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S-Toepliz Operators on Orlicz Spaces

Dr. Shally Gupta
Associate Prof. in Mathematics
Govt. MAM College Jammu
shallyg31@gmail.com

Abstract

In this paper we characterize invariant and reducing subspaces of multiplication operators. The operator S_u on the Hardy Orlicz space which is analogous to the Toeplitz operator on the Hardy space is introduced.

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1 Introduction

Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous convex function such that (i) $\phi(x) = 0$ if and only if x = 0
(ii) $\lim_{x\to\infty} \phi(x) = \infty$.
Such a function ϕ is known as young function. Let (X, S, μ) be a σ-finite measure space and let $L^{\phi}(\mu) = \{f : X \to C$ is measurable : $\int \phi(\in |f|)d\mu < \infty$ for some $\in > 0\}$. If we set $||f||_{\phi} = inf\{\epsilon > 0 : \int \phi(|f|)d\mu \leq 1\}$, then $L^{\phi}(\mu)$ is a Banach space under the norm $||.||_{\phi}$. If $\phi(x) = x^p, 1 \leq p < \infty$, then $L^{\phi}(\mu) = L^p(\mu)$, the well known Banach space of p-integrable functions on X.

A young function $\phi : R \to R^+$ is said to satisfy the $\triangle_2$- condition (globally) if $\phi(2x) < k\phi(x), x \geq x_0 \geq 0 (x_0 = 0)$ for some absolute constant $k > 0$. If $\mu(X) = \infty$ then ϕ is called $\triangle_2$ - regular. With each young function ϕ we can associate another convex function $\psi : R \to R^+$ defined by $\psi(y) = sup\{x|y| - \phi(x) : x \geq 0\}, y \in R$ which have similar properties. The function ψ is called the complementary function to ϕ. In general, simple functions are not dense in $L^{\phi}(\mu)$, but in case ϕ satisfy the $\triangle_2$-condition, then the class of simple function become dense in $L^{\phi}(\mu)$. For more literature concerning orlicz space we refer to Roo[1] Kufner[2] and Hudzik[3].
A young function $\phi : \mathcal{R} \to \mathcal{R}^+$ is called an N-function if it satisfies the following conditions
(i) $\phi(x) = 0$ if and only if x = 0
(ii) $\lim_{x\to 0} \dfrac{\phi(x)}{x} = 0$
(iii) $\lim_{x\to\infty} \dfrac{\phi(x)}{x} = \infty$
and ϕ is said to satisfy the $\triangle_2$- condition (globally) denoted as $\phi \in \triangle_2$ if $\phi(2x) < k\phi(x), x \geq x_0 \geq 0 (x_0 = 0)$ for some absolute constant $k > 0$. If the measure space is infinte then we say that ϕ is $\triangle_2$ regular. The space $L^{\phi}(\mu)$ of all measuable func-

tions $f : \Omega \to \mathcal{R}$ such that $\alpha f \in \widetilde{L}^{\phi}(\mu)$ for some $\alpha > 0$, is called an orlicz space. i.e., $L^{\phi}(\mu) = \{f : \Omega \to \mathcal{R}$, measurable $| \int_{\Omega} \phi(\alpha f) d\mu < \infty$ for some $\alpha > 0\}$. If we set

$$||f||_{\phi} = inf\{\in > 0 : \int_{\Omega} \phi(\frac{|f|}{\in}) d\mu \leq 1\},$$

then $L^{\phi}(\mu)$ is a Banach space under the norm $||.||_{\phi}$. If $\phi(x) = x^p, 1 \leq p < \infty$, then $L^p(\mu) = L^{\phi}(\mu)$. Therefore the orlicz spaces are considered as the generlization of L^p-spaces.The Hardy orlicz class $\widetilde{H}^{\phi}$ is defined as

$$\widetilde{H}^{\phi} = \{f | f : D \to C \text{is analytic and } \sup_{0 \leq r \leq 1} \int_0^{2\pi} (log|f(re^{i\theta})| d\mu(\theta) < \infty\}$$

If $f : D \to C$ is analytic such that $\alpha f \in \widetilde{H}^{\phi}$ for some α , then the Hardy Orlicz class is known as Hardy Orlicz space.

Theorem 1.1 : *Suppose* $L^2(\mu) \subset L^{\phi}(\mu)$ *and* $M_u \in B(L^{\phi}(\mu))$. *Then* $L^2(\mu)$ *is invariant under* M_u *if and only if* $u \in L^{\infty}(\mu)$.

Proof: If M_u is bounded , then $u \in L^{\infty}(\mu)$ and hence $M_u(L^2(\mu)) \subset L^2(u)$.

Theorem 1.2 : *Let* $u \in L^{\infty}(\mu)$. *Then* S_u *is invertible if and only if* M_u *is invertible, where* $S_u = M_u | L^2(\mu)$, *(the restriction of* M_u *to* $L^2(\mu)$*).*

Proof: If S_u is invertible in $L^2(\mu)$, then there exists $\epsilon > 0$ such that $||S_u f||_2 \geq \in ||f||_2$ for all $f \in L^2(\mu)$. This implies that $|u(x)| \geq \in$ for μ- almost all $x \in \Omega$ and so $M_u \in B(L^{\phi}(\mu))$. Thus M_u is invertible with M_u^{-1} as its inverse.

Corollary 1.3 : *The mapping* $\beta : L^{\infty}(\mu) \to B(H^{\phi}(\mu))$ *is an isometry.*

Proof: For $u \in L^{\infty}(\mu)$, we have

$$\begin{aligned} ||u||_{\infty} &= sup\{|\lambda| : \lambda \in ranu\} \\ &= sup\{|\lambda| : \lambda \in \sigma(S_u) \leq ||S_u|| \leq ||u||_{\infty}\} \end{aligned}$$

Hence $||S_u|| \leq ||u||_{\infty}$.

Theorem 1.4 :*If* $u \in L^{\infty}([\theta, 2\pi])$, *then* H^{ϕ} *is invarient under* M_u *if and only if* $u \in H^{\phi}$.

Proof:If $u \in H^{\phi}$, then $u.e_n \in H^{\phi}$ for every $n = 0, 1, 2, ...$ Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial. Since $\int_0^{2\pi} u.e_n du = 0$ for every n=1,2,..., it follows that $\int_0^{2\pi} u(z)p(z)d\mu(z) = 0$. Further since polynomials are dense in H^{ϕ}, we have $uH^{\phi} \subset H^{\phi}$.

Theorem 1.5 : *For each inner function u, the space* $u.H^{\phi}$ *is a closed invariant subspace of* M_{e_1}.

Proof:If u is an inner function, then the mapping $M_u : H^{\phi} \to H^{\phi}$ is an isometry. Therefore its range $u.H^{\phi}$ is a closed subspace of H^{ϕ}. Thus $M_{e_1}(u.f) = M_u(e_1.f) \in H^{\phi}$.

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