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Fixed Point Theorems Under an Expansive Rational Condition in A_b- Metric Spaces

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Abstract: ***We introduce an expansive rational condition for self maps acting on A_b-metric spaces and study its consequence for fixed point existence and uniqueness .While contraction condition are classical in fixed point theory, expansive condition when paired with structural hypotheses such as inevitability or contractive behavior of an inverse or an iterate also yield fixed points in generalized metric setting .We present a main existence and uniqueness theorem that applies when the inverse map satisfies a linear rational contractive :three corollaries treat iterates, commuting maps and a multivalued selection case. Several examples illustrate the hypotheses .The setting generalizes existing results on A_b-metric and related generalized metric spaces.***

Keywords: A_b-metric space, expansive rational condition, fixed point, linear rational contraction, generalized metric spaces.

1. Introduction

Fixed point theory is a central theme in nonlinear analysis with profound applications across mathematics and applied science. The classical Banach contraction principle established a foundation that has since been generalized to various nonstandard distance structures such as b-metric spaces,[1] Fuzzy metric spaces[2,3],partial metric spaces,[4] S-metric spaces,[16] A-metric and A_b-metric spaces [14,6] In these settings,the contraction framework has been refined by adopting Kannan,[8] Reich,[9] and Ciric-type conditions,[10] as well as rational expressions.[11,12,13] Such generalizations allow broader classes of nonlinear operation to admit fixed point.

Recent progress has highlighted the potential of rational contractive conditions in both ordered and unordered generalized metric spaces.[14,15,16,17,18] For instance, multivalued mappings have been studied by Nadler,[19] while Geraghty and rational- type contractions have gained importance in applications .[20] In the specific case of A_b-metric spaces, Ughade et al.[6] and Patel et al.[7] provided Banach –type and rational –type fixed point results.

Despite the focus on contractive behavior, expansive conditions when combined with structural properties such as invertibility or iterate contractivity may still lead to fixed points. Motivated by this, we introduce an expansive rational condition (ERC) in A_b-metric spaces and develop new results extending earlier contributions. Our work complements classical contraction- based approaches by providing fixed point theorems under expansive rational assumption, thereby enriching the framework of A_b-metric fixed point theory.

2. Preliminaries

Definition 2.1 (A_b-metric). Let X be a nonempty set and $N \geq 2$ a fixed point integer.

A function

$$A_b: X^N \to [0, \infty)$$

Is called an A_b-metric on X (with control constant $s \geq 1$) if for all $\alpha_1, \ldots, \alpha_N, \xi \in X$

The following hold:

1) (Positivity and definiteness) $A_b(\alpha_1, \ldots, \alpha_N) = 0$ if and only if $\alpha_1 = \ldots = \alpha_N$.
2) (Symmetry) A_b is invariant under any permutation of its N arguments.
3) (Generalized triangle- type inequality) these exists $S \geq 1$ such that for every choice of points and every auxiliary $\xi \in X$,

$$A_b(\alpha_1, \ldots, \alpha_N) \leq s \sum_{i=1}^{N} A_b(\underbrace{\alpha_i, \ldots, \alpha_i}_{N-1\ times}, \xi).$$

The pair $(X, A_b)(or (X, A_b, s))$ is called an A_b-metric space.

Definition 2.2 (Convergence, Cauchy, completeness). Let (X, A_b) be an A_b-metric space.

- A sequence $(\alpha_n) \subset X$ converges to $\alpha \in X$ if $A_b(\alpha_n, \ldots \alpha_n, \alpha) \to 0$ as $n \to \infty$.
- The sequence (α_n) is Cauchy if $A_b(\alpha_n, \ldots \alpha_n, \alpha_m) \to 0$ as $n, m \to \infty$.
- The space (X, A_b) is complete if every Cauchy sequence converges to a point of X.

Definition 2.3 (Continuity and contraction in the A_b setting), Let (X, A_b) be an A_b-metric space and $\Phi: X \to X$ a map.

- Φ is continuous at $\alpha \in X$ if $\alpha_n, \to \alpha$ implies $\Phi\alpha_n \to \Phi\alpha$.

- Φ is (rational) contraction if there exist parameters making an inequality of rational-contractive type hold when the left-hand side is an A_b distance between Φ-images and the right-hand side is a function of the corresponding A_b distance of the original points.

Remark 2.4. The form of the generalized triangle inequality in Definition 2.1 is the standard one used in recent A_b-metric literature; see for examples [6,7].

In all three examples below let (X, d) be any ordinary metric space with metric d. We define A_b-type function on X^N (With a fixed point $N \geq 2$) and show they satisfy the A_b-metric axioms, giving the explicit control constant s in each case.

Example 1: Maximum pairwise distance

Define
$A_b^{(1)}(x_1, ... , x_n) = max_{1\leq i<j\leq N} d(x_{i,} x_j)$.
Then $A_b^{(1)}$ is an A_b-metric.

Proof: Positivity definiteness. If $(x_1 =, ... , = x_N)$ then every pairwise distance is $0, so$ $A_b^{(1)} = 0$. Conversely, if $A_b^{(1)} = 0$ then every $d(x_{i,} x_j). = 0$, so $x_{\,i} = x_j$ for all i, j; thus al coordinates are equal.

Symmetry. The expression depends only on the unordered pairs $\{i, j\}$, hence it is invariant under permutation of arguments.

Generalized triangle inequality: Fix arbitrary x_1, $. . . . , x_N$, $\xi \in X$. Let

$$M := \max_{1\leq i<j\leq N} d(x_i , x_j) = A_b^{(1)}(x_1 ,. . . . , x_N).$$

For any pair i, j the metric triangle inequality gives

$$d(x_{i,} x_j) \leq d(x_{i,}\, \xi) + d(\xi, x_j).$$

Taking the maximum over pairs on the left and noting the right –hand side is bounded by

$d(x_{i,}\, \xi) + d(x_j, x_j) \leq \sum_{K=1}^{N} d(x_k , \xi) + \sum_{K=1}^{N} d(x_k , \xi) = 2\sum_{K=1}^{N} d(x_k , \xi)$, we obtain

$$M \leq 2\sum_{K=1}^{N} d(x_k , \xi).$$

Observe that for each k,

$$A_b^{(1)}\left(\underbrace{x_k ,. . . . , x_{k,}}_{N-1\ times} \xi)\right) = \max\{d(x_k, \xi), 0\} = d(x_k, \xi).$$

Hence

$$A_b^{(1)}(x_1 ,. . . . , x_{N,}) \leq 2\sum_{K=1}^{N} A_b^{(1)}\left(\underbrace{x_k ,. . . . , x_{k,}}_{N-1\ times} \xi\right).$$

Thus, the generalized triangle inequality holds with control constant $s = 2$. This proves $A_b^{(1)}$ is A_b-metric.

Example 2: Average (mean) of pair wise distance. Define

$$A_b^{(2)}(x_1 ,. . . . , x_{N,}) = \frac{2}{N(N-1)} \sum_{1\leq i\leq j\leq N} d(x_i , x_j),$$

i.e. the arithmetic mean of the $\binom{N}{2}$ pairwise distances. Then $A_b^{(2)}$ is an A_b-metric.

Proof: Positivity /definiteness. If all coordinates coincide then every pair wise distance is zero and hence $A_b^{(2)} = 0$. conversely, if $A_b^{(2)} = 0$ then every pairwise distance zero (0). So, all coordinates coincide.

Symmetry: The sum over unordered pair is permutation invariant.

Generalized triangle inequality. Fix $x_1 ,. . . . , x_{N,}\, \xi \in X$. Using the metric triangle inequality
$d(x_{i,} x_j) \leq d(x_{i,}\, \xi) + d(\xi, x_j)$. and summing over all pairs$1 \leq i < j \leq N$ yields

$$\sum_{i<j} d(x_i , x_j \leq \sum_{i<j}\left(d(x_{i,}\, \xi) + d(\xi, x_j)\right) = (N-1)\sum_{K=1}^{N} d(x_{k,}\, \xi),$$

Because each fixed point k appears in exactly $(N-1)$ unordered pairs. Therefore

$$\begin{aligned} A_b^{(2)}(x_1 ,. . . . , x_{N,}) &= \frac{2}{N(N-1)}\sum_{i<j} d(x_i , x_j) \\ &\leq \frac{2}{N(N-1)} . (N-1)\sum_{K=1}^{N} d(x_{k,}\, \xi) \\ &= \frac{2}{N}\sum_{K=1}^{N} d(x_{k,}\, \xi). \end{aligned}$$

As in example 1 we note $A_b^{(2)}\,\underbrace{(x_k ,. . . . , x_{k,}}_{N-1\ times}\, \xi)$ $= \frac{2}{N(N-1)}\sum_{i<j}(.)$ when all but one argument equal

$x_{k,}$ which simplifies to a quantity proportional to $d(x_k, \xi)$.

$$A_b^{(2)}\left(\underbrace{x_k ,. . . . , x_{k,}}_{N-1\ times} \xi\right) = \frac{2}{N(N-1)}\sum_{i<j} d(.) = \frac{2}{N(N-1)} . (N-1) d(x_k, \xi) = \frac{2}{N} d(x_k, \xi).$$

Combining the inequalities.

$$A_b^{(2)}(x_1 ,. . . . , x_N) \leq \frac{2}{N}\sum_{K=1}^{N} d(x_{k,}\, \xi) = \sum_{K=1}^{N} A_b^{(2)}\left(\underbrace{x_k ,. . . . , x_{k,}}_{N-1\ times} \xi\right).$$

Thu the generalized triangle inequality holds with $s = 1$ in the displayed final form (or, if one prefers the earlier inequality scale, with $s = 1$) Hence $A_b^{(2)}$ is an A_b-metric (with the displayed estimates showing the required control).

Example 3: Sum of all pairwise distances

Define

$$A_b^{(2)}(x_1,\ldots,x_N)=\sum_{1\le i<j\le N} d(x_i,x_j).$$

Then $A_b^{(3)}$ is an A_b-metric.

Proof: Positivity/ definiteness. If all coordinates are equal then every pair wise distance is 0, hence $A_b^{(3)}=0$. conversely, if $A_b^{(3)}=0$ then all pairwise distance are 0, so all coordinates coincide

Symmetry. The sum over unordered pairs is invariant under permutations.

Generalized triangle inequality. Fix $x_1,\ldots,x_N,\xi\in X$. Using $d(x_i,x_j)\le d(x_i,\xi)+d(\xi,x_j)$. and summing over unordered pairs.

$$\sum_{i<j} d(x_i,x_j\le\sum_{i<j}\left(d(x_i,\xi)+d(\xi,x_j)\right)=(N-1)\sum_{K=1}^{N} d(x_k,\xi),$$

Therefore

$$A_b^{(3)}(x_1,\ldots,x_N)\le.(N-1)\sum_{K=1}^{N} d(x_k,\xi).$$

But for each k,

$$A_b^{(3)}\left(\underbrace{x_k,\ldots,x_k}_{N-1\ times},\xi\right)=\sum_{i<j} d(.)=(N-1)\,d(x_k,\xi),$$

Since when all but one slot equals x_k, the only nonzero pairwise distance are those involving the differing slot ξ, of which there are $N-1$. consequently,

$$A_b^{(3)}(x_1,\ldots,x_N)\le\sum_{k=1}^{N} A_b^{(3)}\left(\underbrace{x_k,\ldots,x_k}_{N-1\ times},\xi\right).$$

Thus, the generalized triangle inequality holds with control constant $s=1$ in this final form, and the function $A_b^{(3)}$ satisfies the A_b-metric axioms. (If one prefers to show the inequality in the more conservative form of Definition 2.1 with a single numeric s multiplying the sum on the right, observe the displayed inequality already has $s=1$.

Remark 2.5 Each of the three constructions is natural and widely used to produce multi-argument distance like functions from a base metric d. Example 1-3 are simple to compute and are helpful in building illustrative examples or counter examples in A_b-metric context; similar constructions (and variants using weights or powers of d) appear in the generalized-metric literature. For further exposition and applications in the A_b-metric framework see [6,7].

3. An Expansive Rational Condition

Fix a complete A_b-metric space (X,A_b,s) and consider a self-maps$\Phi: X\to X$.

Definition 3.1 (Expansive rational condition (ERC)). We say Φ satisfies an expansive rational condition (ERC) if there exist constants $p>0, q\ge 0, r\ge 0$ and a nonnegative constant k such that for all distinct $\alpha,\beta\in X$

$$Ab\left(\underbrace{\Phi\alpha,\ldots,\Phi\alpha}_{N-1\text{ times}},\Phi\beta\right)\ge\frac{p\,Ab\left(\underbrace{\alpha,\ldots\alpha}_{N-1\text{ times}},\beta\right)+q}{1+r\,Ab\left(\underbrace{\alpha,\ldots\alpha}_{N-1\text{ times}},\beta\right)}-k,\quad(3.1)$$

With the understanding that the right –hand side is nonnegative for the admissible range of parameters. When $p>1$ we call the condition strictly expansive rational.

Example 3.2 (Linear dilation –strict expansive). Let $X=\mathbb{R}$, fix $N=2$, and define

$$A_b(x,x,y)=|x-y|\qquad(x,y\in\mathbb{R}).$$

Define the map $\Phi:\mathbb{R}\to\mathbb{R}$ by $\Phi(x)=2x$. For distinct α,β put

$$d:=A_b(\alpha,\alpha,\beta)=|\alpha-\beta|>0.$$

Then

$A_b(\Phi\alpha,\Phi\beta)=|\Phi\alpha-\Phi\beta|=|2\alpha-2\beta|=2|\alpha-\beta|=2d.$

Choose parameters $p=2, q=0, r=0, k=0$. Then right –hand side of (3.1)

Becomes

$$\frac{pd+q}{1+rd}-k=\frac{2d+0}{1+0.d}-0=2d.$$

Thus for every distinct α,β we have

$$A_b(\Phi\alpha,\Phi\beta)=2d\ge 2d$$

So Φ satisfies the ERC with $(p,q,r,k)=(2,0,0,0)$. since $p=2>1$ this is a strictly expansive rational example (in the $N=2$ *case*).

Example 3.3: (Affine dilation with rational term). Again let $X=\mathbb{R}$ and $N=2$ with $A_b(x,x,y)=|x-y|$.Define the affine map $\Psi:\mathbb{R}\to\mathbb{R}$ by $\Psi(x)=3x+1$ For distinct α,β set $d=|\alpha-\beta|>0$. Then

$$A_b(\Psi\alpha-\Psi\beta)=|\Psi\alpha-\Psi\beta|=|3\alpha+1-(3\beta+1)|=3|\alpha-\beta|=3d.$$

Take parameters $p=3, q=1, r=1$, and $k=1$. Compute the RHS of (3.1)

$$\frac{pd+q}{1+rd}-k=\frac{3d+1}{1+d}-1=\frac{3d+1-(1+d)}{1+d}=\frac{2d}{1+d}.$$

Since $d>0$ we have $1+d>1$ and therefore

$$3d=A_b(\Psi\alpha-\Psi\beta)\ge\frac{2d}{1+d}=\frac{3d+1}{1+d}-1,$$

So Ψ satisfies the ERC inequality (3.1) with the chosen parameters. Note the RHS is nonnegative for all $d\ge 0$, so the admissibility condition is met. This example exhibits an expansive linear coefficient (3) while the rational correction and subtraction of k make the inequality nontrivial.

Remark 3.4 Both example above use the $N=2$ instantiation $A_b(x,x,y)=|x-y|$ to keep verification transparent. For genuine multi-argument A_b distances one may replace $|\alpha-\beta|$ by the appropriate $A_b(\alpha,\ldots,\alpha,\beta)$ and repeat the same algebraic checks, the pattern of parameter choices carries over. These examples are intended to show explicit maps and parameter tuples (p,q,r,k) that satisfy Definition 3.1 (ERC).

Remark 3.5: When $p < 1, q = 0 \geq 0$, and $k = 0$, the condition reduces to a linear rational contraction. Our interest hear is situations where $p \geq 1$ (expansive) but additional structural hypotheses permit fixed point conclusions. Rational –type conditions have been exploited in metric and generalized metric spaces to broaden the class of admissible operators (see, e.g., [11, 12].

We now give several theorems showing that an expansive rational condition can still yield fixed points once additional hypotheses are enforced.

4. Main Results

Theorem 4.1: (Inverse-Contraction Principle). Let (X, A_b, s) be a complete A_b-metric spaces. Suppose $\Phi : X \to X$ is bijective and its inverse Φ^{-1} satisfies the linear rational contraction condition: there exist $\alpha_0 \in (0,1)$ and $\beta_0 \geq 0$ such that for all $\xi, \eta \in X$,

$$A_b(\underbrace{\Phi^{-1}\xi, \ \ldots\ldots \Phi^{-1}\xi}_{N-1\ times}, \Phi^{-1}\eta) \leq \frac{\alpha_0 A_b\left(\underbrace{\xi,\ldots\ldots\xi}_{N-1\ times}, \ \eta\right)}{1+\beta_0\left(1-A_b\left(\underbrace{\xi,\ldots\ldots\xi}_{N-1\ times}, \ \eta\right)\right)} \tag{4.1}$$

Then Φ has a unique fixed point $\alpha^* \in X$. Moreover, for any $\alpha_0 \in X$ the backward picard iteration $\alpha_{n+1} = \Phi^{-1} \ \alpha_n$ converges to α^*.

Proof: We first observe a standard normalization convenience that avoids cumbersome case distinctions for the rational denominator in (4.1). If the A_b-distance can attain arbitrarily large values one may replace $A_b$by the equivalent bounded metric.

$$\widetilde{A_b}(u_1, \ldots, u_N) = \frac{A_b(u_1, \ldots, u_N)}{1 + A_b((u_1, \ldots, u_N)},$$

Which takes values in [0,1] and preserves the separation, symmetry and generalized triangle

Type inequality up to harmless constants; hence completeness and fixed point assertions for maps expressed in terms of A_b are equivalently expressed in terms of $\widetilde{A_b}$. Thus without loss of generality we assume the A_b-values under consideration lie in [0,1] so that the denominator $1 + \beta_0(1 - A_b\ (;\ .))$ appearing in (4.1) is bounded below by 1 and remains strictly positive.

Choose an arbitrary initial point $\alpha_0 \in X$ and define the backward Picard sequence by

$$\alpha_{n+1} = \Phi^{-1}\alpha_n, \quad n = 0, 1, 2 \ldots .$$

Set

$$d_n = A_b(\underbrace{\alpha_n, \ \ , \ \ldots \ \ , \alpha_n}_{N-1\ times}, \quad \alpha_{n+1}) \qquad (n \geq 0)$$

Applying (4.1) with $\xi = \alpha_n$ and $\eta = \alpha_{n+1}$ and using the fact that $\Phi^{-1} \ \alpha_n = \alpha_{n+1}$ and $\Phi^{-1}\alpha_{n+1} = \alpha_{n+2}$ yields the one –step inequality.

$$d_{n+1} = A_b(\underbrace{\alpha_{n+1}, \ldots . \alpha_{n+1}}_{N-1\ times}, . \alpha_{n+2}) \leq \frac{\alpha_0\, d_n}{1+\beta_0(1-d_n)} = \varphi(d_n). \tag{4.2}$$

The function $\varphi: [0,1] \to [0,1)$ defined by $\varphi(t) = \frac{\alpha_0 t}{1+\beta_0(1-t)}$ is continuous and nonnegative on [0,1]. And satisfies $\varphi(0) = 0$.since $0 < \alpha_0 < 1$ we have $\varphi(t) < t$ for every sufficiently small positive t (indeed $\frac{\varphi(t)}{t} = \frac{\alpha_0 t}{1+\beta_0(1-t)} \leq \alpha_0 < 1$ for all $t \in [0,1]$). The recurrence (4.2) therefore implies the sequence (d_n) is nonnegative and non increasing after the first index at which d_n enters a sufficiently small neighborhood of 0; in particular (d_n) has a limit $L \geq 0$. Passing to the limit in (4.2) and using continuity of φ gives

$$L \leq \varphi(L) = \frac{\alpha_0 L}{1+\beta_0(1-L)}.$$

If $L > 0$ then rearranging yields

$$1 \leq \frac{\alpha_0}{1+\beta_0(1-L)} \implies 1 + \beta_0(1 - L) \leq \alpha_0,$$

Which is impossible because the left-hand side is at least 1 (since $\beta_0 \geq 0$ and $1 - L \geq 0$ while $\alpha_0 < 1$. Hence the only admissible limit $L = 0$. Consequently

$$\lim_{n\to\infty} d_n = 0.$$

We next proves that (α_n) is Cauchy in the A_b-sense. Fix integers $m > n$. Using the generalized triangle-type inequality of A_b-metric. with auxiliary point α_{k+1} repeatedly, one obtains the telescoping estimate

$$A_b\,(\underbrace{\alpha_n, \ \ldots \alpha_n,}_{N-1\ times} \alpha_m) \leq s \ \sum_{k=n}^{m-1} A_b\,(\underbrace{\alpha_k, \ \ldots \alpha_k,}_{N-1\ times} \alpha_{k+1}) = s \ \sum_{k=1}^{m-1} d_k,$$

Where $s \geq 1$ is the control of the A_b-metric. Because $d_k \to 0$ as $k \to \infty$ the tail sums $\sum_{k=n}^{\infty} d_k$ can be made arbitrarily small by choosing n large. Therefore for every $\varepsilon > 0$ there exists N_ε such that for all $m > n \geq N_\varepsilon$,

$$A_b\,(\underbrace{\alpha_n, \ \ldots \alpha_n,}_{N-1\ times} \alpha_m) \leq s \sum_{k=1}^{m-1} d_k < \varepsilon.$$

This shows (α_n) is a Cauchy sequence in (X, A_b). By completeness of (X, A_b) there exist $\alpha^* \in X$ with $\alpha_n \to \alpha^*$.

It remains to show α^* is a fixed point of Φ and that it is unique. Continuity of Φ is not assumed but the contraction property of the inverse suffices. Passing to the limit in the identity $\alpha_n = \Phi_{\alpha_{n+1}}$ (which holds for every n) and using the continuity of the map $\xi \to A_b(\underbrace{\xi, \ldots . \xi}_{N-1\ times}, \Phi\alpha^*)$ with respect to the first arguments (a consequence of the A_b-metric topology) we get

$$A_b\left((\underbrace{\alpha^*, \ldots ., \alpha^*,}_{N-1\ times} \ \Phi\alpha^*)\right) = \lim_{n\to\infty} A_b\,(\underbrace{\alpha_n, \ \ldots \alpha_n,}_{N-1\ times}, \Phi\alpha^*) = \lim_{n\to\infty} A_b\left((\underbrace{\Phi\alpha_{n+1}, \ldots . \Phi\alpha_{n+1},}_{N-1\ times} \ \Phi\alpha^*)\right)$$

Apply the rational contraction (4.1) for Φ^{-1} with $\xi = \Phi\alpha_{n+1}$ and $\eta = \Phi\alpha^*$ to obtain

$$A_b\left(\underbrace{(\alpha_{n+1},\ldots\alpha_{n+1}}_{N-1\ times},\alpha^*)\right) \le \frac{\alpha_0 A_b(\underbrace{\Phi\alpha_{n+1},\ldots\Phi\alpha_{n+1}}_{N-1\ times},\Phi\alpha^*)}{1+\beta_0(1-A_b(\underbrace{\Phi\alpha_{n+1},\ldots\Phi\alpha_{n+1}}_{N-1\ times},\Phi\alpha^*))}.$$

Letting $n \to \infty$ and using $\alpha_{n+1} \to \alpha^*$ together with continuity of the A_b-distance yields

$$A_b\left(\underbrace{(\alpha^*,\ldots,\alpha^*,}_{N-1\ times}\ \alpha^*)\right) \le \frac{\alpha_0 A_b(\underbrace{\Phi\alpha^*,\ldots,\Phi\alpha^*}_{N-1\ \text{times}},\Phi\alpha^*)}{1+\beta_0\left(1-A_b\left(,\underbrace{\Phi\alpha^*,\ldots,\Phi\alpha^*}_{N-1\ \text{times}},\Phi\alpha^*\right)\right)}.$$

By definiteness of the A_b-metric the left-hand side is zero , hence the numerator on the right hand side must vanish and we obtain

$$A_b\left(\underbrace{\Phi\alpha^*,\ldots,\Phi\alpha^*}_{N-1\ \text{times}}\Phi\alpha^*\right) = 0$$

Which by definiteness again implies $\Phi\alpha^* = \alpha^*$. Thus α^* is a fixed point of Φ.

Uniqueness follows from the contraction property of Φ^{-1}: if β^* is any fixed point of Φ then β^* is also a fixed point of Φ^{-1} , and applying (4.1) to the pair $(\Phi\beta^*, \Phi\alpha^*) = (\beta^*, \alpha^*)$ yields

$$A_b(\underbrace{\beta^*,\ldots\beta^*}_{N-1\ times},\alpha^*) \le \frac{A_b(\underbrace{\beta^*,\ldots\beta^*}_{N-1\ times},\alpha^*)}{1+\beta_0(1-A_b\left(\underbrace{\beta^*,\ldots\beta^*}_{N-1\ times},\alpha^*\right))}.$$

If $A_b\left(\underbrace{\beta^*,\ldots\beta^*}_{N-1\ times},\alpha^*\right) > 0$ this would force $1 \le \frac{\alpha_0}{1+\beta_0(1-A_b(.)} < 1$, a contradiction. Hence $A_b(\underbrace{\beta^*,\ldots\beta^*}_{N-1\ times},\alpha^*) = 0$ and $\beta^* = \alpha^*$. Finally, because the initial point α_0 was arbitrary, then backward Picard iteration $\alpha_{n+1} = \Phi^{-1}\alpha_n$ from any starting value converges to the unique fixed point α^*.

Corollary 4.2 (Iterate contraction). If there exists $m \in \mathbb{N}$ such that Φ^m is bijective and $(\Phi^m)^{-1}$ satisfies a linear rational contraction (as in (4.1)), then Φ has a unique fixed point.

Proof: Apply Theorem 4.1 to the bijection Φ^m.If $(\Phi^m)^{-1}$ is contractive in the rational sense, then Φ^m has a unique fixed point α^*. Since $\Phi(\Phi^{m-1}\alpha^*) = \alpha^*$ and Φ^m is bijective, one checks that α^* is invariant under Φ; uniqueness follows analogously.

Theorem 4.3: (Common fixed point for commuting maps with inverse contraction). Let $\Phi, \Psi : X \to X$ be commuting bijection on a complete A_b-metric space (X, A_b). If Φ^{-1} satisfies a linear rational contraction of the form (4.1) and Ψ commutes with Φ^{-1}, then Φ and Ψ have a unique common fixed point.

Proof: By Theorem 4.1 (Inverse-contraction principle) the bijection Φ admits a unique fixed point $\alpha^* \in X$; equivalently α^* is the unique fixed point of Φ^{-1}. We show that this α^* is also fixed point by Ψ and that it is the unique common fixed point of Φ and Ψ.

Since Ψ commutes with Φ^{-1}, the equality

$$\Phi^{-1}(\Psi\alpha^*) = \Psi(\Phi^{-1}\alpha^*)$$

Holds but α^* is a fixed point of Φ^{-1}, so $\Phi^{-1}\alpha^* = \alpha^*$; substituting this into the previous display gives

$$\Phi^{-1}(\Psi\alpha^*) = \Psi\alpha^*,$$

That is, $\Psi\alpha^*$ is a fixed point of Φ^{-1}. Uniqueness of the fixed point of Φ^{-1} then forces $\Psi\alpha^* = \alpha^*$

Thus α^* is a common fixed point of Φ and Ψ.

To see uniqueness of the common fixed point. Let $\beta \in X$ be any point with $\Phi\beta = \beta$ and $\Psi\beta = \beta$. The hypothesis of Theorem 4.1 already guarantees that Φ (and hence Φ^{-1}) has a unique fixed point so β

Must coincide with α^*. Therefore no point other than α^* can be a common fixed point of Φ and Ψ.

This completes the proof : Φ and Ψ. Have exactly one common fixed point, namely α^*.

5. Examples and Discussion

We given two short illustrative example. The details verify the hypotheses in familiar A_b-like constructions.

Example 5.1 Consider the real line $X = \mathbb{R}$ equipped with the A_b-distance defined by

$$.\ A_b(x, x, y) = |x - y|, \quad x, y \in \mathbb{R}.$$

This choice corresponds to the case $N = 2$, where the A_b-metric reduces to the usual absolute value metric. The self-map $\Phi : X \to X$ is defined by $\Phi(x) = 2x$. This map is bijective, since for every $y \in \mathbb{R}$ there exists a unique $x = {}^{y}/_{2}$ such that $\Phi(x) = y$. Its inverse $\Phi^{-1}(x) = {}^{x}/_{2}$.

We now verify that Φ^{-1} satisfies the linear rational contraction condition (4.1). Take arbitrary points $\xi, \eta \in \mathbb{R}$. Then

$$A_b(\Phi^{-1}\xi, \Phi^{-1}\eta) = \left|\frac{\xi}{2} - \frac{\eta}{2}\right| = \frac{1}{2}|\xi - \eta| = \frac{1}{2}A_b(\xi, \eta).$$

Thus the A_b-distance between the images under Φ^{-1} is exactly one-half the A_b-distance between the original points. This is precisely of the form required by the rational contraction condition with parameters $\alpha_0 = \frac{1}{2}$ and $\beta_0 = 0$. Indeed, the inequality

$$A_b(\Phi^{-1}\xi, \Phi^{-1}\eta) \le \frac{A_b(\xi,\eta)\alpha_0}{1+\beta_0(1-A_{(\xi,\eta)}}$$

Becomes

$$\frac{1}{2}A_b(\xi, \eta) \le \frac{\frac{1}{2}A_b(\xi,\eta)}{1+0A_b(\xi,\eta)}.$$

Which holds as an equality for all ξ, η. Hence Φ^{-1} is rational contraction of linear type.

Because Φ^{-1} satisfies the contraction hypothesis a complete A_b-metric space, Theorem 4.1 applies and ensure the existence of unique fixed point of Φ.Denote this point by α^*. To identify it, observe that $\Phi(\alpha^*) = \alpha^*$ implies $2\alpha^* = \alpha^*$,hence $\alpha^* = 0$ Therefore the unique fixed point of Φ. is the origin. Moreover, the backward Picard iteration defined by $\alpha_{n+1} = \Phi^{-1}\alpha_n = \frac{1}{2}\alpha_n$ converges to zero for any initial value $\alpha_0 \in \mathbb{R}$, confirming both existence and uniqueness of the fixed point as asserted.

Example 5.2 Let $X = [0,1]$ and define the A_b-distance for $N = 2$ by

$$.\ A_b(x, x, y) = |x - y|, \quad x, y \in [0,1].$$

This is simply the restriction of the usual absolute value metric to the unit interval which clearly satisfies the axioms of an A_b-metric spaces with constant $s = 1$. Define the map $\Phi : X \to X$ by by $\Phi(x) = 1 - x$. The map Φ is bijective on [0,1], since for every $y \in [0,1]$ there is exactly one $x = 1 - y$ *in* [0,1] such that $\Phi(x) = y$, and the inverse of Φ is itself ,that is, $\Phi^{-1} = \Phi$.

To study the contraction properties of Φ or its iterates, observe that $\Phi^2 = \Phi \circ \Phi$. is the identity map on [0,1]. Indeed, for every $x \in [0,1]$,

$$\Phi^2(x) = \Phi(\Phi(x)) = \Phi(1 - x) = 1 - (1 - x) = x,$$

So $\Phi^2(x) = x$. Therefore $(\Phi^2)^{-1} = \Phi^2 = Id$, the identity map on [0,1]. The identity map trivially satisfies the rational contraction condition (1.1), because for all $\xi, \eta \in [0,1]$ we have

$$A_b((\Phi^{-2})^{-1}\xi, (\Phi^2)^{-1}\eta) = A_b(\xi, \eta),$$

And this inequality holds with equality for all admissible parameters $\alpha_0 = 1$ *and* $\beta_0 = 0$. In particular, the inverse of Φ^2 is contractive in the degenerate sense that it does not enlarge A_b-distance, and Theorem 4.1 applies.

By the iterate version of the theorem, since $(\Phi^2)^{-1}$ satisfies the rational contraction condition, the mapping Φ itself must have a unique fixed point in [0,1]. To determine this fixed point explicitly, suppose $\Phi(\alpha^*) = \alpha^*$.Then $1 - \alpha^* = \alpha^*$, which simplifies to $2\alpha^* = 1$, hence $\alpha^* = \frac{1}{2}$. Thus, the unique fixed point of Φ is $\frac{1}{2}$.

Finally, note that this conclusion is consistent with the iteration of Φ: starting from any initial $\alpha_0 \in [0,1]$, successive applications of Φ alternate between α_0 and $1 - \alpha_0$, and hence the second iterate Φ^2 always returns to the original value. The fixed point $\frac{1}{2}$ is precisely the only point for which $\Phi(\alpha) = \alpha$ and consequently is the unique fixed point guaranteed by the theorem.

Example 5.3. Let (X, d) be any metric space and fix $N \ge 2$.Define

$$A_b(x_{1,}\ldots x_N) = \sqrt{\sum_{1\le i<j\le N}(d(x_i, x_j))^2}$$

We claim that (X, A_b) is an A_b-metric space

First $A_b(x_{1,}\ldots x_N) \ge 0$ by construction. Moreover, $A_b(x_{1,}\ldots x_N) = 0$ if and only if every $d(x_i, x_j) = 0$, which forces $x_{1,}\ldots = \cdots. = x_N$ .Thus the definiteness condition holds.

Next, symmetry is automatic, since the sum $\sum_{i<j} d(x_i, x_j)^2$ is invariant under any permutation of the argument. Therefore, A_b is symmetric in all N variables.

For the generalized triangle –type inequality, fix $\xi \in X$ and consider.
$A_b(x_{1,}\ldots x_N) = \sum_{1\le i<j\le N} d(x_i, x_j)^2$

By the usual triangle inequality in d, one has
$d(x_i, x_j) \le d(x_i, \xi) + d(\xi, x_j)$.

Squaring and applying the inequality $(a + b)^2 \le 2(a^2 + b^2)$ gives.
$d(x_i, x_j)^2 \le 2(d(x_i, \xi) + d(\xi, x_j)$.

Summing over all pairs $1 \le i < j \le N$ yields.
$\sum_{i<j} d((x_i, x_j)2 \le 2(N - 1)\sum_{k=1} d(x_k, \xi)^2$.

Therefore,
$A_b(x_{1,}\ldots x_N) \le \sqrt{2(N-1)}\ \sqrt{\sum_{k=1}^{N} d(x_k, \xi)2}$

Continue so on hence the generalized triangle inequality holds with constant $s = \sqrt{2}$.

By the theorem 4.1, it follows that Φ has a unique fixed point in $(\mathbb{R}, A_b)$. Solving $\Phi(x) = x$ gives $2x = x$, hence $x = 0$. Thus the o rigin is the unique fixed point of Φ under this quadratic A_b-metric.

Moreover, the backward Picard iteration $\alpha_{n+1} = \Phi^{-1}\alpha_n = \frac{1}{2}\alpha_n$ converges to 0 for every initial choice $\alpha_0 \in \mathbb{R}$. Thu s demonstrates explicitly how the main theorem guarantees both existence and uniqueness of the fixed point in a novel A_b-metric environment not previously considered in the literature.

6. Concluding Remarks and Open Problems

In this paper we introduced the expansive rational condition (ERC) in the framework of A_b-metric spaces and established several fixed point theorems ensuring existence and uniqueness of solutions under additional structural hypotheses such as inevitability of the operator, rational contraction of the inverse, commutativity, and the presence of suitable selection for multivalued maps. Our results extended the classical contraction- based principles into expansive regimes where forward behavior is non-contractive but can be compensated by inverse or iterate properties. The illustrative examples, including the naval quadratic A_b-metric, demonstrate the applicability of the theory beyond previously studied max-and sum type constructions. Several directions for further investigation remain open and may form the basis of future research.

1) **Refinement of control functions**. In our results the inverse maps satisfy linear rational contraction. It would be of interest to extend this framework to more general control functions (e.g.,φ-contractions, Geraghty-type conditions, or adaptive rational inequalities) within the A_b-metric context.
2) **Beyond bijectivity:** Theorems currently assume Φ is bijective so that Φ inverse exists globally. An important challenge is to weaken this assumption, for instance by requiring only surjectivity together with local or approximate inverses, while retaining fixed point existence and uniqueness.
3) **Iterative convergence rates:** While we proved convergence of back ward Picard iterations the quantitative rate of convergence remains unexplored. Establishing explicit error estimates in terms of the contraction parameters (α_0, β_0) would strengthen the practical applicability of the results.
4) **Extensions to multivalued and hybrid mappings:** The multivalued selection version demonstrates feasibility, but more general classes such as random operators, hybrid single /multi-valued maps, and non-convex selection deserve systematic study under ERC-type conditions.
5) **Novel A_b-metrics**. The quadratic A_b-metric presented here shows that non-linear aggregations of pairwise distance can be valid. Exploring further new constructions, such as geometric or harmonic mean – based A_b-metric, way provide additional testbeds for ERC-type fixed point theorems.
6) **Applications**. Potential areas of application include iterative algorithms in numerical analysis, dynamic systems with expanding forward maps but contractive inverse, and models in computer science or network theory where multi-argument distances naturally arise. Investigating ERC-type theorems in such applied setting would enhance their impact.

These problems indicate that the expansive rational condition is a flexible and fertile concepts, and its interaction with the still-developing structure of A_b-metric spaces promises several fruitful avenues for future research.

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This work is intended to serve as a foundation for further explorations of novel A_b-metrics and their applications in fixed point theory. The authors dedicate this contribution to researchers working at the interface of metric generalizations and nonlinear analysis, in the hope that it encourages the search for new structural conditions and concepts applications.

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