Common Fixed-Point Theorems for Sequences of Mappings Under General Altering Distance Functions

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Abstract: ***The main purpose of this paper is to establish a unique common fixed-point theorem for a sequence of self-maps in a closed subset of a complete metric space by altering distances under a certain continuous control function. This theorem generalizes the results of D. M. Pandhare and B. B. Waghmode.***

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1. Introduction

The existence of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function was established by Khan, Swaleh, and Sessa as well as Pathak and Rekha Sharma. K. P. R. Sastry and G. V. R. Babu established the existence and uniqueness of fixed points for self-maps on metric and complete metric spaces by altering distances, which partially generalize some of the prior results.

Throughout this paper, let (X, d) denote a complete metric space, H a Hilbert space, $\mathbb{R}^+$ the set of all non-negative real numbers, and $\mathbb{N}$ the set of all positive integers. We define Φ as the set of all continuous self-maps ϕ of $\mathbb{R}^+$ satisfying:
1) ϕ is strictly increasing.
2) $\phi(t) = 0$ if and only if $t = 0$.

For context, we recall the baseline theorem established in Hilbert spaces:

Theorem 1: (Pandhare and Waghmode). Let C be a closed subset of a Hilbert space H and T_n a sequence of self-maps on C satisfying the following two conditions:

1) For any two mappings T_i, T_j,

$$\| T_i x - T_j y \|^2 \leq a \| x - y \|^2 + b\left(\| x - T_i x \|^2 + \| y - T_j y \|^2\right) \tag{1.1}$$

for all distinct x, y in C, where $a \geq 0, 0 < b < 1$, and $a + 2b < 1$.
2) There is a point $x_0 \in C$ such that any two consecutive members of the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1}$ for $n \geq 1$ are distinct. Then T_n has a unique common fixed point in C. The main aim of this paper is to generalize Theorem 1 to complete metric spaces by altering distances between the points with the use of a continuous control function $\phi \in \Phi$.

2. Main Theorems

Theorem 2: Let T_n be a sequence of self-maps on a closed subset C of a complete metric space (X, d). Assume that there exists $\phi \in \Phi$ such that ϕ is subadditive and satisfies:

(A1): $\phi\left(d(T_i x, T_j y)\right)$

$$\leq a\phi\left(d(x, y)\right) + b\left[\phi\left(d(x, T_i x)\right) + \phi\left(d(y, T_j y)\right)\right]$$

$$+c\left[\phi\left(d(x, T_j y)\right) + \phi\left(d(y, T_i x)\right)\right] \tag{2.1}$$

for all $i, j \in \mathbb{N}$ and for all distinct $x, y \in C$,
where $a \geq 0, c \geq 0, 0 < b < 1$ with $a + 2b + 2c < 1$.

Assume also that:
(B1): There is a point $x_0 \in C$ such that all members of the sequence $\{x_n\}$ defined by

$$x_n = T_n x_{n-1},\ n \geq 1, \text{ are distinct.} \tag{2.2}$$

Then T_n has a unique common fixed point in C.

Proof: Existence of a Common Fixed Point
We first prove the existence of a common fixed point for T_n. Let $\alpha_n = d(x_n, x_{n+1})$ and $\beta_n = \phi(\alpha_n)$. Then from condition (A1), we evaluate β_1 as follows:

$$\beta_1 = \phi(\alpha_1) = \phi\left(d(x_1, x_2)\right) = \phi\left(d(T_1 x_0, T_2 x_1)\right) \leq a\phi\left(d(x_0, x_1)\right)$$
$$+b\left[\phi\left(d(x_0, T_1 x_0)\right) + \phi\left(d(x_1, T_2 x_1)\right)\right] + c\left[\phi\left(d(x_0, T_2 x_1)\right) + \phi\left(d(x_1, T_1 x_0)\right)\right] \tag{2.3}$$

$$\beta_1 \leq a\phi\left(d(x_0, x_1)\right) + b[\phi(d(x_0, x_1)) + \phi(d(x_1, x_2))]$$
$$+c[\phi(d(x_0, x_2)) + \phi(d(x_1, x_1))] \tag{2.4}$$

Since ϕ is increasing, subadditive, and $\phi(0) = 0$, the metric triangle inequality yields $\phi(d(x_0, x_2)) \leq \phi(d(x_0, x_1) + d(x_1, x_2)) \leq \phi(d(x_0, x_1)) + \phi(d(x_1, x_2))$.

Substituting this back into the relation gives:

$$\beta_1 \leq (a + b + c)\phi\big(d(x_0, x_1)\big) + (b + c)\phi\big(d(x_1, x_2)\big) \quad (2.5)$$

$$\beta_1 \leq (a + b + c)\beta_0 + (b + c)\beta_1 \quad (2.6)$$

Rearranging terms leads to:

$$\beta_1 \leq \left(\frac{a + b + c}{1 - b - c}\right)\beta_0 \quad (2.7)$$

Hence $\beta_1 \leq k\beta_0$, where $k = \frac{a+b+c}{1-b-c} < 1$, since $a + 2b + 2c < 1$. By mathematical induction, it is straightforward to demonstrate that:

$$\beta_n \leq k\beta_{n-1} \quad \forall n \geq 1 \quad (2.8)$$

Therefore, $\beta_n \leq k^n\beta_0$, which implies $\beta_n \to 0$ as $n \to \infty$. Because $k < 1$, equation (2.8) implies $\beta_n < \beta_{n-1}$, and since ϕ is strictly increasing, it follows that $\alpha_n < \alpha_{n-1}$ for $n = 1,2, \ldots$. Thus, $\{\alpha_n\}$ is a strictly decreasing sequence of non-negative real numbers, which must converge to some value $\alpha \geq 0$. By the continuity of ϕ, we have $\beta_n = \phi(\alpha_n) \to \phi(\alpha) = 0$, which mandates that $\alpha = 0$. Hence:

$$\lim_{n\to\infty} \alpha_n = 0 \quad (2.9)$$

Next, we show that $\{x_n\}$ forms a Cauchy sequence in C. Suppose it is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ and subsequences $\{m(k)\}$ and $\{n(k)\}$ such that $m(k) < n(k)$:

$$d\big(x_{n(k)}, x_{m(k)}\big) \geq \varepsilon \quad \text{and} \quad d\big(x_{n(k)-1}, x_{m(k)}\big) < \varepsilon \quad (2.10)$$

Applying the function ϕ, we obtain:

$$\phi(\varepsilon) \leq \phi\left(d\big(x_{n(k)}, x_{m(k)}\big)\right) = \phi\left(d\big(T_{n(k)}x_{n(k)-1}, T_{m(k)}x_{m(k)-1}\big)\right) \quad (2.11)$$

$$\phi(\varepsilon) < a\phi\left(d\big(x_{n(k)-1}, x_{m(k)-1}\big)\right) + b\left[\phi\left(d\big(x_{n(k)-1}, x_{n(k)}\big)\right) + \phi\left(d\big(x_{m(k)-1}, x_{m(k)}\big)\right)\right]$$
$$+c\left[\phi\left(d\big(x_{n(k)-1}, x_{m(k)}\big)\right) + \phi\left(d\big(x_{m(k)-1}, x_{n(k)}\big)\right)\right] \quad (2.12)$$

From the triangle inequality, we know that:

$$\varepsilon \leq d\big(x_{n(k)}, x_{m(k)}\big) \leq d\big(x_{n(k)}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{m(k)}\big) < d\big(x_{n(k)}, x_{n(k)-1}\big) + \varepsilon \quad (2.13)$$

Taking the limit as $k \to \infty$ and applying equation (2.9) yields $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$. Letting $k \to \infty$ inside inequality (2.12) results in:

$$\phi(\varepsilon) \leq a\phi(\varepsilon) + 2c\phi(\varepsilon) = (a + 2c)\phi(\varepsilon) \quad (2.14)$$

Since $a + 2b + 2c < 1 \Longrightarrow a + 2c < 1$, equation (2.14) reduces to $\phi(\varepsilon) < \phi(\varepsilon)$, which is a contradiction. Thus, $\{x_n\}$ is verified to be a Cauchy sequence in C. Since X is a complete metric space and C is a closed subset, there exists an element $x \in C$ such that $\lim_{n\to\infty} x_n = x$. Now, let m be any arbitrary positive integer. Consider the distance relation:

$$\phi\big(d(x_n, T_m x)\big) = \phi\big(d(T_n x_{n-1}, T_m x)\big) \leq a\phi\big(d(x_{n-1}, x)\big) +$$
$$b\big[\phi\big(d(x_{n-1}, x_n)\big) + \phi\big(d(x, T_m x)\big)\big] + c\big[\phi\big(d(x_{n-1}, T_m x)\big) + \phi\big(d(x, x_n)\big)\big] \quad (2.15)$$

Taking the limit as $n \to \infty$ on both sides yields:

$$\lim_{n\to\infty} \phi(d(x_n, T_m x)) \leq (b + c)\phi(d(x, T_m x))$$
$$\Longrightarrow \phi(d(x, T_m x)) \leq (b + c)\phi(d(x, T_m x)) \quad (2.16)$$

Since $0 < b + c < 1$, the relation forces $\phi(d(x, T_m x)) = 0$, which states that $d(x, T_m x) = 0$. Hence, x is a fixed point for T_m. Because m was chosen arbitrarily, x stands as a common fixed point for the entire sequence $\{T_n\}_{n=1}^{\infty}$.

Uniqueness: Let $y \in C$ $(y \neq x)$ be another common fixed point for $\{T_n\}_{n=1}^{\infty}$. Then $T_i x = x$ and $T_j y = y$ for all $i, j \in \mathbb{N}$. Evaluating their distance under condition (A1) gives:

$$\phi\big(d(x, y)\big) = \phi\left(d\big(T_i x, T_j y\big)\right) \leq a\phi\big(d(x, y)\big) + b\left[\phi\big(d(x, T_i x)\big) + \phi\left(d\big(y, T_j y\big)\right)\right]$$
$$+c\left[\phi\left(d\big(x, T_j y\big)\right) + \phi\big(d(y, T_i x)\big)\right] \quad (2.17)$$

$$\phi\big(d(x, y)\big) \leq a\phi\big(d(x, y)\big) + b[\phi(0) + \phi(0)] + c\big[\phi\big(d(x, y)\big) + \phi\big(d(y, x)\big)\big] \quad (2.18)$$

$$\phi\big(d(x, y)\big) \leq (a + 2c)\phi\big(d(x, y)\big) \quad (2.19)$$

Since $a + 2c < 1$, this demands $\phi(d(x, y)) < \phi(d(x, y))$, a clear contradiction. Thus, we conclude $y = x$, completing the proof of the theorem.

3. Results and Discussion

If we set X as a Banach space and define the identity control function $\phi(t) = t$ for $t > 0$, Theorem 2 transitions into a linear framework:

Theorem 3: Suppose C is a closed subset of a Banach space X and T_n is a sequence of self-maps on C satisfying condition (B1) and the inequality:

$$\text{(A2)}: \| T_i x - T_j y \| \leq a \| x - y \| + b\left(\| x - T_i x \| + \| y - T_j y \|\right) + c\left(\| x - T_j y \| + \| y - T_i x \|\right) \quad (3.1)$$

for all $i, j \in \mathbb{N}$ and all distinct $x, y \in C$, where $a \geq 0, c \geq 0, 0 < b < 1$ with $a + 2b + 2c < 1$. Then T_n has a unique common fixed point in C.

Note: In the following corollary, the property of subadditivity for the mapping $\phi \in \Phi$ is completely omitted from the proof constraints.

Corollary 1: If $c = 0$ in condition (A1) and there exists $\phi \in \Phi$ such that:

$$\text{(A3)}: \phi\left(d(T_i x, T_j y)\right) \leq a\phi\left(d(x, y)\right) + b\left[\phi\left(d(x, T_i x)\right) + \phi\left(d(y, T_j y)\right)\right] \quad (3.2)$$

for all $i, j \in \mathbb{N}$ and all distinct $x, y \in C$ with $a + 2b < 1$, and condition (B1) holds, then T_n has a unique common fixed point in C.

Analytical Remark: Let X be a Hilbert space H. By assigning the quadratic control function $\phi(t) = t^2$ for $t \geq 0$ in Corollary 1, we map directly back to the conditions of Theorem 1. Hence, Corollary 1 represents a valid architectural generalization of Theorem 1 from a localized Hilbert inner-product space to expansive metric fields under shifting distance maps.

4. Conclusion

In this work, we proved a unique common fixed-point theorem for a sequence of self-maps within a closed subset of a complete metric space. By employing a continuous, increasing control function $\phi \in \Phi$ to alter the operating distances between points, we successfully relaxed structural space constraints. Our findings generalize the mapping structures previously established by Pandhare and Waghmode in Hilbert spaces. We demonstrated that the sequence of generated iterates preserves Cauchy convergence under multi-parametric bounding constants ($a + 2b + 2c < 1$). Furthermore, specialized applications within Banach frameworks and reductions under zero cross-product bounds ($c = 0$) confirm the operational consistency of this generalized altering-distance methodology.

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