A Generalized Complex Fuzzy Representation Space: Metric Structures, Fixed-Point Analysis, and Applications to Uncertain Complex Systems

Ravi Shanker Kumar
Department of Mathematics
Bhupendra Narayan Mandal University
Madhepura, Bihar–852113, India
Email: raviraj000pcm@gmail.com

Abstract

The integration of complex analysis with fuzzy set theory provides an effective mathematical framework for representing uncertain complex-valued information arising in engineering, computational intelligence, and decision sciences. This article introduces a new Generalized Complex Fuzzy Representation Operator (GCFRO) that transforms each complex number into a fuzzy structure characterized by a nonlinear distance-based membership mechanism.

A Generalized Complex Fuzzy Representation Space (GCFRS) is constructed, and a novel fuzzy metric is proposed to quantify the similarity between complex fuzzy objects. Theoretical investigations establish fundamental properties including boundedness, invariance, continuity, stability, completeness, compactness, and convergence.

Furthermore, a generalized contraction operator and a Banach-type fixed-point theorem are developed to guarantee the existence and uniqueness of stable fuzzy transformations. Numerical examples and applications to uncertain signal processing, noise-resilient information representation, and complex-valued intelligent systems demonstrate the effectiveness of the proposed framework.

The obtained results provide a new mathematical direction for the integration of complex numbers and fuzzy uncertainty models.

Keywords: Complex fuzzy representation, Fuzzy metric space, Fixed-point theory, Complex uncertainty, Intelligent fuzzy systems
Mathematics Subject Classification (2020): 03E72, 54E35, 47H10, 54H25, 46S40, 68T27.

1 Introduction

Complex numbers are fundamental mathematical objects that play a significant role in many branches of science and engineering, including electromagnetic theory, quantum computation, control systems, signal processing, and image analysis. A complex quantity is generally expressed in the form

$$z = x + iy,$$

where x and y represent the real and imaginary components, respectively.

Although classical complex models accurately represent magnitude and phase information, they are inadequate for handling uncertainty caused by incomplete observations, environmental disturbances, and measurement errors.

Fuzzy set theory provides a powerful mathematical approach for handling imprecise information by assigning each element a degree of membership between zero and one. The combination of fuzzy concepts with complex information has attracted considerable attention because many modern applications involve both oscillatory behavior and uncertainty.

Several generalized fuzzy structures, such as intuitionistic fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, and complex fuzzy systems, have been introduced to model different types of uncertain information. However, there remains a need for a rigorous mathematical framework that directly converts ordinary complex numbers into a structured fuzzy space with well-defined metric, topological, and algebraic properties.

Therefore, this work proposes a new Generalized Complex Fuzzy Representation Space (GCFRS). The main contributions of this paper are summarized as follows:

1. A new complex-to-fuzzy representation operator is introduced.
2. A generalized fuzzy metric is constructed to analyze the distance between complex fuzzy representations.
3. Several new mathematical results involving continuity, stability, completeness, compactness, and convergence are established.
4. A generalized Banach-type fixed-point theorem is developed for complex fuzzy contraction operators.
5. Practical applications in signal processing and intelligent complex-valued systems are presented.

2 Literature Review

The development of fuzzy mathematics began with the introduction of classical fuzzy sets, which provided a framework for representing gradual uncertainty. Later developments expanded fuzzy models into various generalized structures including intuitionistic, Pythagorean, spherical, and complex fuzzy environments.

Recent studies in complex fuzzy systems have demonstrated their importance in decision-making, artificial intelligence, pattern recognition, and computational intelligence. However, most existing approaches focus on aggregation and decision models, whereas the direct transformation of a classical complex number into a complete fuzzy metric structure has received comparatively less attention.

The proposed GCFRS addresses this gap by establishing a mathematically consistent transformation framework with new metric, algebraic, and fixed-point properties.

3 Mathematical Preliminaries

This section presents the basic mathematical concepts required for the development of the proposed Generalized Complex Fuzzy Representation Space (GCFRS).

Definition 3.1 (Complex Plane)**.** *The complex plane is denoted by*

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i^2 = -1\},$$

where x is the real component and y is the imaginary component of the complex number $z = x + iy$. The modulus of z is defined by

$$|z| = \sqrt{x^2 + y^2}.$$

Definition 3.2 (Classical Fuzzy Set)**.** *Let X be a non-empty universe of discourse. A fuzzy set A on X is characterized by a membership function*

$$\mu_A : X \to [0, 1],$$

where the value $\mu_A(x)$ represents the degree of belonging of an element x to the fuzzy set A.

Definition 3.3 (Euclidean Distance on the Complex Plane)**.** *For any two complex numbers $z_1, z_2 \in \mathbb{C}$, the distance between them is given by*

$$d(z_1, z_2) = |z_1 - z_2|.$$

4 Generalized Complex Fuzzy Representation Operator

We now introduce a new mathematical operator that transforms an exact complex number into a generalized fuzzy representation.

Definition 4.1 (Generalized Complex Fuzzy Representation Operator)**.** *Let $F(\mathbb{C})$ denote the collection of all fuzzy subsets of the complex plane. For a parameter $\lambda > 0$, define the operator*

$$\mathcal{T}_\lambda : \mathbb{C} \to F(\mathbb{C})$$

such that

$$\mathcal{T}_\lambda(z) = \mu_z(\xi) = \exp\left(-\lambda|\xi - z|^2\right), \quad \xi \in \mathbb{C}.$$

The fuzzy object generated by $\mathcal{T}_\lambda$ is called a Generalized Complex Fuzzy Representation (GCFR) of the complex number z.

Definition 4.2 (Generalized Complex Fuzzy Representation Space)**.** *The collection*

$$\mathcal{G}_\lambda = \{\mathcal{T}_\lambda(z) : z \in \mathbb{C}\}$$

is called the Generalized Complex Fuzzy Representation Space (GCFRS).

5 Fundamental Properties of the Proposed GCFRS

Lemma 5.1 (Boundedness Property)**.** *For every $z, \xi \in \mathbb{C}$, the membership function satisfies*

$$0 < \mu_z(\xi) \leq 1.$$

Proof. Since

$$|\xi - z|^2 \geq 0$$

and $\lambda > 0$, we obtain

$$-\lambda|\xi - z|^2 \leq 0.$$

Therefore,

$$0 < \exp(-\lambda|\xi - z|^2) \leq 1.$$

Hence,

$$0 < \mu_z(\xi) \leq 1.$$

Thus, the membership function is a valid fuzzy representation.

□

Lemma 5.2 (Exponential Decay Property)**.** *The membership degree decreases continuously as the distance between the observation point ξ and the original complex number z increases.*

Proof. Let

$$r = |\xi - z|.$$

Then the membership function can be written as

$$\mu(r) = e^{-\lambda r^2}.$$

Differentiating with respect to r,

$$\frac{d\mu}{dr} = -2\lambda r e^{-\lambda r^2}.$$

Since

$$\lambda > 0 \quad \text{and} \quad r \geq 0,$$

it follows that

$$\frac{d\mu}{dr} \leq 0.$$

Therefore, the membership function is monotonically decreasing with respect to the distance. □

Theorem 5.3 (Maximum Representation Principle). *For every complex number $z \in \mathbb{C}$,*

$$\mu_z(z) = 1.$$

Therefore, the original complex number possesses complete certainty within its fuzzy representation.

Proof. By choosing

$$\xi = z,$$

we have

$$|\xi - z|^2 = 0.$$

Consequently,

$$\mu_z(z) = e^{-\lambda(0)} = 1.$$

Hence, the theorem is established. □

Theorem 5.4 (Translation Symmetry). *For any complex numbers $z, \xi, a \in \mathbb{C}$,*

$$\mu_{z+a}(\xi + a) = \mu_z(\xi).$$

Thus, the generalized complex fuzzy representation is invariant under complex translation.

Proof. Using the definition of the membership function,

$$\mu_{z+a}(\xi + a) = \exp\left(-\lambda|(\xi + a) - (z + a)|^2\right).$$

Since

$$(\xi + a) - (z + a) = \xi - z,$$

we obtain

$$\mu_{z+a}(\xi + a) = \exp\left(-\lambda|\xi - z|^2\right) = \mu_z(\xi).$$

Therefore, the representation preserves translational geometry. □

Theorem 5.5 (Rotational Symmetry). *Let $e^{i\theta}$ be a complex rotation with $\theta \in \mathbb{R}$. Then,*

$$\mu_{e^{i\theta}z}(e^{i\theta}\xi) = \mu_z(\xi).$$

Hence, the proposed GCFRS remains invariant under rotation.

Proof. From the definition,

$$\mu_{e^{i\theta}z}(e^{i\theta}\xi) = \exp\left(-\lambda|e^{i\theta}\xi - e^{i\theta}z|^2\right).$$

Factoring $e^{i\theta}$, we obtain

$$= \exp\left(-\lambda|e^{i\theta}(\xi - z)|^2\right).$$

Because the modulus of a complex rotation satisfies

$$|e^{i\theta}| = 1,$$

we have

$$|e^{i\theta}(\xi - z)| = |\xi - z|.$$

Therefore,

$$\mu_{e^{i\theta}z}(e^{i\theta}\xi) = \exp(-\lambda|\xi - z|^2) = \mu_z(\xi).$$

Hence, the proposed representation is rotationally invariant.

□

6 Generalized Complex Fuzzy Metric and Topological Properties

In this section, we construct a new distance measure on the Generalized Complex Fuzzy Representation Space (GCFRS) and investigate its topological and stability characteristics. The proposed metric provides a mathematical tool for measuring the dissimilarity between two complex fuzzy representations.

Definition 6.1 (Generalized Complex Fuzzy Metric)**.** *Let*

$$\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2) \in \mathcal{G}_\lambda.$$

The generalized complex fuzzy distance between two fuzzy representations is defined by

$$D_G(\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2)) = \sup_{\xi\in\mathbb{C}} |\mu_{z_1}(\xi) - \mu_{z_2}(\xi)|.$$

The pair

$$(\mathcal{G}_\lambda, D_G)$$

is called the Generalized Complex Fuzzy Metric Space (GCFMS).

Lemma 6.2 (Basic Properties of the Generalized Fuzzy Metric)**.** *For all*

$$A, B \in \mathcal{G}_\lambda,$$

the distance D_G satisfies

$$D_G(A, B) \geq 0,$$

and

$$D_G(A, B) = D_G(B, A).$$

Paper ID: SR26624164731 DOI: https://dx.doi.org/10.21275/SR26624164731

Proof. By the property of the absolute value,

$$|\mu_A(\xi) - \mu_B(\xi)| \geq 0,$$

for every $\xi \in \mathbb{C}$. Therefore,

$$D_G(A, B) \geq 0.$$

Moreover,

$$|\mu_A(\xi) - \mu_B(\xi)| = |\mu_B(\xi) - \mu_A(\xi)|.$$

Taking the supremum over all $\xi \in \mathbb{C}$ gives

$$D_G(A, B) = D_G(B, A).$$

Hence, the required properties hold.

□

Theorem 6.3 (Metric Characterization of GCFMS)**.** *The function D_G defines a metric on the Generalized Complex Fuzzy Representation Space $\mathcal{G}_\lambda$.*

Proof. The non-negativity and symmetry follow from the previous lemma.

Assume

$$D_G(\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2)) = 0.$$

Then,

$$\mu_{z_1}(\xi) = \mu_{z_2}(\xi)$$

for every $\xi \in \mathbb{C}$.

Hence,

$$e^{-\lambda|\xi - z_1|^2} = e^{-\lambda|\xi - z_2|^2}.$$

Taking logarithms yields

$$|\xi - z_1| = |\xi - z_2|.$$

Choosing $\xi = z_1$, we obtain

$$|z_1 - z_2| = 0,$$

which gives

$$z_1 = z_2.$$

Thus,

$$\mathcal{T}_\lambda(z_1) = \mathcal{T}_\lambda(z_2).$$

For the triangle inequality,

$$|\mu_1 - \mu_3| \leq |\mu_1 - \mu_2| + |\mu_2 - \mu_3|.$$

By taking the supremum over $\xi \in \mathbb{C}$,

$$D_G(T_1, T_3) \leq D_G(T_1, T_2) + D_G(T_2, T_3).$$

Therefore, D_G satisfies all metric axioms.

□

Theorem 6.4 (Continuity of the Generalized Representation Operator)**.** *The operator*

$$\mathcal{T}_\lambda : \mathbb{C} \to \mathcal{G}_\lambda$$

is continuous with respect to the Euclidean metric on $\mathbb{C}$ *and the metric* D_G*.*

Proof. Let

$$z_n \to z$$

in $\mathbb{C}$.
For every $\xi \in \mathbb{C}$,

$$|\xi - z_n| \to |\xi - z|.$$

Since the exponential function is continuous,

$$\mu_{z_n}(\xi) = e^{-\lambda|\xi - z_n|^2} \to e^{-\lambda|\xi - z|^2} = \mu_z(\xi).$$

Hence,

$$D_G(\mathcal{T}_\lambda(z_n), \mathcal{T}_\lambda(z)) \to 0.$$

Therefore, the transformation operator is continuous.

□

Theorem 6.5 (Exponential Stability Property)**.** *There exists a positive constant* $L > 0$ *such that*

$$D_G(\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2)) \leq L|z_1 - z_2|$$

for all

$$z_1, z_2 \in \mathbb{C}.$$

Proof. Consider the function

$$f(r) = e^{-\lambda r^2}.$$

By the Mean Value Theorem,

$$|f(r_1) - f(r_2)| \leq \max |f'(r)||r_1 - r_2|.$$

Since

$$f'(r) = -2\lambda r e^{-\lambda r^2},$$

the derivative is bounded on $[0, \infty)$. Let

$$L = \sup_{r \geq 0} 2\lambda r e^{-\lambda r^2}.$$

Then,

$$|\mu_{z_1}(\xi) - \mu_{z_2}(\xi)| \leq L \left| |\xi - z_1| - |\xi - z_2| \right|.$$

Using the reverse triangle inequality,

$$||\xi - z_1| - |\xi - z_2|| \leq |z_1 - z_2|.$$

Thus,

$$D_G(\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2)) \leq L|z_1 - z_2|.$$

Therefore, the GCFRS possesses exponential stability.

□

Corollary 6.6 (Robustness Under Perturbation). *Let a perturbed complex value be*

$$z^* = z + \epsilon,$$

where $\epsilon \in \mathbb{C}$ represents measurement noise. Then,

$$D_G(\mathcal{T}_\lambda(z^*), \mathcal{T}_\lambda(z)) \leq L|\epsilon|.$$

Hence, small disturbances in the original complex signal produce controlled variations in the fuzzy representation.

Theorem 6.7 (Completeness of the Generalized Complex Fuzzy Metric Space). *The metric space*

$$(\mathcal{G}_\lambda, D_G)$$

is complete.

Proof. Let

$$\{\mathcal{T}_\lambda(z_n)\}$$

be a Cauchy sequence in the metric D_G.
By the stability inequality,

$$D_G(\mathcal{T}_\lambda(z_n), \mathcal{T}_\lambda(z_m)) \leq L|z_n - z_m|.$$

Thus, the corresponding sequence $\{z_n\}$ is Cauchy in $\mathbb{C}$.
Since the complex plane is complete, there exists

$$z \in \mathbb{C}$$

such that

$$z_n \to z.$$

By continuity of the operator $\mathcal{T}_\lambda$,

$$\mathcal{T}_\lambda(z_n) \to \mathcal{T}_\lambda(z).$$

Since

$$\mathcal{T}_\lambda(z) \in \mathcal{G}_\lambda,$$

the limit belongs to the space itself.
Therefore,

$$(\mathcal{G}_\lambda, D_G)$$

is a complete generalized complex fuzzy metric space.

□

7 Generalized Complex Fuzzy Contraction Mappings and Fixed-Point Theory

In this section, we establish a new fixed-point framework in the Generalized Complex Fuzzy Metric Space (GCFMS). The proposed results extend the classical Banach contraction principle to complex fuzzy representations and provide conditions for the existence, uniqueness, and stability of fixed transformations.

Definition 7.1 (Generalized Complex Fuzzy Contraction Operator). *Let*

$$\Phi : \mathcal{G}_\lambda \to \mathcal{G}_\lambda$$

be an operator. The mapping Φ *is called a generalized complex fuzzy contraction if there exists a constant* $0 < k < 1$ *such that*

$$D_G(\Phi(T_1), \Phi(T_2)) \leq k D_G(T_1, T_2)$$

for every

$$T_1, T_2 \in \mathcal{G}_\lambda.$$

Theorem 7.2 (Generalized Banach Fixed-Point Theorem). *Let*

$$(\mathcal{G}_\lambda, D_G)$$

be a complete Generalized Complex Fuzzy Metric Space and let

$$\Phi : \mathcal{G}_\lambda \to \mathcal{G}_\lambda$$

be a generalized complex fuzzy contraction operator. Then, there exists a unique transformation

$$T^* \in \mathcal{G}_\lambda$$

such that

$$\Phi(T^*) = T^*.$$

Furthermore, the sequence generated by

$$T_{n+1} = \Phi(T_n),$$

converges to T^* *for every initial approximation* $T_0 \in \mathcal{G}_\lambda$.

Proof. Choose an arbitrary initial transformation $T_0 \in \mathcal{G}_\lambda$ and define the iterative sequence

$$T_{n+1} = \Phi(T_n).$$

By the contraction property,

$$D_G(T_{n+1}, T_n) \leq k D_G(T_n, T_{n-1}).$$

Applying the inequality repeatedly gives

$$D_G(T_{n+1}, T_n) \leq k^n D_G(T_1, T_0).$$

For $m > n$, using the triangle inequality,

$$D_G(T_m, T_n) \leq \sum_{j=n}^{m-1} D_G(T_{j+1}, T_j).$$

Therefore,

$$D_G(T_m, T_n) \leq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) D_G(T_1, T_0).$$

Because the geometric series converges,

$$D_G(T_m, T_n) \leq \frac{k^n}{1-k} D_G(T_1, T_0).$$

As $n \to \infty$,

$$D_G(T_m, T_n) \to 0.$$

Hence, $\{T_n\}$ is a Cauchy sequence. Since GCFMS is complete, there exists a transformation

$$T^* \in \mathcal{G}_\lambda$$

such that

$$T_n \to T^*.$$

By continuity of Φ,

$$\Phi(T^*) = \lim_{n\to\infty} \Phi(T_n) = \lim_{n\to\infty} T_{n+1} = T^*.$$

Therefore, T^* is a fixed point.
For uniqueness, assume another fixed point S^* exists. Then,

$$D_G(T^*, S^*) = D_G(\Phi(T^*), \Phi(S^*))$$

which implies

$$D_G(T^*, S^*) \leq k D_G(T^*, S^*).$$

Since $0 < k < 1$,

$$(1-k) D_G(T^*, S^*) \leq 0.$$

Hence,

$$D_G(T^*, S^*) = 0,$$

which gives

$$T^* = S^*.$$

Thus, the fixed point is unique.

□

8 Algebraic Structure of the GCFRS

We now introduce algebraic operations that preserve the generalized complex fuzzy structure.

Definition 8.1 (Generalized Fuzzy Addition)**.** *For two transformations*

$$\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2) \in \mathcal{G}_\lambda,$$

define their generalized fuzzy addition by

$$\mathcal{T}_\lambda(z_1) \oplus \mathcal{T}_\lambda(z_2) = \mathcal{T}_\lambda(z_1 + z_2).$$

Definition 8.2 (Complex Scalar Transformation)**.** *For any scalar*

$$\beta \in \mathbb{C},$$

define

$$\beta \odot \mathcal{T}_\lambda(z) = \mathcal{T}_\lambda(\beta z).$$

Theorem 8.3 (Closure Property of GCFRS)**.** *The Generalized Complex Fuzzy Representation Space*

$$\mathcal{G}_\lambda$$

is closed under generalized fuzzy addition and complex scalar transformation.

Proof. Let

$$\mathcal{T}_\lambda(z_1), \mathcal{T}_\lambda(z_2) \in \mathcal{G}_\lambda.$$

Since the complex number system is closed under addition,

$$z_1 + z_2 \in \mathbb{C}.$$

Therefore,

$$\mathcal{T}_\lambda(z_1 + z_2) \in \mathcal{G}_\lambda.$$

Hence,

$$\mathcal{T}_\lambda(z_1) \oplus \mathcal{T}_\lambda(z_2) \in \mathcal{G}_\lambda.$$

Similarly, for any

$$\beta \in \mathbb{C},$$

the product

$$\beta z_1 \in \mathbb{C}.$$

Therefore,

$$\mathcal{T}_\lambda(\beta z_1) \in \mathcal{G}_\lambda.$$

Thus,

$$\beta \odot \mathcal{T}_\lambda(z_1) \in \mathcal{G}_\lambda.$$

Hence, GCFRS is algebraically closed.

□

Corollary 8.4 (Associative Property of Fuzzy Addition)**.** *For any three elements*

$$T_1, T_2, T_3 \in \mathcal{G}_\lambda,$$

we have

$$(T_1 \oplus T_2) \oplus T_3 = T_1 \oplus (T_2 \oplus T_3).$$

Proof. Let

$$T_i = \mathcal{T}_\lambda(z_i), \quad i = 1, 2, 3.$$

Then,

$$(T_1 \oplus T_2) \oplus T_3 = \mathcal{T}_\lambda((z_1 + z_2) + z_3).$$

By associativity of complex addition,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

Hence,

$$\mathcal{T}_\lambda((z_1 + z_2) + z_3) = \mathcal{T}_\lambda(z_1 + (z_2 + z_3)),$$

which gives

$$(T_1 \oplus T_2) \oplus T_3 = T_1 \oplus (T_2 \oplus T_3).$$

Therefore, generalized fuzzy addition is associative.

□

Remark 8.5. *The established algebraic properties indicate that the proposed GCFRS is not merely a collection of fuzzy images of complex numbers; rather, it possesses a well-defined mathematical structure suitable for the analysis of complex uncertain systems.*

The generalized Banach fixed-point theorem guarantees the existence and uniqueness of stable fuzzy transformations, which can be utilized in iterative algorithms, complex-valued optimization, and intelligent computational models.

9 Compactness, Convergence, and Computational Applications

In this section, we investigate additional topological properties of the Generalized Complex Fuzzy Representation Space (GCFRS). Furthermore, numerical examples and a computational algorithm are provided to demonstrate the practical implementation of the proposed framework.

9.1 Sequential Convergence and Compactness Properties

Theorem 9.1 (Sequential Convergence of GCFRS)**.** *Let*

$$\{\mathcal{T}_\lambda(z_n)\}_{n=1}^{\infty} \subseteq \mathcal{G}_\lambda$$

be a sequence of generalized complex fuzzy representations. If

$$z_n \to z \quad in \quad \mathbb{C},$$

then

$$\mathcal{T}_\lambda(z_n) \to \mathcal{T}_\lambda(z)$$

with respect to the generalized complex fuzzy metric D_G.

Proof. Since

$$z_n \to z,$$

we obtain

$$|\xi - z_n| \to |\xi - z|$$

Paper ID: SR26624164731 DOI: https://dx.doi.org/10.21275/SR26624164731

for every $\xi \in \mathbb{C}$.
The membership function

$$\mu_z(\xi) = e^{-\lambda|\xi - z|^2}$$

is continuous; therefore,

$$\mu_{z_n}(\xi) \to \mu_z(\xi).$$

Taking the supremum over all $\xi \in \mathbb{C}$, we have

$$D_G(\mathcal{T}_\lambda(z_n), \mathcal{T}_\lambda(z)) \to 0.$$

Thus, the sequence converges in GCFRS.

□

Theorem 9.2 (Compactness Preservation)**.** *Let*

$$K \subseteq \mathbb{C}$$

be a compact set. Then the transformed set

$$\mathcal{T}_\lambda(K) = \{\mathcal{T}_\lambda(z) : z \in K\}$$

is compact in the generalized complex fuzzy metric space.

Proof. The representation operator

$$\mathcal{T}_\lambda : \mathbb{C} \to \mathcal{G}_\lambda$$

has already been proved to be continuous. Since the continuous image of a compact set is compact, it follows that

$$\mathcal{T}_\lambda(K)$$

is compact in the GCFRS.

□

9.2 Numerical Illustrations

Example 9.3 (Fuzzy Representation of a Complex Point)**.** *Consider the complex number*

$$z = 2 + 3i$$

and choose the parameter

$$\lambda = 0.5.$$

Let the observation point be

$$\xi = 3 + 3i.$$

The complex distance is

$$|\xi - z| = 1.$$

Therefore,

$$\mu_z(\xi) = e^{-0.5(1)^2} = e^{-0.5} \approx 0.6065.$$

Thus, the point ξ belongs to the fuzzy representation of z with a membership degree of approximately 60.65%.

Example 9.4 (Effect of Distance on Membership Degree). *Let*

$$z = 0 + 0i, \qquad \lambda = 1.$$

For different distances $r = |\xi - z|$, the corresponding membership values are computed by

$$\mu(r) = e^{-r^2}.$$

The obtained results are shown in Table 1.

Table 1: Effect of distance on fuzzy membership value

Distance r	Membership value $\mu(r)$
0	1.0000
1	0.3679
2	0.0183
3	0.0001

The table demonstrates that the membership degree decreases exponentially as the distance from the original complex point increases.

9.3 Algorithm for Generalized Complex Fuzzy Representation

The computational procedure for transforming a complex number into a generalized fuzzy representation is summarized as follows.

Algorithm 1: Generalized Complex-to-Fuzzy Transformation

1. Input a complex number $z = x + iy$.
2. Select the sensitivity parameter $\lambda > 0$.
3. Choose an observation point $\xi \in \mathbb{C}$.
4. Compute the complex distance

$$r = |\xi - z|.$$

5. Evaluate the fuzzy membership value

$$\mu_z(\xi) = e^{-\lambda r^2}.$$

6. Output the generalized fuzzy representation $\mathcal{T}_\lambda(z)$.

9.4 Applications of the Proposed GCFRS

9.4.1 Application in Uncertain Complex Signal Processing

A complex signal can be represented as

$$s(t) = A(t)e^{i\phi(t)},$$

where $A(t)$ denotes the amplitude and $\phi(t)$ denotes the phase.

In practical systems, disturbances and measurement errors introduce uncertainty into the signal. The proposed representation converts the signal into a fuzzy complex model:

$$\mu_{s(t)}(\xi) = e^{-\lambda|\xi - s(t)|^2}.$$

Therefore, uncertain amplitude and phase information can be represented through gradual fuzzy memberships.

9.4.2 Application in Noise-Resistant Information Modeling

Suppose the observed complex data is

$$z^* = z + \epsilon,$$

where ϵ represents external noise.

From the stability theorem,

$$D_G(\mathcal{T}_\lambda(z^*), \mathcal{T}_\lambda(z)) \leq L|\epsilon|.$$

Hence, small perturbations produce only limited variations in the fuzzy representation, demonstrating robustness against noise.

9.4.3 Application in Complex-Valued Intelligent Systems

Modern intelligent systems, such as communication networks, image processing, quantum-inspired computation, and pattern recognition, frequently involve complex-valued data.

The proposed GCFRS provides a unified mathematical framework that converts exact complex information into fuzzy structures while preserving metric, algebraic, and topological properties.

Therefore, the proposed model can be employed in future complex fuzzy decision systems, artificial intelligence models, and uncertainty-aware computational methods.

Remark 9.5. *The numerical experiments and theoretical investigations demonstrate that the proposed generalized complex fuzzy representation offers a stable and flexible mechanism for handling uncertain complex-valued information. The exponential membership model provides smooth uncertainty transition, while the metric and algebraic structures ensure mathematical consistency.*

10 Comparative Analysis with Existing Fuzzy Models

The proposed Generalized Complex Fuzzy Representation Space (GCFRS) extends the existing complex fuzzy frameworks by providing a direct mathematical transformation from classical complex numbers into a structured fuzzy metric environment. Unlike conventional fuzzy models, which mainly focus on uncertainty representation or aggregation operators, the proposed approach develops a complete analytical framework involving geometric invariance, metric topology, fixed-point theory, and algebraic operations.

A comparative summary of different fuzzy models is presented in Table 2.

Table 2: Comparison between existing fuzzy structures and proposed GCFRS

Model	Representation Domain	Main Characteristics	Limitations
Classical Fuzzy Set	Real-valued data	Membership uncertainty	Does not represent complex information
Intuitionistic Fuzzy Set	Membership and non-membership degrees	Additional uncertainty modeling	Limited direct complex representation
Complex Fuzzy Set	Complex-valued membership grades	Phase and amplitude information	Limited metric and fixed-point structures
Spherical Fuzzy Set	Three-dimensional uncertainty representation	Independent membership parameters	Designed mainly for decision environments
Proposed GCFRS	Complex-to-fuzzy transformation space	Metric, topology, fixed points, and stability	Requires further computational extensions

11 Results and Discussion

The theoretical investigation demonstrates that the proposed GCFRS creates a mathematically consistent bridge between complex analysis and fuzzy mathematics. The Gaussian-type representation operator

$$\mathcal{T}_\lambda(z) = \exp(-\lambda|\xi - z|^2)$$

allows a smooth transformation of an exact complex point into a fuzzy structure where uncertainty increases with the distance from the original complex value.

The principal mathematical achievements obtained in this work are summarized as follows:

1. A new Generalized Complex Fuzzy Representation Operator was introduced for transforming complex information into fuzzy objects.
2. A new Generalized Complex Fuzzy Metric was constructed, which allowed the development of a complete topological structure.
3. Two fundamental lemmas were established to prove boundedness and exponential decay properties.
4. Eleven theoretical results, including invariance, continuity, stability, convergence, compactness, algebraic closure, and Banach-type fixed-point properties, were developed.
5. Numerical examples verified the practical behavior of the proposed membership model and demonstrated the reduction of membership values with increasing complex distance.
6. The proposed framework showed potential applicability in complex signal processing, artificial intelligence, and uncertainty modeling.

The overall mathematical structure of the proposed model is summarized in Table 3.

Table 3: Summary of theoretical developments in GCFRS

Result	Mathematical Property	Contribution
Lemma 1	Boundedness	Validity of fuzzy membership
Lemma 2	Exponential decay	Distance-dependent uncertainty
Theorem 1	Maximum principle	Preservation of original complex information
Theorem 2	Translation invariance	Geometric consistency
Theorem 3	Rotational invariance	Phase preservation
Theorem 4	Metric construction	Formation of GCFMS
Theorem 5	Continuity	Smooth complex-fuzzy transformation
Theorem 6	Stability	Resistance against perturbations
Theorem 7	Completeness	Existence of limits in GCFMS
Theorem 8	Fixed-point theorem	Unique stable transformations
Theorem 9	Algebraic closure	Preservation under operations
Theorem 10	Sequential convergence	Convergence of fuzzy sequences
Theorem 11	Compactness preservation	Topological consistency

12 Conclusion and Future Research

This article proposed a new Generalized Complex Fuzzy Representation Space (GCFRS) that provides a rigorous mathematical framework for converting classical complex numbers into generalized fuzzy representations. A Gaussian distance-based membership operator was introduced, and its analytical behavior was studied through various mathematical tools.

The developed theory established boundedness, geometric invariance, metric properties, continuity, stability, completeness, compactness, and fixed-point results. The introduction of a generalized complex fuzzy metric enabled the investigation of convergence and topological properties of fuzzy transformations.

The numerical illustrations confirmed that the proposed representation smoothly decreases with increasing distance and effectively models uncertainty around complex-valued data.

Future studies may extend the proposed framework toward complex intuitionistic fuzzy spaces, spherical complex fuzzy environments, neural-network-based complex fuzzy learning systems, and advanced optimization algorithms under uncertain complex information.

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