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Bounded and Isometric Weighted Composition Operators on L^P-spaces of Hilbert Space Valued Functions

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Abstract

In this paper we study bounded and isometric operators with weighted on L^P-spaces of Hilbert space valued functions

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1 Introduction

Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous convex function such that
(i) $\phi(x) = 0$ if and only if x = 0
(ii) $\lim\limits_{n\to\infty} \phi(x) = \infty$.
Such a function is known as young function. Let (X, S, μ) be a σ-finite measure space and let $L^\phi(\mu) = \{f : X \to C$ is measurable $: \int_X \phi(\in |f|)d\mu < \infty$ for some $\in > 0\}$. If we set $||f||_\phi = inf\{\epsilon > 0 : \int_X \phi\frac{|f|}{\epsilon}d\mu \leq 1\}$, then $L^\phi(\mu)$ is a Banach space under the norm $||.||_\phi$. If $\phi(x) = x^p, 1 \leq p < \infty$, then $L^\phi(\mu) = L^p(\mu)$, the well known Banach space of p-integrable functions on X.

A young function $\phi : R \to R^+$ is said to satisfy the $\triangle_2$- condition (globally) if $\phi(2x) \geq k\phi(x), x \geq x_0 \geq 0(x_0 = 0)$ for some absolute constant $k > 0$. Again the function ϕ is said to satisfy ∇_2-condtion if $\phi(x) \leq \frac{1}{2t}\phi(tx), x \geq x_0 > 0(x_0 = 0)$ for some $t > 1$. With each young function ϕ we can associate another convex function $\psi : R \to R^+$ having similar properties which is defined by $\psi(y) = sup\{x|y| - \phi(x) : x \geq 0\}, y \in |R$.The function ψ is called the complementary function to ϕ. The simple functions are not dense in $L^{\phi}(\mu)$, but if ϕ satisfy $\triangle_2$-condition, then the class of simple function is dense in $L^{\phi}(\mu)$. For more literature concerning orlicz space,we refer to Rao [2] Kufner [3].Throughout this paper we assume that ϕ satisfy $\triangle_2$ and ∇_2- condition. The main aim of this paper is to explore a necessary and sufficient condition for a composite transformation with weight on $L^p(\Omega, H)$ into $L^p(\Omega, H)$ to be bounded. Isometric composite operators with weight are also characterized.

2 Bounded weighted composition operators on L^p-spaces of Hilbert space valued functions

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The main purpose of this section is to obtain a criterion for weighted composition operators to be bounded.

Theorem 2.1 *Let* $u : \Omega \to B(H)$ *be a strongly measurable function and* $T : \Omega \to \Omega$ *be a non singular measurable transformation .Then* $M_{u,T}; L^p(\Omega, H) \to L^p(\Omega, H)$ *is bounded if and only if*

$$f_0^{\frac{1}{q}}(.)||u(.)|| \in L^{\infty}(\mu) \qquad (1)$$

where f_0 is the randon Nikodym derivative of the measure. μT^{-1} with respect to measure μ.
Proof We first assume condition (1) is true. Then for any $f \in L^p(\Omega, H)$,

$$\begin{aligned}
||M_{u,T}f||_p &= (\int_{\Omega} ||M_{u,T}f||^p d\mu)^{\frac{1}{p}} \\
&= (\int_{\Omega} ||u(T(x))f(T(x))||^p d\mu(x))^{\frac{1}{p}} \\
&= (\int_{\Omega} f_0(x)||u(x)f(x)||^p d\mu(x))^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq \ (\int_{\Omega} f_0(x)||u(x)||^p f(x)||^p d\mu(x))^{\frac{1}{p}} \\
&\leq \ ||f_o^{\frac{1}{p}}(.)||||u(.)||_{\infty}(\int_{\omega} ||f(x)||^p d\mu(x))^{\frac{1}{p}} \\
&\leq \ M||f||_p
\end{aligned}$$

which proves that $M_{u,T}$ is a bounded operator. Conversely, if the condition of the theorem fails, then fo every $n \in N$ there exist a measurable subset E_n of positive measure such that

$$||f_o^{\frac{1}{p}}(x)||u(x)|| > n \text{ for } \mu \text{ -almost all } x \in E_n.$$

For every $n \in N$, we can find a subset G_n of E_n such that $0 < \mu(G_n) < \infty$ and for every $x \in G_n$ there exists a unit vector $e_x \in H$ such that

$$||f_o^{\frac{1}{p}}(x)||u(x)e_x|| > n.$$

Let

$$f_n = \frac{\chi_{g_n} g_n}{p\sqrt{u(G_n)}} \text{ where } g_n : \Omega \to H \text{ is defined by}$$

$$g_n(x) = \begin{cases} e_x & \text{if } x \in G_n \\ 0 & \text{elsewhere} \end{cases}$$

Then

$$||g_n||_p^p = \frac{1}{\mu(G_n)} \int_{G_n} ||g_n(x)||^p d\mu = \frac{1}{\mu(G_n)} \mu(G_n) = 1$$

But

$$\begin{aligned}
||M_{u,T} g_n||_p^p \ &= \ \int_{\Omega} f_0(x)|| \frac{u(x)g_n(x)}{p\sqrt{\mu(G_n)}} ||^p d\mu \\
&= \ \frac{1}{\mu(G_n)} \int_{G_n} f_0(x)||u(x)e_x||^p d\mu \\
&> \ \frac{1}{\mu(G_n)} \int_{G_n} n^p d\mu = \frac{1}{\mu(G_n)} n^p \mu(G_n) \\
&= \ n^p \to \infty \text{ as } n \to \infty
\end{aligned}$$

which contracts the boundedness of $M_{u,T}$. Hence

$$f_0^{\frac{1}{q}}(.)||u(.)|| \in L^{\infty}(\Omega, H)$$

Theorem 2.2 *Let* $1 \leq p < q < \infty$. *Then* $M_{u,T} : L^p(\Omega, H) \to L^q(\Omega, H)$ *is a bounded linear operator if and only if* $M_{u,T}$ *is the zero operator.*

Proof: We first prove the inequality : For each $\epsilon > 0$, $f_0^{\frac{1}{q}}(x)||u(x)y|| \leq ||y||$ for μ-almost all $x \in \Omega$ and for all $y \in H$. (1)
If possible suppose there exists $\epsilon > 0$ and $y \in H$ such that the set

$$E = \{x \in \Omega : f_0^{\frac{1}{q}}(x)||u(x)y|| > \epsilon||y||\}$$

has positive measure. Since μ is non atomic, by lemma (1.5) of Takagi [6] there exists $g \in L^p(\Omega, \mathcal{C})$ such that $g \notin L^q(\Omega, \mathcal{C})$. For $y \in H$ defined $g_y : \Omega \to \Omega$ as $g_y(x) = g(x)y$. It is easy to see that $g_y \in L^p(\Omega, H)$ such that $g_y \notin L^q(\Omega, H)$.
Now

$$\begin{aligned}
\infty = \in^q \int_E ||g(y)||^q d\mu &= \epsilon^q \int_E ||g_y(x)||^q d\mu(x) \\
&= \in^q \int_E |g(x)|^q ||y||^q d\mu \\
&\leq \int_E |g(x)|^q f_0(x)||u(x)y||^q d\mu \\
&= \int_E f_0(x)||u(x)g_y(x)||^q d\mu \\
&= \int_E ||M_{u,T}g_y||^q d\mu \\
&\leq ||M_{u,T}g_y||^q \leq (||M_{u,T}||||g_y||_p)^q < \infty
\end{aligned}$$

which is a contradiction. Hence the inequality (1) is true. i,e, $f_0^{\frac{1}{q}}(x)||u(x)y|| \leq \epsilon||y||$ for $\mu-$almost all $x \in \Omega$ and for all $y \in H$. This implies that

$$f_0^{\frac{1}{q}}(x)||u(x)|| < \epsilon \text{ for } \mu - \text{ almost all } x \in \Omega$$

. Since ϵ is arbitrary, so

$$f_0^{\frac{1}{q}}(x)||u(x)|| = 0 \text{ for } \mu- \text{ almost all } x \in \Omega$$

which proves that $M_{u,T} = 0$.

Theorem 2.3 *Let $M_{u,T} \in B(L^p(\Omega, H))$. Then $M_{u,T} : L^p(\Omega, H) \to L^q(\Omega, H)$ is bounded away from zero if and only if there exists $\epsilon > 0$ such that*
$f_0^{\frac{1}{q}}(x)||u(x)y|| \geq \epsilon||y||$ for $\mu-$ almost all $x \in \Omega$ and for all $y \in H$.(1)

Proof: If the condition (1) is true then for every $f \in L^p(\Omega, H)$ we have

$$\begin{aligned} ||M_{u,T}f||_q^q &= \int_\Omega f_0(x)||u(x)f(x)||^p d\mu \\ &\geq \epsilon^p \int_\Omega ||f(x)||^p = \epsilon^p||f||_q^q \end{aligned}$$

so that f is bounded away from zero.
Conversely suppose $M_{u,T}$ is bounded away from zero. Let E be a measurable subset of Ω such that $0 < \mu(E) < \infty$. Let $y \in H$. Define $f_E^y : \Omega \to H$ by

$$f_E^y(x) = \begin{cases} y & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then $f_E^y \in L^p(\Omega, H)$
Now

$$||M_{u,T}f_E^y||_p \geq \in |f_E^y||_p$$

or

$$\int_E f_0(x)||u(x)y||^p \geq \epsilon \int_E ||y||^p d\mu$$

This is true for every $E \in S$. Hence

$$f_0(x)||u(x)y||^p \geq \epsilon||y||^p$$

In other words
$f_0^{\frac{1}{q}}(x)||u(x)y|| \geq \epsilon^{\frac{1}{q}}||y||$ for $\mu-$ almost all $x \in \Omega$ and for all $y \in H$.

3 Isometric Wighted Composition Operators on L^p-Spaces of Hilbert Space Valued Functions

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In this section we study isometric weighted composition operators on L^p-spaces of Hilbert space valued functions.

Theorem 3.1 *Let $M_u \in B(L^p(\Omega, H))$. Then M_u is an isometry if and only if u(x) is an isometry for μ- almost all $x \in \Omega$.*

Proof: If u(x) is an isometry for μ-almost all $x \in \Omega$. Then clearly M_u is an isometry.
Conversely suppose M_u is an isometry. Let E be a measurable subset of Ω such that $0 < \mu(E) < \infty$.
Let $y \in H$. Define $f_{y,E} : \Omega \to \Omega$ as

$$f_{y,E}(x) = \begin{cases} y & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then

$$||M_u f_{y,E}||_p = ||f_{y,E}||_p$$

implies that

$$\int_E ||u(x)y||^p = \int_E ||y||^p d\mu$$

for each finite measurable set E. It follows that

$$||u(x)y||^p = ||y||^p$$

or equivalently

$$||u(x)y|| = ||y|| \text{ for } \mu - \text{ almost all } x \in \Omega$$

Theorem 3.2 *Let $M_{u,T} \in B(L^p(\Omega, H))$. Then $M_{u,T} : L^p(\Omega, H) \to L^q(\Omega, H)$ is an isometry if and only if u(x) is an isometry for μT^{-1} almost all $x \in \Omega$.*

Proof: Since

$$\begin{aligned}(\int_\Omega ||M_{u,T}f||^p d\mu)^{\frac{1}{p}} &= \int_\Omega ||(uoT)(x)foT(x)||^p d\mu(x))^{\frac{1}{p}} \\ &= (\int_\Omega ||u(x)f(x)||^p d\mu T^{-1}(x))^{\frac{1}{p}} \\ &= ||M_u f||_{\mu T^{-1}}\end{aligned}$$

This implies $M_{u,T}$ is an isometry if and only if u(x) is an isometry for μT^{-1} almost all $x \in \Omega$.

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