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Hilbert Space and Midpoint Convex Set with Reference to Projection and Trijection

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Abstract: ***In this paper, we explore the geometric structural properties of a specialized subset within a Hilbert space defined via the inner product and the set of dyadic rational numbers. For a fixed element, we define the set. We demonstrate three key topological and algebraic results: first, that constitutes a midpoint convex set of; second, that for any two subsets and of such that, the containment relation holds strictly; and third, that the operator set of is invariant under its dyadic convex hull, satisfying, where represents the midpoint convex hull of. These findings provide fresh structural insights into sequence-to-set maps within inner product spaces.***

Keywords: Hilbert space, Midpoint convexity, Dyadic rational numbers, Inner product spaces, Dyadic convex hull

1. Introduction

Suppose H be a Hilbert space. Let Qd be the set of all dyadic rational number Clearly $Qd \subseteq [0,1]$. Let X be an element of H. Then we define a set Sx as

$$Sx = \{y \mid (x, y) \in Qd\}.$$

Since $(x, 0) = 0 \in Qd$ it is clear that 0 is an element of Sx.

Also $y \in S_x \Leftrightarrow (x, y) \in Qd$
$\Leftrightarrow (y, x) \in Qd$
$\Leftrightarrow x \in S_y$

Moreover $y\{x\}\}^{\perp} \Rightarrow (y, x) = 0$
$\Rightarrow (x, y) = 0$
$\Rightarrow y \in Sx$

Thus $\{x\}^{\perp} \subseteq S_x$.

2. Objectives

The present investigation would be reveled with different objectives, as followed:

1) To deal H be a Hilbert space and x be an element of H, then Sx is midpoint convex set of H;
2) To deal H be a Hilbert space and A, B be its subsets, then $S(B) \subseteq S(A)$;
3) To deal H be a Hilbert space and A be its subsets, then $S(A) = S(C_m(A))$.

3. Results and Discussions

(a) Suppose z_1 and z_2 be element of S_x.
Then $\left(X, \frac{1}{2}(z_1 + z_2)\right) = \frac{1}{2}(X, z_1) + \frac{1}{2}(X, z_2)$.
Now (X, z_1) and (X, z_2) are elements of Qd.

Thus, we can write $(X, z_1) = m/2^n$, where m is zero or a positive integer less than or equal to Z^n an n is a positive integer.

Similarly $(X, z_2) = k/2$, where $0 \leqslant k \leqslant 2^L$ and k and L are integer with $L > 0$,
Hence $\frac{1}{2}(x, z_1) + \frac{1}{2}(x, z_2) = \frac{1}{2} \cdot \frac{m}{2^n} + \frac{1}{2} \cdot \frac{k}{2^e} = \frac{m2^L + k \cdot 2^n}{2^{n+L+1}}$

Now $0 \leqslant m \leqslant 2^n \Rightarrow 0 \leqslant m \cdot 2^L \leqslant 2^{L+n}$ and $0 \leqslant k \leqslant 2^L \Rightarrow 0 \leqslant k \cdot 2^n \leqslant 2^{L+n}$

Adding the above two inequalities $0 \leqslant m \cdot 2^L + k \cdot 2^n \leqslant 2^{L+n} + 2^{L+n} = 2^{L+n+1}$

Hence $0 \leqslant \frac{m \cdot 2^L + k \cdot 2^n}{2^{n++1}} < 1$
Thus $\frac{1}{2}(X, z_1) + \frac{1}{2}(X, z_2)$ also belongs to Qd.
Hence $\frac{1}{2}(z_1 + z_2)$ is an element of S_x.
Therefore S_x is a midpoint convex set of H.
(b) We have $Z \in S(B) \Rightarrow (X, Z) \in Qd$ for all X in B
$\Rightarrow (X, Z) \in Qd$ for all X in A since $A \subseteq B \Rightarrow z \in S(A)$.
Hence $S(B) \subseteq S(A)$

(c) Suppose X be an element of $S(A)$ then for all Z in A, (Z, X) belongs to Qd.
Let γ belong to $c_m(A)$ then we can write $\gamma = t_1 a_1 + t_2 a_2 + \cdots \ldots + t_n a_n$
Where $a_i \in A, t_i$ dyadic rationals, $t_i \geqslant 0$ and $\Sigma t_i = 1$.

Hence $(\gamma, X) = (t_1 a_1 + \cdots \ldots + tna_n, X) = t_1(a_1, X) + \cdots + t_n(an, X)$.

Since (a_i, X) belongs to Qd, it is dyadic rational number and $0 \leqslant (a_i, Z) < 1$.

Since t_i is a dyadic rational it follows that (X, X) is a dyadic rational number.

Let $\max\{a_i, X), \ldots \ldots \ldots, (a_n, X)\} = (a_k, X) \leqslant 1$.

Where K is an element of the set $\{1, 2, \ldots \ldots n\}$.

Therefore $(X, X) \leqslant (t_1 + \cdots \ldots \ldots + t_n)(a_k, X) = (a_k, X) \leqslant 1$.

Thus (γ, X) is an element of Qd. Since Y is an arbitrary element of $C_m(A)$.

We see that $(\gamma, X) \in Qd$, for all Y in $C_m(A)$.

Hence X belongs to $S(C_m(A))$. Thus, $X \in S(A) \Rightarrow X \in S(Cm(A))$.

Therefore

$$S(A) \subseteq S\big(C_m(A)\big) \qquad (i)$$

Also, due to previous result

$$A \subseteq C_m(A) \Rightarrow S\big(C_m(A)\big) \subseteq S(A) \qquad (ii)$$

From (i) and (ii), $S(A) = S(Cm(A))$.

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