International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Impact Factor 2025: 7.089

Symmetric Erdélyi-Kober Operators and their Applications to Generalized Weber-Orr Transforms

Dr. Haresh Gambhir Chaudhari

Department of Mathematics, Mgsm Dadasaheb Suresh G Patil College, Chopda (India)
Email: *hareshgchaudhari[at]gmail.com*

Abstract: ***In this paper, we establish formal integral transforms that generalize the classical Weber-Orr transform and its inverse by utilizing symmetric generalized Erdélyi-Kober fractional integral operators. Building upon asymmetric frameworks, we introduce modifications of the arbitrary order (μ, ν) transform and systematically investigate two distinct parametric operational regimes: $\mu = \nu - \alpha$ and $\mu = \nu + \alpha$ for $\alpha > 0$. Through a series of technical lemmas and core inversion theorems, we derive exact solutions for unknown function mappings. Special cases, including reductions to the classical Weber-Orr transforms and formulations under zero-parameter kernels ($k = 0$), are evaluated to demonstrate the consistency of this overarching symmetric operator framework.***

Keywords: Weber-Orr transform, Erdélyi-Kober fractional integrals, Integral transforms, Inversion theorem, Symmetric operator framework

1. Introduction

The utilization of fractional integral operators to solve dual integral equations has wide-ranging applications in mathematical physics, particularly within the theory of elasticity and boundary value crack problems. Historically, Patil and Choudhary (1993) introduced un-symmetric fractional integration operators I^K and I_Π, defined explicitly as:

$$I^K(\eta,\alpha)\phi = \frac{2^{\alpha-1}k^{1-\alpha}}{x^{2\alpha+2\eta}} \int_x^\infty t^{-\eta+1}\,(t^2 - x^2)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(k\sqrt{t^2-x^2}\right)\phi(t)\,dt I_\Pi(\eta,\alpha)\phi = \frac{2^{\alpha-1}k^{1-\alpha}}{x^{-\alpha+\eta}} \int_0^x t^{\eta+\alpha-1}\,(x^2 - t^2)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(k\sqrt{x^2-t^2}\right)\phi(t)\,dt \quad (1.1)$$

These asymmetric maps were shown to behave as automorphisms of the spaces $T(\lambda, \mu)$, ensuring that their respective adjoint operators operate as automorphisms of the dual spaces $T'(\lambda, \mu)$. While these formulations relied heavily on non-symmetric modifications pioneered by Lowndes (1985), this paper expands the domain by generalizing the Weber-Orr integral transform and its corresponding inversion pathways through Lowndes' fractional integral operators cast in strict symmetric forms. The generalized Weber-Orr transform $W^k_{\mu,\nu}[\cdot]$ of arbitrary order (μ, ν) and its native inverse are written through the following integral expressions:

$$f(t) = W^k_{\mu,\nu}[f(s);t] = \int_a^\infty R^k_{\mu,\nu}(t;s,a)\,sf(s)\,ds \quad (1.2)$$

$$f(s) = W^k_{\mu,\nu}[f(t);s] = \int_0^\infty R^k_{\mu,\nu}(t;s,a)[J_\nu^2(ta) + Y_\nu^2(ta)]tf(t)\,dt \quad (1.3)$$

where the continuous kernel on (a, ∞) satisfies the restriction $\int_a^\infty x^{1/2} f(x)\,dx < \infty$, with the kernel function defined as:

$$R^k_{\mu,\nu}(t;s,a) = [J_\mu(ts)Y_\nu(ta) - J_\nu(ta)Y_\mu(ts)]_0 F_1\left(\nu-\mu; \frac{k^2(x^2-s^2)}{4}\right) \quad (1.4)$$

In the subsequent sections, we analyze this transform under specialized shifting conditions for parameter μ relative to values $(\nu - \alpha)$ and $(\nu + \alpha)$.

2. Results and Discussion

2.1 The Transform Regime for $\mu = \nu - \alpha$ $(\alpha > 0)$We begin by establishing structural behavior when the parameter is diminished by a positive scalar factor α.

Lemma 1: If $0 < \alpha < \frac{1}{2}\nu + \frac{3}{4}$, then the kernel function satisfies the decomposition:

$$R^k_{\nu-\alpha,\nu}(t;s,a) = s^{\nu-\alpha}t^\alpha k^{1-\alpha} \int_s^\infty x^{1-\nu}\,(x^2 - t^2)^{\frac{\alpha-1}{2}} R^k_{\nu,\nu}(t;x,a) J_{\alpha-1}\left(k\sqrt{x^2-t^2}\right)dx \quad (2.1)$$

This identity is derived directly by applying standard infinite integral forms of Bessel functions J_ν and Y_ν given by Erdélyi (1954).

Lemma 2: If $0 < \alpha < \frac{1}{2}\nu + \frac{3}{4}$, $\nu > -\frac{1}{2}$, $\int_a^\infty x^{x+\frac{1}{2}} f(x)\,dx < \infty$, and we define:

$$F(x) = x^{\alpha-\nu}k^{1-\alpha} \int_a^x s^{1+\nu-\alpha}\,(x^2 - s^2)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(k\sqrt{x^2-s^2}\right) f(s)\,ds \quad (2.2)$$

then,

$$\int_a^\infty x^{\frac{1}{2}-\alpha} F(x)\,dx = 2^{-\alpha}\Gamma(\alpha)B_k\left(\frac{1}{2}\nu+\frac{1}{4}-\alpha,\alpha\right) \int_0^\infty s^{\frac{1}{2}+\alpha}\,{}_0F_1\left(\alpha;\frac{k^2(x^2-s^2)}{4}\right)f(s)\,ds \qquad (2.3)$$

where the beta-like component expands to

$$B_k\left(\frac{1}{2}\nu+\frac{1}{4}-\alpha,\alpha\right)=\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{4}+\alpha)\Gamma(\alpha)}{\Gamma(\frac{1}{2}\nu+\frac{1}{4})}.$$

By absolute convergence, changing the order of integration confirms the result for $0<\alpha<\frac{1}{2}\nu+\frac{1}{4}$, which analytically extends to the full range via analytic continuation. This induces the following boundedness property:

$$\int_a^\infty x^{\frac{1}{2}-k} F(x)\,dx \le K\int_a^\infty s^{\frac{1}{2}+\alpha} f(s)\,ds < \infty \qquad (2.4)$$

Theorem 1: If $0<\alpha<\frac{1}{2}\nu+\frac{3}{4}$, $\nu>\frac{1}{2}$, $\int_a^\infty x^{\frac{1}{2}+\alpha} f(x)\,dx<\infty$, and $f(t)=W^k_{\nu-\alpha,\nu}[f(s);t]$, then:

$$x^{-\alpha}F(x)=W^k_{\nu-\alpha,\nu}[t^{-\alpha}f(t);x] \qquad (2.5)$$

Proof: Expanding $f(t)$ via equation (1.2) and invoking Lemma 1 yields the double integral:

$$f(t)=t^\alpha k^{1-\alpha}\int_a^\infty s^{1+\nu-\alpha} f(s)\left[\int_s^\infty x^{1-\nu}\,(x^2-s^2)^{\frac{\alpha-1}{2}}R^k_{\nu,\nu}(t;x,a)J_{\alpha-1}\left(k\sqrt{x^2-s^2}\right)dx\right]ds \qquad (2.6)$$

Reversing the integration order through absolute convergence definitions maps the identity directly to:

$$t^{-\alpha}f(t)=\int_a^\infty x^{1-\alpha}\,R^k_{\nu,\nu}(t;x,a)F(x)\,dx=W^k_{\nu,\nu}[x^{-\alpha}F(x);t] \qquad (2.7)$$

Applying the inversion formula under the conditions of Corollary 2.4 generates the targeted result:

$$x^{-\alpha}F(x)=W^k_{\nu-\alpha,\nu}[t^{-\alpha}f(t);x]$$

We now present the generalized Erdélyi-Kober fractional operators due to Lowndes (1970):

$${}_aA^k_x(\eta,\alpha)f(u) = \frac{2^\alpha x^{-2\eta-2\alpha}k^{1-\alpha}}{\Gamma(\alpha)}\int_a^x u^{2\eta+1}\,(x^2-u^2)^{\frac{\alpha-1}{2}}J_{\alpha-1}\left(k\sqrt{x^2-u^2}\right)f(u)\,du \qquad (2.8)$$

For negative values ($\alpha<0$), utilizing $D_x=\frac{1}{2}\left(\frac{1}{x}\frac{d}{dx}\right)$, the operator becomes:

$${}_aA^k_x(\eta,\alpha)f(u)=x^{-1-2\eta-2\alpha}D^m_x[x^{2m+2\alpha+2\eta+1}\cdot {}_aA^k_x(\eta,\alpha+m)f(u)] \qquad (2.9)$$

where ${}_aA^k_x(\eta,0)$ behaves as the identity operator, possessing the inverse structure ${}_aA^k_x(\eta,\alpha)^{-1}={}_aA^k_x(\eta+\alpha,-\alpha)$. When $k=0$, these reduce to Sneddon's (1974) representations used by Nasim (1989).

Lemma 3: If $F(x)$ satisfies the relation in Lemma 2, then for $\alpha>0$ and $0\le m-\alpha<1$:

$$f(x)=\frac{2^{1+\alpha}}{\Gamma(m-\alpha)}x^{\alpha-\nu-1}D^m_x\int_a^x u^{\nu-\alpha+1}\,(x^2-u^2)^{m-\alpha-1}{}_0F_1\left(m-\alpha;\frac{k^2(x^2-u^2)}{4}\right)f(u)\,du \qquad (2.10)$$

Proof: Expressing $F(x)$ via the Lowndes operators gives $F(x)=x^{2\alpha}\cdot{}_aA^k_x\left(\frac{\nu-\alpha}{2},\alpha\right)f(u)$. Inverting this map according to properties (2.9) leads directly to the differential integration model of equation (2.10). For the specific case where $\alpha=m$, this collapses into:

$$f(x)=x^{m-\nu}\left(\frac{1}{x}\frac{d}{dx}\right)[x^{\nu-m}F(x)] \qquad (2.11)$$

This enables the formulation of the following dual properties under Watson's (1944) differential Bessel identities:

Lemma 4: $x^{\nu-m}f(x)=\left(\frac{1}{x}\frac{d}{dx}\right)^m[x^{\nu-m}F(x)]$

Lemma 5: $\left(\frac{1}{x}\frac{d}{dx}\right)^m[x^\nu R^k_{\nu,\nu}(t;x,a)]=t^m x^{\nu-m}R^k_{\nu-m,\nu}(t;x,a)$

Theorem 2: If $0<m<\frac{1}{2}\nu+\frac{3}{4}$, $m\in\mathbb{N}$, $\nu>-\frac{1}{4}$, $\int_a^\infty x^{m+\frac{1}{2}}f(x)\,dx<\infty$, and $f(t)=W^k_{\nu-m,\nu}[f(s);t]$, then the inverse transform is given by:

$$f(s)=W^k_{\nu-m,\nu}[f(t);s] \qquad (2.12)$$

Proof: By Theorem 1, we write $x^{-m}F(x)=W^k_{\nu-m,\nu}[t^{-m}f(t);x]$. Applying the explicit base definition of the $W^k_{\nu,\nu}$ operator gives:

$$x^{-m}F(x)=\int_0^\infty R^k_{\nu,\nu}\,(t;x,a)[J^2_\nu(ta)+Y^2_\nu(ta)]t^{1-m}f(t)\,dt \qquad (2.13)$$

Applying the differential operator $\left(\frac{1}{x}\frac{d}{dx}\right)^m x^\nu$ to both sides, passing the derivative operator inside the integral via the uniform convergence bounds of Lemma 5, and matching with Lemma 4 on the left side yields:

$$f(x)=\int_0^\infty R^k_{\nu-m,\nu}\,(t;x,a)[J^2_\nu(ta)+Y^2_\nu(ta)]tf(t)\,dt \qquad (2.14)$$

which matches the transform format $f(x)=W^k_{\nu-m,\nu}[f(t);x]$.

2.2 The Transform Regime for $\mu=\nu+\alpha$ ($\alpha>0$)

Next, we evaluate the system behavior when the parameter is shifted upward.

Lemma 6: If $0<\alpha<\nu+\frac{3}{2}$, then the kernel functions satisfies:

$$R^k_{\nu,\nu}(t;x,a) = k^{1-\alpha}t^{\alpha}x^{\nu}\int_x^{\infty} s^{1-\nu-\alpha}\,(s^2 - x^2)^{\frac{\alpha-1}{2}}R^k_{\nu+\alpha,\nu}(t;s,a)J_{\alpha-1}\left(k\sqrt{x^2-s^2}\right)ds \quad (2.15)$$

This matching baseline is established via standard integrals involving Bessel functions J_μ and Y_μ found in Erdélyi (1954).

Lemma 7: If $\alpha > 0$, $\alpha \neq n + \frac{3}{4} - \frac{1}{2}\nu$ ($n \in \mathbb{N}_0$), $\int_a^{\infty} x^{\frac{1}{2}-\alpha} f(x)\,dx < \infty$, and we define $f(x)$ as a function of $f(s)$ bounded by the kernel components, then:

$$\int_a^{\infty} x^{\frac{1}{2}-\alpha} f(x)\,dx = 2^{-\alpha}\Gamma(\alpha)B_k\left(\alpha, \frac{\nu}{2} - \frac{1}{4}\right)\int_a^{\infty} s^{\alpha+\frac{3}{2}}F(s)\,{}_0F_1\left(\alpha; \frac{k^2(x^2-s^2)}{4}\right)ds \quad (2.16)$$

This yields the finite constraint property $\int_a^{\infty} s^{\alpha+\frac{3}{2}}F(s)\,ds \leq K\int_a^{\infty} x^{\frac{1}{2}-\alpha} f(x)\,dx < \infty$.

Theorem 3: If $0 < \alpha < \nu + \frac{3}{2}$, $\alpha \neq n + \frac{3}{4} - \frac{1}{2}\nu$, $\int_a^{\infty} x^{\frac{1}{2}-\alpha} f(x)\,dx < \infty$ and $f(t) = W^k_{\nu+\alpha,\nu}[f(s);t]$, then:

$$x^{\alpha}F(x) = W^k_{\nu,\nu}[t^{\alpha}f(t);x] \quad (2.17)$$

Proof: Expressing $f(t)$ via equation (1.2) and substituting the operational expansion of $f(s)$ given in the theorem statement yields:

$$f(t) = k^{1-\alpha}\int_a^{\infty} x^{1+\nu+\alpha}F(x)\left[\int_x^{\infty} s^{1-\nu-\alpha}\,(s^2-x^2)^{\frac{\alpha-1}{2}}R^k_{\nu+\alpha,\nu}(t;s,a)\,J_{\alpha-1}\left(k\sqrt{s^2-x^2}\right)ds\right]dx \quad (2.18)$$

Invoking Lemma 6 reduces this expression directly to:

$$t^{\alpha}f(t) = \int_a^{\infty} R^k_{\nu,\nu}\,(t;x,a)x^{\alpha+1}F(x)\,dx = W^k_{\nu,\nu}[x^{\alpha}F(x);t] \quad (2.19)$$

Applying the standard inversion formula valid via Lemma 7 gives the desired mapping condition: $x^{\alpha}F(x) = W^k_{\nu,\nu}[t^{\alpha}f(t);x]$.

2.3 General Structural Inversion Equation:

By substituting our fractional operators directly back into the results of Theorem 3, we can isolate the unknown function $f(s)$ as:

$$f(s) = s^{2\alpha}\cdot {}_aA^k_s\left(\frac{\nu}{2},\alpha\right)[x^{\alpha}F(x)] = s^{2\alpha}\cdot {}_aA^k_s\left(\frac{\nu}{2},\alpha\right)W^k_{\nu,\nu}[t^{\alpha}f(t);x] \quad (2.20)$$

This expands into the generalized symmetric representation:

$$f(s) = \int_0^{\infty} R^k\,(t;s,a)[J_\nu^2(ta) + Y_\nu^2(ta)]t^{1+\alpha}f(t)\,dt \quad (2.21)$$

where the operational kernel satisfies $R^k(t;s,a) = s^{2\alpha}\cdot {}_aA^k_s\left(\frac{\nu}{2},\alpha\right)R^k_{\nu,\nu}(t;x,a)$.

Theorem 4: If the baseline assumptions of Theorem 1 are satisfied, the unknown function f is given explicitly by equation (2.21). When analyzing the special case where $\alpha = m$ ($m \in \mathbb{N}$), evaluating $R^k(t;s,a)$ via m-times integration by parts yields the series expression:

$$f(s) = s^{-(\nu+m)}\sum_{n=1}^{m}\frac{2^{n-m}}{(m-n)!}a^{\nu+n}(s^2 - a^2)^{m-n}W^k_{\nu+n,\nu}[t^{m-n}f(t);a]\,{}_0F_1\left(m;\frac{k^2(s^2-x^2)}{4}\right) + W^k_{\nu+m,\nu}[f(t);s]\,{}_0F_1\left(m;\frac{k^2(s^2-x^2)}{4}\right) \quad (2.22)$$

Setting $m = 1$ simplifies the unknown function mapping to:

$$f(s) = -as^{\nu+1}W^k_{\nu+1,\nu}[f(t);a]\,{}_0F_1\left(m;\frac{k^2(s^2-x^2)}{4}\right) + W^k_{\nu+1,\nu}[f(t);s]\,{}_0F_1\left(m;\frac{k^2(s^2-x^2)}{4}\right) \quad (2.23)$$

where $f(t) = W^k_{\nu+1,\nu}[f(s);t]$.

2.4 Analysis of Special Cases:

1) The $k = 0$ Reduction: Setting the parameter $k = 0$ eliminates the hyper-geometric function component since ${}_0F_1(m;0) = 1$. Under these conditions, equations (2.22) and (2.23) reduce directly to the classical fractional transform configurations derived by Nasim (1989), Sneddon, and the early works of Weber and Orr in Erdélyi (1940).
2) The $\mu = \nu - m$ Reduction: Setting $k = 0$ and $\mu = \nu - m$ for $m = 1,2$ maps the system onto the standard Weber-Orr transforms introduced by Krajewski and Olesiak.

3. Conclusion

In this work, we successfully expanded the theory of dual integral transforms by presenting a symmetric generalization of the Weber-Orr transform using Erdélyi-Kober operators. By investigating the shifting behaviors of parameter μ at boundaries $(\nu - \alpha)$ and $(\nu + \alpha)$, we proved that the symmetric formulations preserve inverse mapping conditions under definite boundary integrations. The resulting multi-step integration equations offer a systematic framework for tracking function values across continuous bounds. Finally, testing the specialized threshold where $k = 0$ showed that our model aligns with classical formulations by Nasim and Sneddon, confirming its structural accuracy and potential utility in solving dual-string elasticity problems in mathematical physics.

References

[1] Erdelyi, A. (1950-51). On some functional transformations. Univ. Politec. Torino Rend. Scin. Mat., 10, 217-231.
[2] Erdelyi, A. (1954). Tables of Integral Transforms, Vol. II. McGraw-Hill, New York.
[3] Lowndes, J. S. (1970). A generalization of Erdélyi-Kober operators. Proceedings of the Edinburgh Mathematical Society, 17, 139-148.
[4] Lowndes, J. S. (1985). On two operators of fractional integration. In A. C. McBride & G. F. Roach (Eds.), Fractional Calculus. Research Notes in Mathematics.
[5] Nasim, C. (1989). Associated Weber integral transforms of arbitrary orders. Indian Journal of Pure and Applied Mathematics, 20, 1126-1138.
[6] Sneddon, I. N. (1974). A survey of the use of operators of fractional integration in applied mathematics. Research Paper No. 258, Yellow Series, The University of Calgary, Canada.
[7] Watson, G. N. (1928). Theorems stated by Ramanujan IV: Theorems on approximate integration and summation of series. Journal of the London Mathematical Society, Vol. 3, 282 p.
[8] Weber, H. (1968). Über eine bestimmte Integral. Crelles Journal, Vol. 69, 222 p.