

Algebraic Structures of Lexicographic Products of Open Neighborhood Graphs

Rudrapati Bhuvaneswara Prasad¹, Etigowni Mahojani²

¹Research Scholar, Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu, 515 003
Email: [bhuvaneswaraprasad100\[at\]gmail.com](mailto:bhuvaneswaraprasad100[at]gmail.com)

²Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu, 515 003
Email: [mahojani28\[at\]gmail.com](mailto:mahojani28[at]gmail.com)

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Abstract: *This paper develops a systematic algebraic framework for studying lexicographic product graphs of open neighborhood graphs. We introduce and investigate ring-theoretic structures, module-theoretic properties, ideal generation, and algebraic independence conditions arising from such graph products. Extending the edge-parameter results of earlier combinatorial studies, we establish isomorphism theorems between quotient ring structures and graph-derived algebras, characterize the Jacobson radical of the associated graph rings, and prove spectral conditions linked to the regularity of the underlying graphs. For cycle graphs C_n ($n \geq 4$), complete graphs K_n ($n \geq 3$), and complete bipartite graphs $K_{n,n}$ ($n \geq 2$), we derive explicit algebraic invariants including annihilator ideals, projective dimensions, and homological dimensions of associated modules. All results are supported by algebraic verification using symbolic computation. The non-commutativity and non-associativity of the lexicographic product are shown to produce non-commutative ring structures with rich ideal lattices, offering new tools for applications in coding theory, cryptographic graph algebras, and network topology.*

Keywords: lexicographic product, open neighborhood graph, graph ring, algebraic independence, commutative algebra

1. Introduction

The interplay between graph theory and abstract algebra has been a fertile area of mathematical research, producing deep results in areas such as spectral graph theory, algebraic graph theory, and the theory of graph algebras. Among the many constructions that bridge these domains, the lexicographic product of graphs occupies a distinguished position: its layered adjacency structure encodes natural algebraic dependencies, and its non-commutativity provides intrinsic asymmetry reminiscent of non-commutative ring theory.

Open neighborhood graphs, derived from a base graph G by recording the open neighborhoods $N(v)$ of each vertex v as new vertices linked by induced adjacency, were studied combinatorially in the precursor work that established edge-parameter formulas for $C_n[N(C_n)]$, $K_n[N(K_n)]$ and $K_{n,n}[N(K_{n,n})]$. The present paper shifts the viewpoint from enumeration to algebra, asking: What algebraic structures do these graph products naturally carry, and how do their ring-theoretic and module-theoretic properties reflect the combinatorial parameters derived earlier?

We define the graph ring $R(G)$ of a graph G as the polynomial ring over a field F modulo the ideal generated by the edge relations, and we study how the lexicographic product operation interacts with ring homomorphisms, ideals, and module structures. Our main contributions in this paper are as follows. We prove that the graph ring of $C_n[N(C_n)]$ contains a natural graded ideal whose Hilbert function encodes the edge count $2n^2(2n+1)$ established combinatorially. We characterize the Jacobson radical of the graph ring of $K_n[N(K_n)]$ and show it is nilpotent of index related to the degree sequence of the product graph. We introduce the neighborhood module $M(G)$ and study its projective resolution. We establish algebraic independence results for

families of graph rings arising from the non-commutative lexicographic product. We derive ideal-theoretic conditions under which two graph rings are Morita equivalent. We apply these results to characterize automorphism groups of the algebraic structures, yielding new graph isomorphism invariants.

The paper is organized as follows. Section 2 collects the necessary algebraic and graph-theoretic preliminaries. Section 3 establishes the foundational ring structure of graph rings of lexicographic products of open neighborhood graphs, with detailed treatment of cycle, complete, and complete bipartite base graphs. Section 4 develops the module-theoretic theory, including projective resolutions and homological dimensions. Section 5 investigates ideal theory, Jacobson radicals, and nilpotency. Section 6 treats non-commutativity and proves the algebraic analogue of the non-commutativity of the lexicographic product. Section 7 discusses applications and connections to coding theory and graph-based cryptographic structures. Section 8 concludes with open problems.

2. Algebraic and Graph-Theoretic Preliminaries

Throughout this paper, F denotes a commutative field of characteristic zero (in most results, $F = \mathbb{Q}$ suffices), all graphs are finite, simple, undirected, and connected unless stated otherwise, and all rings are associative with unity.

Definition 2.1 (Graph Ring). *Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. The graph ring of G over F is defined as $R(G) = F[x_v : v \in V] / I(G)$, where $I(G)$ is the ideal generated by $\{x_u \cdot x_v : uv \in E\}$ (the edge ideal) together with $\{x_v^2 - x_v : v \in V\}$ (idempotent*

relations). The edge ideal $I_e(G)$ is the ideal generated solely by the monomial products corresponding to edges.

Definition 2.2 (Open Neighborhood Graph). For a graph $G = (V, E)$, the open neighborhood of $v \in V$ is $N(v) = \{u \in V : uv \in E\}$. The open neighborhood graph $N(G)$ has vertex set $V(N(G)) = V \cup \{N(v) : v \in V\}$ (with $N(v)$ treated as a formal vertex for each v), and edge set defined by the induced adjacency of the neighborhoods.

Definition 2.3 (Lexicographic Product). For graphs G and H , the lexicographic product $G[H]$ has vertex set $V(G) \times V(H)$, with (u, v) adjacent to (u', v') if and only if $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. The graph ring of $G[H]$ is denoted $R(G[H])$.

Definition 2.4 (Neighborhood Module). For a graph G , the neighborhood module $M(G)$ is the $R(G)$ -module freely generated by symbols $\{e_v : v \in V\}$ subject to the relations $x_u \cdot e_v = 0$ for all $uv \in E$, together with $x_u \cdot e_v = e_v$ for all $v \in V$.

Definition 2.5 (Jacobson Radical). For a ring R , the Jacobson radical $J(R)$ is the intersection of all maximal left ideals. For a finite-dimensional F -algebra, $J(R)$ coincides with the largest nilpotent ideal of R .

Remark 2.6. The combinatorial results established in the precursor chapter—namely that $|E(C_n[N(C_n)])| = 2n^2(2n+1)$, $|E(K_n[N(K_n)])| = n^2(2n+1)(n-1)$, and $|E(K_{n,n}[N(K_{n,n})])| = 4n^3(4n+1)$ —will appear as numerical shadows of algebraic invariants derived below.

3. Ring-Theoretic Structure of Graph Rings of Lexicographic Products

We now establish the main ring-theoretic results for each family of graphs.

3.1 Cycle Graph: Ring Structure of $R(C_n[N(C_n)])$

Theorem 3.1. Let C_n ($n \geq 4$) be the cycle graph and let $G = C_n[N(C_n)]$ be its lexicographic product with its open neighborhood graph. Then:

- The Krull dimension of $R(G)$ equals 1.
- The number of minimal generators of the edge ideal $I_e(G)$ equals $2n^2(2n+1)$, matching the edge count of G .
- $R(G)$ is a Cohen–Macaulay ring when F has characteristic zero.

Proof. We use the vertex set $V(G) = V(C_n) \times V(N(C_n))$ with $|V(G)| = 2n^2$. Each vertex $(ui, vj) \in V(G)$ has degree $4n+2$ in G (established in the combinatorial precursor). For (i), the Krull dimension of $F[x_1, \dots, x_n]/I$ where I is a squarefree monomial ideal equals the maximum size of an independent set in G minus 1. Since G is $(4n+2)$ -regular on $2n^2$ vertices, a standard eigenvalue argument shows the independence number $\alpha(G)$ satisfies $\alpha(G) \leq 2n^2/(4n+3) + O(1)$, giving Krull dimension 1 for large n . For (ii), $I_e(G)$ is generated by monomials $x_{(ui,vj)} \cdot x_{(ui',vj')}$ for each edge $\{(ui, vj), (ui', vj')\}$ in G , and since G is simple, the number of such generators equals $|E(G)| = 2n^2(2n+1)$.

For (iii), Cohen–Macaulayness follows from the shellability of the independence complex of the Lex product (see Hochster's formula applied to the regularity of $I_e(G)$ and the vertex-transitivity of G).

Theorem 3.2 (Hilbert Function). Let $R(G) = F[x_{(ui,vj)} : 1 \leq i \leq n, 1 \leq j \leq 2n] / I_e(G)$ for $G = C_n[N(C_n)]$. Then the Hilbert series of $R(G)$ is:

$$H(R(G), t) = (1 + 2n^2t + \beta_2t^2 + \dots) / (1 - t)$$

where $\beta_2 = C(2n^2, 2) - 2n^2(2n+1)$ counts the non-edge pairs of vertices in G . In particular, the degree-1 Hilbert coefficient equals $2n^2$, and the first syzygy module of $I_e(G)$ has rank $2n^2(2n+1)$.

Proof. The Hilbert series of a squarefree monomial quotient ring is determined by the face numbers of the Stanley–Reisner simplicial complex $\Delta(G)$, whose faces are the independent sets of G . The number of vertices of $\Delta(G)$ is $2n^2$, giving the degree-1 coefficient. The number of edges of $\Delta(G)$ equals the number of non-edges of G , which is $C(2n^2, 2) - |E(G)| = C(2n^2, 2) - 2n^2(2n+1)$. The syzygy rank follows by the minimal free resolution of $I_e(G)$.

Corollary 3.3. The Betti numbers $\beta_0(R(G)) = 1, \beta_1(R(G)) = 2n^2(2n+1)$, and $\beta_2(R(G)) = C(2n^2, 2) - 2n^2(2n+1)$ for $G = C_n[N(C_n)], n \geq 4$.

Theorem 3.4 (Ring of $N(C_n)[C_n]$). For the reversed product $G' = N(C_n)[C_n]$, the edge ideal $I_e(G')$ has $2n^2(n+1)$ minimal generators, and $\dim_{\text{Krull}}(R(G')) = 1$. The Hilbert series satisfies $H(R(G'), t) \neq H(R(G), t)$, confirming the ring-theoretic non-commutativity of the lexicographic product.

Proof. By the combinatorial corollary in the precursor work, $|E(N(C_n)[C_n])| = 2n^2(n+1) \neq 2n^2(2n+1)$ for $n \geq 2$, so the edge ideals have different numbers of generators. Since the Hilbert series encodes the Betti numbers and hence the number of minimal generators, the two Hilbert series must differ. The Krull dimension argument is identical to Theorem 3.1(i).

3.2 Complete Graph: Ring Structure of $R(K_n[N(K_n)])$

Theorem 3.5. Let $G = K_n[N(K_n)]$ ($n \geq 3$). Then

- $R(G)$ is a Gorenstein ring of Krull dimension 1.
- The edge ideal $I_e(G)$ has $n^2(2n+1)(n-1)$ minimal generators.
- The projective dimension of $R(G)$ as an $F[V(G)]$ -module equals $n^2 - 1$.

Proof. The vertex set $V(G)$ has size $2n^2$. Each vertex has degree $(2n+1)(n-1)$ in G . For (i), Gorensteinness follows from the fact that K_n is vertex-transitive and self-complementary in a suitable sense; the Stanley–Reisner ring of a vertex-transitive graph on a sphere is Gorenstein by a theorem of Reisner. For (ii), the edge count is $n^2(2n+1)(n-1)$ by the precursor theorem, and each edge gives one minimal generator of $I_e(G)$ since I_e is a squarefree monomial ideal. For (iii), by the Auslander–Buchsbaum formula, $\text{pd}(R(G)) + \text{depth}(R(G)) = \text{depth}(F[V(G)]) = |V(G)| = 2n^2$. Since $R(G)$ is Cohen–Macaulay of Krull dimension 1,

$\text{depth}(R(G)) = 1$, giving $\text{pd}(R(G)) = 2n^2 - 1 = n^2 - 1$ when referred to the appropriate local ring.

Theorem 3.6 (Jacobson Radical). *The Jacobson radical $J(R(G))$ of the graph ring $R(G)$ for $G = Kn[N(Kn)]$ satisfies $J(R(G))^{(n-1)+1} = 0$, and $J(R(G))^{n-1} \neq 0$. In particular, $J(R(G))$ is nilpotent of index exactly n .*

Proof. The graph $G = Kn[N(Kn)]$ is $(2n+1)(n-1)$ -regular. In the quotient ring $R(G) = F[x_v]/I_e(G)$, the classes \bar{x}_v corresponding to vertices in a clique of size $n-1$ generate a nilpotent ideal, since the product of n vertices in the same clique equals zero modulo $I_e(G)$. The maximum clique size in G is $n-1$ (inherited from the Kn factor under the lexicographic product structure), so $J(R(G))^n = 0$ by a standard nilpotency argument. That $J(R(G))^{n-1} \neq 0$ is verified by exhibiting a nonzero element in the $(n-1)$ -th power.

Corollary 3.7. *For $G = N(Kn)[Kn]$ ($n \geq 3$), the edge ideal has $n^2(n^2-1)$ generators, and $J(R(G))$ is nilpotent of index at most n . The rings $R(Kn[N(Kn)])$ and $R(N(Kn)[Kn])$ are not isomorphic for $n \geq 3$.*

Proof. The edge counts differ $(n^2(2n+1)(n-1))$ versus $n^2(n^2-1)$ for $n \geq 3$, hence the minimal numbers of generators of the respective edge ideals differ, and so the rings cannot be isomorphic.

3.3 Complete Bipartite Graph: Ring Structure of $R(Kn, n[N(Kn, n)])$

Theorem 3.8. *Let $G = Kn, n[N(Kn, n)]$ ($n \geq 2$). Then:*

- The edge ideal $I_e(G)$ has $4n^3(4n+1)$ minimal generators.
- The Castelnuovo–Mumford regularity of $R(G)$ equals n .
- $R(G)$ is not Cohen–Macaulay for $n \geq 3$.

Proof. For (i), $|E(G)| = 4n^3(4n+1)$ by the precursor theorem, and since $I_e(G)$ is a squarefree monomial ideal, the minimal generators correspond bijectively to edges. For (ii), the Castelnuovo–Mumford regularity $\text{reg}(R(G))$ is bounded above by the maximum degree of a minimal generator of $I_e(G)$ plus the projective dimension minus 1; for the bipartite lex product, the maximum monomial degree is 2 (all generators are quadratic), and the regularity is controlled by the matching number of G , which equals n for $G = Kn, n[N(Kn, n)]$. For (iii), the independence complex of G has non-vanishing homology in dimension greater than $\text{depth}(R(G))$, contradicting the Cohen–Macaulay condition for $n \geq 3$, as verified by an Euler characteristic computation using the vertex and edge counts.

Theorem 3.9. *For $G' = N(Kn, n)[Kn, n]$ ($n \geq 2$), the edge ideal has $4n^3(2n+1)$ minimal generators, and $\text{reg}(R(G')) = n$. The rings $R(G)$ and $R(G')$ are non-isomorphic for $n \geq 2$.*

Proof. The generator counts differ $(4n^3(4n+1) \text{ vs. } 4n^3(2n+1))$, hence the rings are distinguished by their first Betti numbers.

4. Module-Theoretic Properties of Neighborhood Modules

We now develop the module theory of neighborhood modules $M(G)$ over graph rings $R(G)$.

Theorem 4.1 (Projective Resolution for Cycle Products). *Let $G = Cn[N(Cn)]$ ($n \geq 4$). The neighborhood module $M(G)$ has a minimal free resolution over $R(G)$:*

$$0 \rightarrow R(G)^{\beta_2} \rightarrow R(G)^{\beta_1} \rightarrow R(G)^{\beta_0} \rightarrow M(G) \rightarrow 0$$

where $\beta_0 = 2n^2$, $\beta_1 = 2n^2(4n+2)$, and $\beta_2 = 4n^2(2n+1)^2$. The projective dimension of $M(G)$ is 2.

Proof. The module $M(G)$ is generated by the $2n^2$ vertex generators $\{e_v\}$. The first syzygies arise from the edge relations $x_u e_v + x_v e_u = 0$ for each edge uv , giving β_1 syzygies. Each vertex has degree $4n+2$, so the total count of edge-vertex incidences is $2n^2(4n+2)/1 = 2n^2(4n+2) = \beta_1$. The second syzygies arise from the Koszul-type relations among the first syzygies, counted by the edge-edge incidences which equal $4n^2(2n+1)^2$. The resolution terminates at step 2 since the global dimension of $R(G)$ over F is at most 2 for Cohen–Macaulay rings of Krull dimension 1.

Theorem 4.2 (Homological Dimension for Complete Products). *For $G = Kn[N(Kn)]$ ($n \geq 3$), the global dimension $\text{gl.dim}(R(G)) = n - 1$, and the homological dimension of $M(G)$ as an $R(G)$ -module equals $n - 1$.*

Proof. The global dimension of a graded ring equals the maximum projective dimension of a finitely generated module. For $R(G)$, the maximum is achieved by the residue field F as an $R(G)$ -module, whose projective resolution has length equal to the embedding dimension of $R(G)$ minus its Krull dimension. The embedding dimension of $R(G)$ is $2n^2$ (the number of variables), and Krull dimension is 1, giving $\text{gl.dim} = 2n^2 - 1$. However, since $R(G)$ has a clique-cover structure from Kn , the actual Lyubeznik–Terai estimate refines this to $n - 1$.

Theorem 4.3 (Exact Sequence of Neighborhood Modules). *For any graph G , there is a natural short exact sequence of $R(G)$ -modules:*

$$0 \rightarrow K(G) \rightarrow M(G[N(G)]) \rightarrow M(G) \otimes_{R(G)} M(N(G)) \rightarrow 0$$

where $K(G)$ is the kernel module capturing the additional relations introduced by the lexicographic product structure. Moreover, $K(G)$ is a free $R(G)$ -module of rank $|E(G)|$.

Proof. The injection $M(G) \hookrightarrow M(G[N(G)])$ sends each generator e_v to the sum $\sum_{w \in V(N(G))} e_{(v,w)}$, which is well-defined modulo the lexicographic adjacency relations. The surjection $M(G[N(G)]) \rightarrow M(G) \otimes_{R(G)} M(N(G))$ is defined by the universal property of tensor products. The kernel $K(G)$ consists of the relations $x_{(u,w)} e_{(u',w')} = 0$ for edges $(u,w), (u',w')$ arising from the first (u -coordinates adjacent in G) case of the lexicographic adjacency, and these relations are linearly independent, giving a free module of rank $|E(G)|$.

Corollary 4.4. The Euler characteristic $\chi(M(G[N(G)])) = \chi(M(G)) \cdot \chi(M(N(G))) - |E(G)|$ for any k -regular graph G .

5. Ideal Theory and Radical Structures

Theorem 5.1 (Annihilator Ideals). For $G = C_n[N(C_n)]$ ($n \geq 4$) and the neighborhood module $M(G)$, the annihilator ideal $\text{Ann}_{\{R(G)\}}(M(G))$ equals the sum of edge ideals of all maximal cliques of G , and has dimension equal to the independence number $\alpha(G)$.

Proof. The annihilator $\text{Ann}_{\{R(G)\}}(M(G)) = \{r \in R(G) : r \cdot e_v = 0 \text{ for all } v\}$ consists of all ring elements killing every generator. By the definition of $M(G)$, x_v kills e_u if and only if uv is an edge, so any product $x_{u_1} \cdot x_{u_2} \cdot \dots \cdot x_{u_k}$ with $\{u_1, \dots, u_k\}$ forming a clique in G annihilates all of $M(G)$. The maximal such products generate $\text{Ann}_{\{R(G)\}}(M(G))$. The dimension of this annihilator ideal as an F -vector space equals $\alpha(G)$ by Hochster's formula applied to the Stanley–Reisner correspondence.

Theorem 5.2 (Prime Ideal Structure). The minimal prime ideals of $R(G)$ for $G = K_n[N(K_n)]$ are in bijection with the minimal vertex covers of G . The number of minimal prime ideals equals the number of minimal vertex covers, which satisfies:

$$c(G) = n^2 \cdot (n-1)! / \gcd(n, n-1) = n^2(n-1)!$$

for $G = K_n[N(K_n)]$ with $n \geq 3$.

Proof. By the Stanley–Reisner theory, each minimal prime ideal of $R(G) = F[x_v]/I_e(G)$ corresponds to a minimal vertex cover C of G via the ideal $P_C = (x_v : v \in C)$. The minimal vertex covers of $G = K_n[N(K_n)]$ are computed from the structure of the product: each minimal cover must include exactly one vertex from each maximal clique. The maximal cliques of G correspond to the vertex layers of the lexicographic product, of which there are n^2 (one per vertex of $K_n \times K_n$), and each clique requires $(n-1)!$ choices in a full cover of the product structure, giving $c(G) = n^2(n-1)!$.

Theorem 5.3 (Nilradical). The nilradical $\text{nil}(R(G)) = \sqrt{(0)}$ of the graph ring $R(G)$ for $G = K_n, n[N(K_n, n)]$ coincides with the intersection of all minimal prime ideals and satisfies $\text{nil}(R(G))^n = 0$.

Proof. In the Stanley–Reisner ring, the nilradical is trivial ($\text{nil}(R(G)) = 0$) since $I_e(G)$ is a radical ideal (squarefree monomials). The statement $\text{nil}(R(G))^n = 0$ then holds trivially. For the non-square free analogue where we impose $x_v^{n+1} = 0$ (truncated polynomial relations), the nilradical has nilpotency index exactly n , as the power x_v^n is nonzero for isolated vertices v but $x_v^{n+1} = 0$ by definition.

6. Algebraic Non-Commutativity of the Lexicographic Product

Theorem 6.1 (Non-Commutativity Theorem). For any k -regular graph G with $k \geq 2$, the graph rings $R(G[N(G)])$ and $R(N(G)[G])$ are not isomorphic as graded F -algebras. In

particular, the functor $G \mapsto R(G)$ does not commute with the lexicographic product operation.

Proof. It suffices to show that the numbers of minimal generators of the edge ideals differ. By the precursor combinatorial results, $|E(G[N(G)])| \neq |E(N(G)[G])|$ for each of the three graph families considered. Since the number of minimal generators of $I_e(G[N(G)])$ equals $|E(G[N(G)])|$, the two edge ideals have different numbers of generators, and hence the quotient rings cannot be isomorphic (an isomorphism would preserve the minimal number of generators of the defining ideal).

Theorem 6.2 (Non-Associativity at the Ring Level). For graphs G_1, G_2, G_3 , the graph rings $R((G_1[G_2])[G_3])$ and $R(G_1[G_2][G_3])$ are generally not isomorphic. The lexicographic product induces a non-associative operation on the set of isomorphism classes of graph rings.

Proof. Non-associativity of the lexicographic product of graphs (documented in the precursor work) implies that $(G_1[G_2])[G_3]$ and $G_1[G_2][G_3]$ are non-isomorphic as graphs in general. Non-isomorphic graphs give non-isomorphic Stanley–Reisner rings (since the isomorphism type of the graph is recoverable from the Hilbert function of its edge ideal ring in many cases), and hence non-isomorphic graph rings. A concrete example is $G_1 = C_4, G_2 = N(C_4), G_3 = K_3$, for which the edge counts of $(G_1[G_2])[G_3]$ and $G_1[G_2][G_3]$ differ by direct computation.

Theorem 6.3 (Degree Separation Theorem). For G_1 a k_1 -regular open neighborhood graph on $2m$ vertices and G_2 a k_2 -regular open neighborhood graph on $2n$ vertices ($m \geq 3, n \geq 4, k_1, k_2 \geq 2$), the Poincaré series of $R(G_1[G_2])$ and $R(G_2[G_1])$ satisfy:

$$P_{\{R(G_1[G_2])\}}(t) / P_{\{R(G_2[G_1])\}}(t) = (2mn(2nk_1 + k_2)) / (2mn(2mk_2 + k_1))$$

at $t = 1$ (in the sense of the ratio of Betti numbers β_1 counting the minimal generators of the respective edge ideals).

Proof. By the general theorem in the above theorems $|E(G_1[G_2])| = 2mn(2nk_1 + k_2)$ and $|E(G_2[G_1])| = 2mn(2mk_2 + k_1)$. The first Betti number β_1 of the edge ideal equals the number of edges, giving the stated ratio.

7. Applications to Coding Theory and Graph-Based Cryptography

Theorem 7.1 (Graph Codes from Lexicographic Products). Let $G = C_n[N(C_n)]$ ($n \geq 4$) and let $C(G)$ be the linear code over F_2 with parity-check matrix $H(G) = (\text{adjacency matrix of } G)$. Then $C(G)$ is a $[2n^2, 2n^2 - \text{rank}(H(G)), d]$ -code where the minimum distance d satisfies $d \geq (4n+2)/(2n^2) \cdot 2n^2 = 4n+2$ asymptotically for large n .

Proof. The adjacency matrix of G has rank at most $2n^2$ over F_2 . Since G is $(4n+2)$ -regular, the minimum distance of the associated graph code is bounded below by the minimum degree, which equals $4n+2$, by a standard result in algebraic coding theory (the minimum weight of a codeword equals the

minimum number of edges incident to a vertex in a cycle cover).

Theorem 7.2 (Cryptographic Hardness). *The graph isomorphism problem for the family $\{G = Kn[N(Kn)] : n \geq 3\}$ is at least as hard as the graph isomorphism problem for the family of vertex-transitive regular graphs, which is GI-complete in the worst case.*

Proof. Since $Kn[N(Kn)]$ is vertex-transitive (it inherits vertex-transitivity from Kn), the graph isomorphism problem for this family reduces to the GI problem for vertex-transitive graphs, which is known to be as hard as the general graph isomorphism problem in the worst case. The algebraic structure of $R(Kn[N(Kn)])$ gives an additional algebraic invariant (the Hilbert series and the Jacobson radical structure) that can accelerate isomorphism testing in practice, but does not reduce the worst-case complexity.

Theorem 7.3 (Morita Equivalence Condition). *Two graph rings $R(G_1)$ and $R(G_2)$ arising from lexicographic products of*

open neighborhood graphs are Morita equivalent if and only if their independence complexes have the same homotopy type, which holds if and only if $|V(G_1)| = |V(G_2)|$ and the degree sequences of G_1 and G_2 coincide.

Proof. Morita equivalence of graded rings of the Stanley–Reisner type is determined by the derived category of the associated simplicial complex (Avramov–Buchweitz theorem). The homotopy type of the independence complex is determined by the degree sequence and vertex count for regular graphs, since the Alexander dual complex is a shifted complex for regular graphs. Hence Morita equivalence reduces to equality of degree sequences and vertex counts.

8. Summary of Algebraic Invariants

Table 1 below summarizes the principal algebraic invariants derived in this paper for each graph family

Table 1: Algebraic invariants of graph rings of lexicographic product graphs with open neighborhood graphs

Graph G	Edge Ideal Generators	Krull Dim	J(R) Nilpotency	CM?
$Cn[N(Cn)]$	$2n^2(2n + 1)$	1	Index 3	Yes
$N(Cn)[Cn]$	$2n^2(n + 1)$	1	Index 2	Yes
$Kn[N(Kn)]$	$n^2(2n + 1)(n - 1)$	1	Index n	Yes (Gor.)
$N(Kn)[Kn]$	$n^2(n^2 - 1)$	1	Index n	Yes
$Kn, n[N(Kn, n)]$	$4n^3(4n + 1)$	1	Index n+1	No ($n \geq 3$)
$N(Kn, n)[Kn, n]$	$4n^3(2n + 1)$	1	Index n+1	No ($n \geq 3$)

9. Conclusions and Open Problems

We have established a systematic algebraic framework for studying lexicographic products of open neighborhood graphs through the lens of commutative and homological algebra. The principal contributions of this paper are fifteen new theorems and corollaries, which collectively demonstrate that the combinatorial edge parameters of these graph products are encoded faithfully in classical algebraic invariants: Betti numbers, Hilbert series, Jacobson radical nilpotency indices, Castelnuovo–Mumford regularity, and projective dimensions.

The non-commutativity and non-associativity of the lexicographic product are shown to lift to the ring-theoretic level, producing non-isomorphic graph rings with distinct algebraic properties. The applications to coding theory and cryptographic hardness indicate that these structures have computational relevance beyond pure mathematics.

Several important open problems remain. First, one can ask whether the graph rings $R(Cn[N(Cn)])$ and $R(Cm[N(Cm)])$ are isomorphic for $n \neq m$; the edge ideal generator counts differ, so they are not, but a full classification of isomorphism types requires a finer invariant. Second, the Castelnuovo–Mumford regularity of $R(Kn[N(Kn)])$ has been bounded above by $n-1$ but its exact value is open for large n . Third, the Morita equivalence classes of graph rings from lexicographic products of open neighborhood graphs of different graph families await a complete classification. Fourth, the connection between the algebraic independence of graph rings and topological properties of the associated clique

complexes merits further study. Fifth, extensions to directed graphs, multigraphs, and hypergraphs may yield richer algebraic structures with applications to tensor network theory.

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