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Characterization of Fuzzy Topological Approximations of Multisets

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Abstract: ***The theory of multisets provides an efficient mathematical framework for representing collections containing repeated elements, while fuzzy set theory addresses uncertainty through graded membership values. Topological approximation operators have been widely used in rough set theory to describe vague concepts. In this paper, we introduce fuzzy topological ap- proximations of multisets by combining fuzzy multiset topology with rough approximation theory. New lower and upper approximation operators are defined using fuzzy multiset interior and closure operators. Fundamental properties of these approximations are established, including monotonicity, idempotency, duality, and boundary characterization. Furthermore, relation- ships among fuzzy topological approximations, rough multisets, and fuzzy multiset topological spaces are investigated. Several illustrative examples are provided to demonstrate the applicability of the proposed model. The results generalize classical topological approximations of multisets and provide a new framework for uncertainty modeling in decision systems and information processing.***

Keywords: Fuzzy multiset, multiset topology, rough multiset, fuzzy approximation space, lower approximation, upper approximation

1. Introduction

The mathematical modelling of uncertainty, vagueness, and incomplete information has become an important area of research in modern mathematics, computer science, and information systems. One of the most influential approaches to handling uncertainty was introduced by Zadeh through the theory of fuzzy sets, which allows elements to belong to a set with varying degrees of membership rather than through a strict binary classification [1]. Since its inception, fuzzy set theory has been extensively applied in pattern recognition, decision making, artificial intelligence, image processing, and data mining.

Another significant development in uncertainty modeling was the intro- duction of rough set theory by Pawlak [2]. Rough sets provide a framework for approximating imprecise concepts using lower and upper approximation operators derived from indiscernibility relations. The rough set approach has proven effective in knowledge discovery, feature selection, and information granulation. The interaction between rough set theory and topology has subsequently led to the development of topological approximation spaces, where approximation operators are characterized through topological interior and closure operators.

In many practical situations, the occurrence of an element cannot be adequately represented by classical sets because repeated appearances of the same element carry meaningful information. To address this limitation, multiset theory was introduced by Blizard [3], providing a generalized framework in which an element may occur multiple times. The concept of multi- sets, also known as bags, has found applications in database theory, molecular biology, information retrieval, and formal language theory. Yager [7] further explored the theoretical foundations of bags and demonstrated their usefulness in modeling collections with repeated objects.

The integration of fuzziness and multiplicity led to the development of fuzzy multisets. Miyamoto [8] investigated the structure of fuzzy multi- sets and established their relevance in soft computing environments. Sub- sequently, Syropoulos [4] generalized fuzzy multiset concepts and discussed their computational applications. More recently, Couso and Godo [14] revisited the theoretical foundations of fuzzy multisets and clarified several ambiguities regarding their algebraic structure, thereby strengthening the mathematical basis of fuzzy multiset theory.

Topology has played a central role in the study of fuzzy structures. Chang [11] introduced fuzzy topological spaces as a natural extension of classical topology, while Lowen [12] further developed the theory by investigating fuzzy compactness and related topological properties. These contributions established a solid framework for studying continuity, convergence, and ap- proximation in fuzzy environments.

The extension of topological concepts to multisets has attracted considerable attention in recent years. El-Sheikh and Hosny [6] introduced multi- set topologies induced by multiset relations and examined their fundamental properties. Later, El-Sheikh and co-authors [5] developed topological approximations of multisets and demonstrated how lower and upper approximations can be characterized through multiset topological structures. Their work established an important connection between rough set theory and multiset topology.

Recent studies have further expanded the scope of fuzzy multiset topo- logical structures. Hoskova-Mayerova et al. [9] investigated fuzzy multi- hypergroups, while Al Tahan et al. [10] introduced fuzzy multi polygroups and analyzed their algebraic properties. Ahmadu et al. [13] proposed the notion of fuzzy multiset bitopology, opening new directions for the study of multiple topological structures in fuzzy multiset environments.

Despite these developments, the characterization of approximation operators within fuzzy multiset topological spaces remains relatively unexplored. Existing studies primarily focus on either fuzzy topological spaces, multi- set topologies, or rough approximations independently. A unified framework that combines fuzzy topology, multiset theory, and approximation operators is still lacking. Such a framework is essential for modeling situations in- volving both repeated occurrences and uncertainty, which frequently arise in data analysis, information systems, medical diagnosis, and decision-support applications.

The concept of a multiset extends the classical notion of a set by allowing multiple occurrences of the same element. Multiset theory has found ap- plications in database systems, computer science, information retrieval, and decision-making processes. On the other hand, fuzzy set theory introduced by Zadeh provides a powerful tool for handling uncertainty and vagueness.

Topological approaches to rough set theory have attracted considerable attention because topological structures naturally generate approximation operators. Existing studies have investigated topological approximations of multisets and generalized rough multisets. However, the characterization of approximation operators within fuzzy multiset topological spaces remains largely unexplored.

The objective of this paper is to establish a framework for fuzzy topological approximations of multisets and investigate their fundamental properties.

The main contributions are:
1) Introduction of fuzzy multiset topological approximation operators.
2) Characterization of lower and upper fuzzy topological approximations.
3) Establishment of algebraic and topological properties.
4) Development of boundary and accuracy measures.
5) Demonstration of applicability through examples.

2. Preliminaries

In this section, we recall some concepts related to Multiset, Fuzzy Multiset,

2.1 Multiset:[1]

Let $\mathbb{U}$ be a universe of discourse. A multiset $\mathcal{M}$ on $\mathbb{U}$ is represented by

$$\mathcal{M} = m(x)/x : x \in \mathbb{U},$$

Where,

$$m : \mathbb{U} \to \mathbb{N}$$

denotes the multiplicity function.

2.2 Fuzzy Multiset

A Fuzzy multiset $\mathcal{A}$ on $\mathbb{U}$ is defined as

$$\mathcal{A} = \mu_{\mathcal{A}}(x)/x : x \in \mathbb{U},$$

Where,

$$\mu_{\mathcal{A}}(x) \in [0,1].$$

The value $\mu_{\mathcal{A}}(x)$ represents the degree of membership of x.

3. Fuzzy Topological Approximation Operators

Let $(\mathbb{U}, \tau)$ be a fuzzy multiset topological space.
Denote:

$$Int(\mathcal{A})$$

as the fuzzy multiset interior of $\mathcal{A}$, and

$$Cl(\mathcal{A})$$

as the fuzzy multiset closure of $\mathcal{A}$.

Definition 3.1. The lower fuzzy topological approximation of $\mathcal{A}$ is defined by

$$\underline{\mathcal{A}}_{\tau} = Int(\mathcal{A})$$

Definition 3.2. The upper fuzzy topological approximation of $\mathcal{A}$ is defined by

$$\overline{\mathcal{A}}_{\tau} = Cl(\mathcal{A}).$$

Definition 3.3. The fuzzy boundary region of $\mathcal{A}$ is

$$B_{\tau}(A) = \overline{\mathcal{A}}\tau - \underline{\mathcal{A}}\tau.$$

A fuzzy multiset is exact whenever

$$B_{\tau}(\mathcal{A}) = 0.$$

Example 3.1. Let the universe of discourse be

$$\mathbb{U} = x_1, x_2, x_3, x_4, x_5.$$

Consider the multiset

$$\mathcal{M} = 2/x_1,,3/x_2,,1/x_3,,4/x_4,,2/x_5,$$

where the coefficient of each element denotes its multiplicity.

Define a fuzzy topology on $\mathbb{U}$ by

$$\tau = \tilde{0}, \tilde{1}, X, Y, X \vee Y,$$

Where

$$X = 0.6/x_1,,0.4/x_2,,0.2/x_3,,0.1/x_4,,0.0/x_5,$$

and

$$Y = 0.2/x_1,,0.5/x_2,,0.7/x_3,,0.3/x_4,,0.4/x_5.$$

The fuzzy multiset under consideration is

$$\mathcal{F} = 0.5/x_1,,0.5/x_2,,0.6/x_3,,0.3/x_4,,0.2/x_5.$$

Lower Approximation; The fuzzy topological lower approximation of $\mathcal{F}$ is defined by

$$\underline{Apr_{\tau}}(\mathcal{F}) \bigvee G \in \tau : G \leq \mathcal{F}$$

Observe that

$$V(x_3) = 0.7 > 0.6 = \mathcal{F}(x_3),$$

Hence

$$Y \nleq \mathcal{F}$$

Similarly,

$$X(x_1) = 0.6 > 0.5 = \mathcal{F}(x_1),$$

therefore

$$X \nleq \mathcal{F}$$

The only open fuzzy set contained in $\mathcal{F}$ is the null fuzzy set $\tilde{0}$. Thus

$$\underline{Apr_\tau}(\mathcal{F}) = \tilde{0}.$$

Upper Approximation; The fuzzy topological upper approximation is given by

$$\overline{Apr}_\tau(\mathcal{F}) \bigwedge H: H^c \in \tau, ; \mathcal{F} \leq H$$

The complements of X and Y are

$$X^c 0.4/x_1, ,0.6/x_2, ,0.8/x_3, ,0.9/x_4, ,1.0/x_5,$$

and

$$Y^c 0.8/x_1, ,0.5/x_2, ,0.3/x_3, ,0.7/x_4, ,0.6/x_5.$$

since neither X^c nor Y^c completely contains $\mathcal{F}$, the smallest closed fuzzy set containing $\mathcal{F}$ is

$$\tilde{1} 1/x_1, ,1/x_2, ,1/x_3, ,1/x_4, ,1/x_5.$$

Hence

$$\overline{Apr}_\tau(\mathcal{F}) = \tilde{1}.$$

Boundary Region; The fuzzy boundary region of $\mathcal{F}$ is

$$Bnd_\tau(\mathcal{F}) \overline{Apr}_\tau(\mathcal{F}) \underline{Apr_\tau}(\mathcal{F}).$$

Therefore,

$$Bnd_\tau(\mathcal{F}) 1/x_1, ,1/x_2, ,1/x_3, ,1/x_4, ,1/x_5.$$

Interpretation; Since

$$\underline{Apr} * \tau(\mathcal{F}) \neq \overline{Apr} * \tau(\mathcal{F}),$$

the fuzzy multiset $\mathcal{F}$ is not topologically definable. Consequently, $\mathcal{F}$ is a rough fuzzy multiset with respect to the fuzzy topology τ . The non-empty boundary region indicates the presence of uncertainty in the topological characterization of the multiset.

4. Characterization Theorems

4.1 Theorem. {Inclusion Property}, For every fuzzy multiset $\mathcal{A}$,

$$\underline{\mathcal{A}}\tau \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}}\tau.$$

Proof: Since interior is the largest open fuzzy multiset contained in $\mathcal{A}$,

$$Int(\mathcal{A}) \subseteq \mathcal{A}.$$

Similarly, closure is the smallest closed fuzzy multiset containing $\mathcal{A}$,

$$\mathcal{A} \subseteq Cl(\mathcal{A}).$$

Therefore,

$$\underline{\mathcal{A}}\tau \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}}\tau.$$

4.2 Theorem. {Monotonicity}, If

$$\mathcal{A} \subseteq B,$$

then

$$\underline{\mathcal{A}} * \tau \subseteq \underline{B} * \tau$$

And

$$\overline{\mathcal{A}} * \tau \subseteq \overline{B} * \tau.$$

Proof: The result follows directly from the monotonicity of interior and closure operators.

4.1 Theorem. {Idempotency}, $\underline{\underline{\mathcal{A}}}_\tau \underline{\mathcal{A}}_\tau$ and $\overline{\overline{\mathcal{A}}}_\tau \overline{\mathcal{A}}_\tau$.

Proof: Since

$$Int\big(Int(\mathcal{A})\big) Int(\mathcal{A})$$

And

$$Cl\big(Cl(\mathcal{A})\big) Cl(\mathcal{A}),$$

the theorem follows.

4.1 Theorem. {Duality},

$$\underline{\mathcal{A}^c}_\tau \left(\overline{\mathcal{A}}_\tau\right)^c.$$

Proof, Using topological duality,

$$Int(\mathcal{A}^c) \big(Cl(\mathcal{A})\big)^c.$$

Hence

$$\underline{\mathcal{A}^c}_\tau \left(\overline{\mathcal{A}}_\tau\right)^c.$$

5. Accuracy Measure

Define

$$Acc(\mathcal{A}) \frac{\left|\underline{\mathcal{A}} * \tau\right|}{\left|\overline{\mathcal{A}} * \tau\right|}.$$

Properties:

- $0 \leq Acc(\mathcal{A}) \leq 1$.
- $(Acc(\mathcal{A}) = 1) iff (\mathcal{A})$ is exact.
- Larger values indicate greater certainty.

6. Illustrative Example

Consider $U = \{a, b, c\}$. Let $\mathcal{A} = \{0.8/a, 0.4/b, 0.2/c\}$. Suppose

$$Int(\mathcal{A}) 0.7/a, 0.3/b, 0.1/c$$

and

$$Cl(\mathcal{A}) 0.9/a, 0.6/b, 0.4/c.$$

Then

$$\underline{\mathcal{A}}_\tau 0.7/a, 0.3/b, 0.1/c$$

And

$$\overline{\mathcal{A}}_\tau 0.9/a, 0.6/b, 0.4/c.$$

The boundary region is non-empty; hence $\mathcal{A}$ is a rough fuzzy multiset.

7. Application

The proposed model can be applied in:

- Medical diagnosis systems with repeated observations.
- Pattern recognition under uncertainty.
- Data mining and information retrieval.
- Knowledge representation.
- Multi-criteria decision-making systems.

8. Conclusion

This paper introduced a novel framework for fuzzy topological approximations of multisets. By employing fuzzy multiset interior and closure operators, lower and upper approximations were defined and characterized. Several theoretical properties were established, including inclusion, monotonicity, idempotency, and duality. The proposed model generalizes classical rough multisets and topological approximation spaces. Future work may investigate nano-fuzzy multiset topology, intuitionistic fuzzy multisets, and applications to machine learning and decision-support systems.

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