. Armchair Polyhex Nanotubes

Carbon polyhex nanotubes exist in nature with remarkable stability and possess very interesting electrical, thermal and mechanical properties. The molecular graph of armchair polyhex nanotube $TUAC_6$ [p, q] is shown in the below graph.

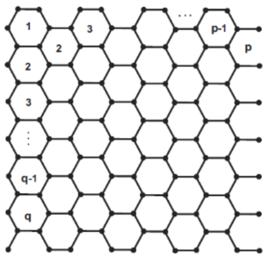

Figure 1

The graphs of armchair polyhex nanotubes have $2p(q+1)$ vertices and $3pq + 2p$ edges are shown in the above graph. Let $G= TUAC_6[p, q]$.

We obtain that $\{d(u), d(v): uv \in E(G)\}$ has three edge set partitions.

$d(u), d(v) \setminus uv \in E(G)$	(2, 2)	(2, 3)	(3, 3)
Number of edges	p	$2p$	$3pq - p$

Theorem 1: The Euler hyperbolic directrix Sombor index of $TUAC_6$ [p, q] is given by

$$EHDSO(G)=\sqrt{3}pq+\frac{\sqrt{3}}{2}p+\frac{\sqrt{19}}{2}p-\frac{1}{\sqrt{3}}p.$$

Proof: Applying definition and edge partition of $TUAC_6$ [p, q], we conclude

$$EHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}$$
$$=p\frac{\sqrt{2^2+2^2+2\times 2}}{2^2}+2p\frac{\sqrt{2^2+3^2+2\times 3}}{2^2}$$
$$+(3pq-p)\frac{\sqrt{3^2+3^2+3\times 3}}{3^2}.$$

By solving the above equation, we get the desired result.

Theorem 2. The Euler hyperbolic directrix Sombor exponential of $TUAC_6$ [p, q] is

$$EHDSO(G,x)=px^{\frac{\sqrt{3}}{2}}+2px^{\frac{\sqrt{19}}{4}}+(3pq-p)x^{\frac{\sqrt{3}}{3}}.$$

Proof: Applying definition and edge partition of $TUAC_6$ [p, q], we conclude

$$EHDSO(G,x)=\sum_{uv\in E(G)}x^{\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}}$$
$$=px^{\frac{\sqrt{2^2+2^2+2\times 2}}{2^2}}+2px^{\frac{\sqrt{2^2+3^2+2\times 3}}{2^2}}$$
$$+(3pq-p)x^{\frac{\sqrt{3^2+3^2+3\times 3}}{3^2}}.$$

By solving the above equation, we get the desired result.

Theorem 3. The modified Euler hyperbolic directrix Sombor index of $TUAC_6$ [p, q] is

$$^{m}EHDSO(G)=\frac{9}{\sqrt{3}}pq+\frac{8}{\sqrt{19}}p-\frac{1}{\sqrt{3}}p.$$

Proof: Applying definition and edge partition of $TUAC_6$ [p, q], we conclude

$$^{m}EHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\min\{d_G(u)^2,d_G(v)^2\}}{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}$$
$$=p\frac{2^2}{\sqrt{2^2+2^2+2\times 2}}+2p\frac{2^2}{\sqrt{2^2+3^2+2\times 3}}$$
$$+(3pq-p)\frac{3^2}{\sqrt{3^2+3^2+3\times 3}}.$$

By solving the above equation, we get the desired result.

Theorem 4. The modified Euler hyperbolic directrix Sombor exponential of $TUAC_6$ [p, q] is

$$^{m}EHDSO(G,x)=px^{\frac{2}{\sqrt{3}}}+2px^{\frac{4}{\sqrt{19}}}+(3pq-p)x^{\frac{3}{\sqrt{3}}}.$$

Proof: Applying definition and edge partition of $TUAC_6$ [p, q], we conclude

$$^{m}EHDSO(G,x)=\sum_{uv\in E(G)}x^{\frac{\min\{d_G(u)^2,d_G(v)^2\}}{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}}$$
$$=px^{\frac{2^2}{\sqrt{2^2+2^2+2\times 2}}}+2px^{\frac{2^2}{\sqrt{2^2+3^2+2\times 3}}}$$

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$$+(3pq-p)x^{\frac{3^2}{\sqrt{3^2+3^2+3\times3}}}.$$

By solving the above equation, we get the desired result.

4. ZigZag Polyhex Nanotubes

The molecular graph of zigzag polyhex nanotube $TUZC_6$ [p, q] is depicted in below graph.

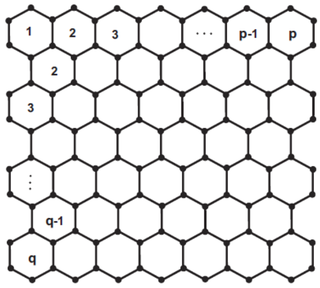

Figure 2

The graphs of zigzag polyhex nanotubes have $2p(q+1)$ vertices and $3pq + 2p$ edges are shown in the above graph. Let $G = TUZC_6[p, q]$.

We obtain that $\{d(u), d(v): uv \in E(G)\}$ has three edge set partitions.

$d(u), d(v) \setminus uv \in E(G)$	(2, 3)	(3, 3)
Number of edges	$4p$	$3pq - 2p$

Theorem 5. The Euler hyperbolic directrix Sombor index of $TUZC_6$ [p, q] is given by

$$EHDSO(G)=\sqrt{3}pq+\sqrt{19}p-\frac{2}{\sqrt{3}}p.$$

Proof: Applying definition and edge partition of $TUZC_6$ [p, q], we conclude

$$EHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}$$
$$=4p\frac{\sqrt{2^2+3^2+2\times3}}{2^2}$$
$$+(3pq-2p)\frac{\sqrt{3^2+3^2+3\times3}}{3^2}.$$

By solving the above equation, we get the desired result.

Theorem 6. The Euler hyperbolic directrix Sombor exponential of $TUZC_6$ [p, q] is

$$EHDSO(G,x)=4px^{\frac{\sqrt{19}}{4}}+(3pq-2p)x^{\frac{\sqrt{3}}{3}}.$$

Proof: Applying definition and edge partition of $TUZC_6$ [p, q], we conclude

$$EHDSO(G,x)=\sum_{uv\in E(G)}x^{\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}}$$
$$=4px^{\frac{\sqrt{2^2+3^2+2\times3}}{2^2}}+(3pq-2p)x^{\frac{\sqrt{3^2+3^2+3\times3}}{3^2}}$$

By solving the above equation, we get the desired result.

Theorem 7. The modified Euler hyperbolic directrix Sombor index of $TUZC_6$ [p, q] is

$${}^mEHDSO(G)=\frac{9}{\sqrt{3}}pq+\frac{8}{\sqrt{19}}p-\frac{6}{\sqrt{3}}p.$$

Proof: Applying definition and edge partition of $TUZC_6$ [p, q], we conclude

$${}^mEHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\min\{d_G(u)^2,d_G(v)^2\}}{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}$$
$$=4p\frac{2^2}{\sqrt{2^2+3^2+2\times3}}+(3pq-2p)\frac{3^2}{\sqrt{3^2+3^2+3\times3}}.$$

By solving the above equation, we get the desired result.

Theorem 8. The modified Euler hyperbolic directrix Sombor exponential of $TUZC_6$ [p, q] is

$${}^mEHDSO(G,x)=4px^{\frac{4}{\sqrt{19}}}+(3pq-2p)x^{\frac{3}{\sqrt{3}}}.$$

Proof: Applying definition and edge partition of $TUZC_6$ [p, q], we conclude

$${}^mEHDSO(G,x)=\sum_{uv\in E(G)}x^{\frac{\min\{d_G(u)^2,d_G(v)^2\}}{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}}$$
$$=4px^{\frac{2^2}{\sqrt{2^2+3^2+2\times3}}}+(3pq-2p)x^{\frac{3^2}{\sqrt{3^2+3^2+3\times3}}}$$

By solving the above equation, we get the desired result.

5. Carbon Nanocone Networks

The molecular graph of pentagonal nanocone network CNC_5 [n] is depicted in below graph.

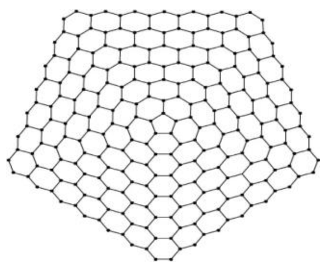

Figure 3

The graphs of pentagonal nanocone networks have $5(n+1)^2$ vertices and $\frac{15}{2}n^2+\frac{25}{2}n+5$ edges. Let $G = CNC_5[n]$.

We obtain that $\{d(u), d(v): uv \in E(G)\}$ has three edge set partitions.

$d(u), d(v) \backslash uv\in E(G)$	(2, 2)	(2, 3)	(3, 3)
Number of edges	5	$10n$	$\frac{15}{2}n^2+\frac{5}{2}n$

Theorem 9. The Euler hyperbolic directrix Sombor index of $CNC_5[n]$ is given by

$$EHDSO(G)=\frac{5\sqrt{3}}{2}n^2+\frac{5\sqrt{19}}{2}n+\frac{5\sqrt{3}}{6}n+\frac{5\sqrt{3}}{2}.$$

Proof: Applying definition and edge partition of $CNC_5[n]$, we conclude

$$EHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}$$
$$=5\frac{\sqrt{2^2+2^2+2\times 2}}{2^2}+10n\frac{\sqrt{2^2+3^2+2\times 3}}{2^2}$$
$$+\left(\frac{15}{2}n^2+\frac{5}{2}n\right)\frac{\sqrt{3^2+3^2+3\times 3}}{3^2}.$$

By solving the above equation, we get the desired result.

Theorem 10. The Euler hyperbolic directrix Sombor exponential of $CNC_5[n]$ is

$$EHDSO(G,x)=5x^{\frac{\sqrt{3}}{2}}+10nx^{\frac{\sqrt{19}}{4}}+\left(\frac{15}{2}n^2+\frac{5}{2}n\right)x^{\frac{\sqrt{3}}{3}}.$$

Proof: Applying definition and edge partition of $CNC_5[n]$, we conclude

$$EHDSO(G,x)=\sum_{uv\in E(G)}x^{\frac{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}{\min\{d_G(u)^2,d_G(v)^2\}}}$$
$$=5x^{\frac{\sqrt{2^2+2^2+2\times2}}{2^2}}+10nx^{\frac{\sqrt{2^2+3^2+2\times3}}{2^2}}$$
$$+\left(\frac{15}{2}n^2+\frac{5}{2}n\right)x^{\frac{\sqrt{3^2+3^2+3\times3}}{3^2}}.$$

By solving the above equation, we get the desired result.

Theorem 11. The modified Euler hyperbolic directrix Sombor index of $CNC_5[n]$ is

$$^{m}EHDSO(G)=\frac{45}{2\sqrt{3}}n^2+\frac{40}{\sqrt{19}}n+\frac{15}{2\sqrt{3}}n+\frac{10}{\sqrt{3}}.$$

Proof: Applying definition and edge partition of $CNC_5[n]$, we conclude

$$^{m}EHDSO(G)$$
$$=\sum_{uv\in E(G)}\frac{\min\{d_G(u)^2,d_G(v)^2\}}{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}$$
$$=5\frac{2^2}{\sqrt{2^2+2^2+2\times 2}}+10n\frac{2^2}{\sqrt{2^2+3^2+2\times 3}}$$
$$+\left(\frac{15}{2}n^2+\frac{5}{2}n\right)\frac{3^2}{\sqrt{3^2+3^2+3\times 3}}.$$

By solving the above equation, we get the desired result.

Theorem 12. The modified Euler hyperbolic directrix Sombor exponential of $CNC_5[n]$ is

$$^{m}EHDSO(G,x)$$
$$=5x^{\frac{2}{\sqrt{3}}}+10nx^{\frac{4}{\sqrt{19}}}+\left(\frac{15}{2}n^2+\frac{5}{2}n\right)x^{\frac{3}{\sqrt{3}}}.$$

Proof: Applying definition and edge partition of $CNC_5[n]$, we conclude

$$^{m}EHDSO(G,x)$$
$$=\sum_{uv\in E(G)}x^{\frac{\min\{d_G(u)^2,d_G(v)^2\}}{\sqrt{d_G(u)^2+d_G(v)^2+d_G(u)d_G(v)}}}$$
$$=5x^{\frac{2^2}{\sqrt{2^2+2^2+2\times2}}}+10nx^{\frac{2^2}{\sqrt{2^2+3^2+2\times3}}}$$
$$+\left(\frac{15}{2}n^2+\frac{5}{2}n\right)x^{\frac{3^2}{\sqrt{3^2+3^2+3\times3}}}.$$

By solving the above equation, we get the desired result.

6. Conclusion

In this research, we have introduced the Euler hyperbolic directrix Sombor and modified Euler hyperbolic directrix Sombor indices of a graph. Also we have determined these newly defined the Euler hyperbolic directrix Sombor indices of armchair polyhex nanotubes, zigzag polyhex nanotubes and carbon nanocone networks.

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