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A Convex Characterization of Bertrand's Postulate via a Legendre-Fenchel Type Transform

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Abstract: ***We establish an equivalence between Bertrand's postulate and a convex-analytic condition involving a Legendre-Fenchel-type transform derived from Euler's totient function. By encoding arithmetic information into a piecewise-affine function and analyzing its dual transform and subdifferential, we obtain a novel interpretation of the existence of primes in the interval*** $(\mathbf{n}, \mathbf{2n}]$***. This approach reveals an intrinsic connection between number theory and convex analysis.***

Keywords: Bertrand's postulate; Euler's totient function; Legendre-Fenchel transform; subdifferential; convex analysis

1. Introduction

Bertrand's postulate. It states that for every integer $n \geq 1$, there exists a prime number p such that

$$n < p \leq 2n.$$

This classical result was first established by Chebyshev in 1850. Later, Erdős provided an elementary proof by estimating the binomial coefficient $\binom{2n}{n}$ and arguing by contradiction. Ramanujan subsequently gave a significantly simpler proof, based on a delicate analysis involving the gamma function and Stirling's approximation, which makes use of the binomial coefficients $\binom{\lfloor x \rfloor}{\lfloor x/2 \rfloor}$ for $x \geq 1$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Meher and Murty later presented a refined version of Ramanujan's argument, elegantly eliminating the use of Stirling's formula.

The proof of Ramanujan relies on convexity properties of the logarithm of the gamma function, see, suggesting a deeper interplay between convexity and prime distribution. Motivated by this observation, we explore Bertrand's postulate through the framework of convex analysis, for instance, Rockafellar.

Let $\phi(n)$ denote Euler's totient function. If

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k},$$

then

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

We use the following classical properties, see Burton:

a) $\phi(n) = n$ if and only if $n = 1$,
b) $\phi(n) \leq n$ for all n,
c) $\phi(n) \leq n - \sqrt{n}$ for composite n.

A Legendre–Fenchel-Type Transform

For each integer $n \geq 2$, define

$$G_n(y) := \min_{m \in \{n, n+1, \ldots, 2n\}} (my - \phi(m)), \quad y \in \mathbb{R}.$$

Since the minimum of affine functions is concave, so is G_n. Moreover, this function can be viewed as a discrete analog of the Legendre-Fenchel transform.

Definition 1. *Let* $h: \mathbb{R} \to \mathbb{R}$ *be convex. A number* $\xi \in \mathbb{R}$ *is a subgradient of* h *at* x *if*

$$h(y) \geq h(x) + \xi(y - x), \quad \forall y \in \mathbb{R}.$$

The set of all such ξ *is called the subdifferential of* h *at* x*, denoted* $\partial h(x)$*.*

2. Main Result

Theorem 1. *For each integer* $n \geq 2$*, the following statements are equivalent:*

1) *There exists a prime* p *such that* $n < p \leq 2n$*.*
2) $G_n(1) = 1$.

Moreover, if p *and* q *denote the smallest and largest primes in* $(n, 2n]$*, then in a neighborhood of* $y = 1$*,*

$$G_n(y) = \begin{cases} qy - \phi(q), & y \leq 1, \\ py - \phi(p), & y \geq 1, \end{cases}$$

and consequently, $\partial(-G_n)(1) = [-q, -p]$.

Proof. We use the identity

$$m - \phi(m) = m\left(1 - \prod_{r|m}\left(1 - \frac{1}{r}\right)\right),$$

where the product is over distinct prime divisors of m.

For every integer $m \geq 2$,

$$m - \phi(m) = 1 \Leftrightarrow m \text{ is prime.}$$

Indeed, if $m = p$ is prime, then $\phi(p) = p - 1$. Conversely, if m is composite, then it has a prime divisor $r \leq m/2$, so

$$\phi(m) \leq m - \frac{m}{r} \leq m - 2,$$

hence, $m - \phi(m) \geq 2$.

Thus,

$$m - \phi(m) \geq 1 \quad \forall m \geq 2,$$

with equality if and only if m is prime.

Now,

$$G_n(1) = \min_{m \in \{n, \ldots, 2n\}} (m - \phi(m)).$$

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Therefore, $G_n(1) = 1$ if and only if there exists $m \in [n, 2n]$ such that $m - \phi(m) = 1$, which is equivalent to the existence of a prime in $(n, 2n]$.

For the second part, note that for primes r,

$$ry - \phi(r) = r(y - 1) + 1.$$

Among such affine functions, the smallest slope corresponds to p and the largest to q. Hence, for $y \geq 1$, the minimum is attained at p, while for $y \leq 1$, it is attained at q.

Thus, $-G_n$ is convex with one-sided derivatives $-q$ and $-p$ at $y = 1$, giving

$$\partial(-G_n)(1) = [-q, -p].$$

3. Conclusion

We have shown that Bertrand's postulate admits a natural convex-analytic formulation via a Legendre-Fenchel type transform. This approach provides a new perspective on prime distribution and suggests further applications of convex analysis in number theory.

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