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A Generalized Finite Difference Structure on Arithmetic Power Arrays

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Abstract: ***We present a finite-difference framework, kind of built from a two-dimensional arithmetic power table. Fix integers a and d and also a nonnegative integer n; form a table so that the (k, j)-entry equals $(a + (j-1)d)^n$, meaning the k^{th} row just consists of the n^{th} powers of an arithmetic progression or something close to that. Let $S_n(k)$ be the sum of the entries in the k^{th} row. Elementary algebra shows that $S_n(k)$ ends up as a polynomial in k, with degree n. Then, using classical finite-difference theory and a bunch of combinatorial identities, we show that the $(n+1)^{th}$ forward difference $\Delta^{(n+1)} S_n(k)$ vanishes identically for all integer k, and we also write down explicit formulas for the lower-order differences using binomial sums. The paper then supplies precise definitions, plus illustrative examples for small n, and some structural observations about how the progression parameters a and d influence the coefficients. Finally, there is a complete self-contained proof of the main theorem, with everything laid out in a fairly direct way, not too hand-wavy.***

Keywords: Arithmetic Progression, Finite Difference, Polynomial Sequence, Difference Table

1. Introduction

Finite difference calculus, in a way, is kind of a basic corner of discrete mathematics and numerical analysis, with a lot of use in interpolation theory sequence analysis and numerical approximation, plus the investigation of discrete structures. One of the classical claims in this area says that if a sequence is built from a polynomial of degree n then the $(n+1)^{th}$ order finite difference of the sequence becomes identically zero. That bit works like a test, for whether something behaves like a polynomial, and it also ends up behind several numerical and analytical methods, for sure. Here, we present a new summation framework, made by taking arithmetic progressions and mixing them with power-based transformations. The idea builds a collection of row wise summation sequences, and each row shows its own numerical rhythm, along with some neat algebraic behavior. We then look into the finite difference behavior of those row sums, and check how the structure changes when we apply successive difference operations. Another part of the study is to figure out the exact order where the finite differences either stabilize or they just disappear, and also to search for possible links between this summation construction and the old polynomial based finite difference theory. Overall the findings offer a wider viewpoint on discrete summation patterns and might support generalized approaches in finite difference analysis, and in neighboring math applications too, somehow.

2. Basic Definition

1) *Definition 1: <u>Arithmetic Progression (AP)</u>*
A sequence of the form:
a, a+d, a+2d, ...
is called an arithmetic progression, where d is the common difference.

2) *Definition 2: <u>Finite Difference</u>*
If $\{M_r\}$ is a sequence, then the first finite difference is defined by: $\Delta M_r = M_{r+1} - M_r$.

Higher order differences are recursively defined as:
$\Delta^2 M_r = \Delta(\Delta M_r)$, $\Delta^3 M_r = \Delta(\Delta^2 M\_r)$, etc.

3) *Definition 3: <u>Polynomial Sequence</u>*
A sequence generated by a polynomial of degree n is called a polynomial sequence of degree n.

3. Construction of Difference Table

Define the table entries by:
$T(r, k) = \{\{a + (r-1) b\} + kd]^n$, where r, k = 0,1,2. . . . . (n−1)

The general table structure is:
Row 1:
$a^n + (a+d)^n + . . . + [a+(n-1)d]^n = M_1$

Row 2:
$(a+b)^n + [(a+b)+d]^n + ... + [(a+b)+(n-1)d]^n = M_2$

Row 3:
$(a+2b)^n + [a+2b)+d]^n + ... + [(a+2b)+(n-1)d]^n = M_3$
. . .
Last Row:
$[a+(n-1)b]^n + [\{a+(n-1)b\}+d]^n + ... + [\{a+(n-1)b\}+(n-1)d]^n = M_n$

<u>Row Sum Definition</u>
The row sum sequence is defined as:
$M_r = \sum_{k=0}^{n-1}[\{a + (r-1)b\} = kd]^n$
Each row is obtained by shifting the previous row by b units.

<u>Difference Table Structure</u>
The first finite differences are:
$D_1 = M_2 - M_1$
$D_2 = M_3 - M_2$
$D_3 = M_4 - M_3$

The second finite differences are:
$\Delta^2 M_r = \Delta(D_r)$

Continuing this process generates higher order finite difference tables.

<u>Main Theorem</u>
Theorem:
Let,

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$M_r = \sum_{k=0}^{n-1}[\{a + (r - 1)b\} = kd]^n$
Then M_r is a polynomial in r of degree n.
Consequently,
$\Delta^{(n+1)} M_r = 0$.

Proof of the Theorem
Each term:
$[\{a+(r-1)b\} + kd]^n$
Is a polynomial in r of degree n.
Since a finite sum of degree n polynomials remains a polynomial of degree n, the row sum M_r is also a polynomial of degree n.
By the classical finite difference theorem:
The (n+1)-th finite difference of a polynomial of degree n is identically zero.
Therefore, $\Delta^{(n+1)} M_r = 0$.
Hence proved.

Example:
for n=2, Then: $M_r = [a + (r−1)b]^2 + [a + (r−1)b + d]^2$.
After expansion, M_r becomes a quadratic polynomial in r.
Therefore: $\Delta^3 M_r = 0$.

Example:
for n=3, Then: $M_r = [a + (r−1)b]^3 + [a + (r−1)b + d]^3 + [a + (r−1)b + 2d]^3$.
After expansion, M_r becomes a cubic polynomial in r.
Therefore: $\Delta^4 M_r = 0$.

4. Observations

Observation 1:
The horizontal movement in the table follows common difference d.

Observation 2:
The vertical movement follows common difference b.

Observation 3:
The row sums form a polynomial sequence.

Observation 4:
The finite difference table stabilizes at order n and becomes zero at order (n+1).

5. Relation with Classical Finite Difference Theory

This construction is connected to Newton's finite difference theorem. The present framework provides a structured summation interpretation of polynomial finite difference behavior.

6. Conclusion

We constructed a two-dimensional arithmetic power table and analyzed the finite difference behavior of its row sums. The row sums generate polynomial sequences of degree n, implying that the $(n+1)^{th}$ finite difference always vanishes. This framework establishes a direct connection between arithmetic progressions, power sums, and finite difference calculus.